Characterization
of tracer cascade
induced by geostrophic eddies

I Evolution of tracer gradient in simple flows

- Basic Mechanisms

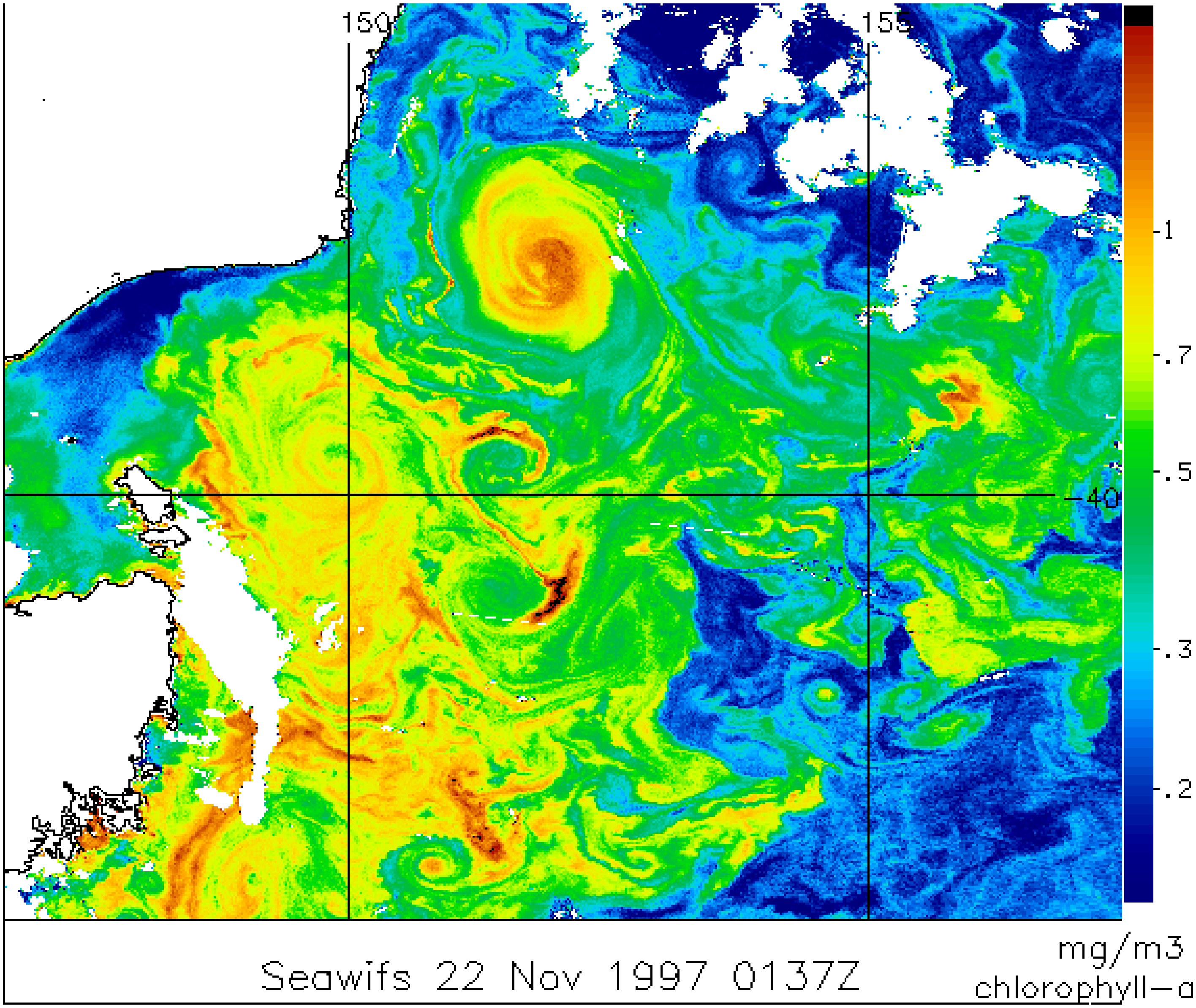
II Dynamics of tracer gradient

- Alignment with specific orientations

  which depend on $\nabla\vec{u}$ and $\nabla\frac{D\vec{u}}{Dt}$

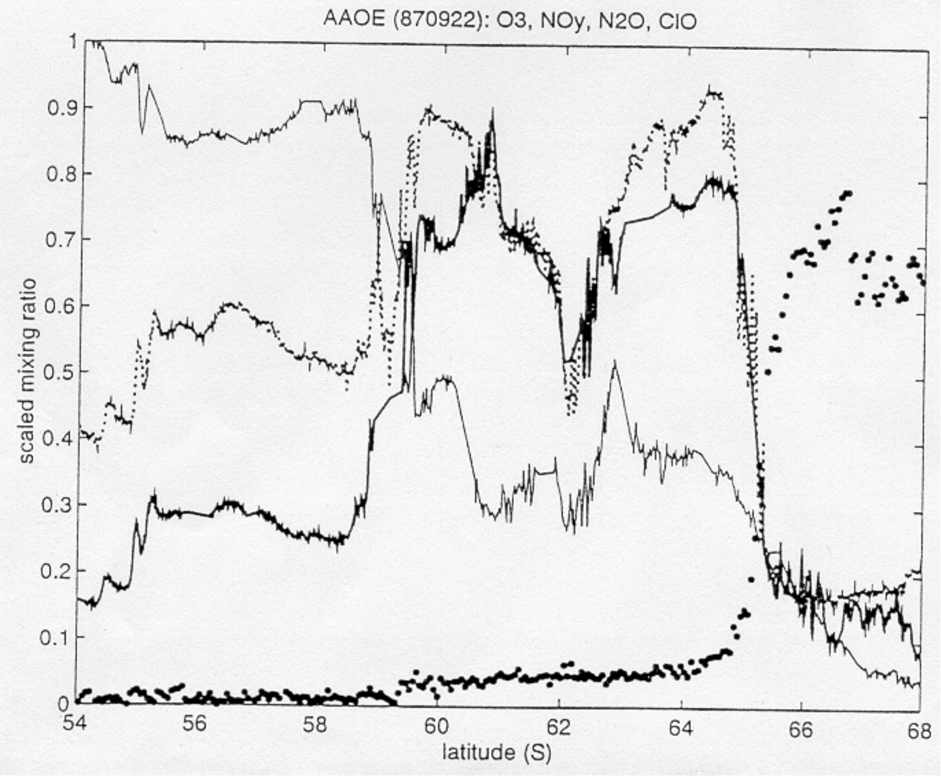
III Numerical simulations

- Statistical validation of alignement properties
- Cascade in physical space: **Palinstrophy production**



Tracer Cascade toward small spatial scales:

- The tracer field results from the stirring by the mesoscale eddies
- Production of small scales ⇒ strong horizontal gradients



From Tuck (1989)

- Horizontal gradients of different tracer fields at **same locations in physical space**

  ⇒ result from the flow topology

Approach :

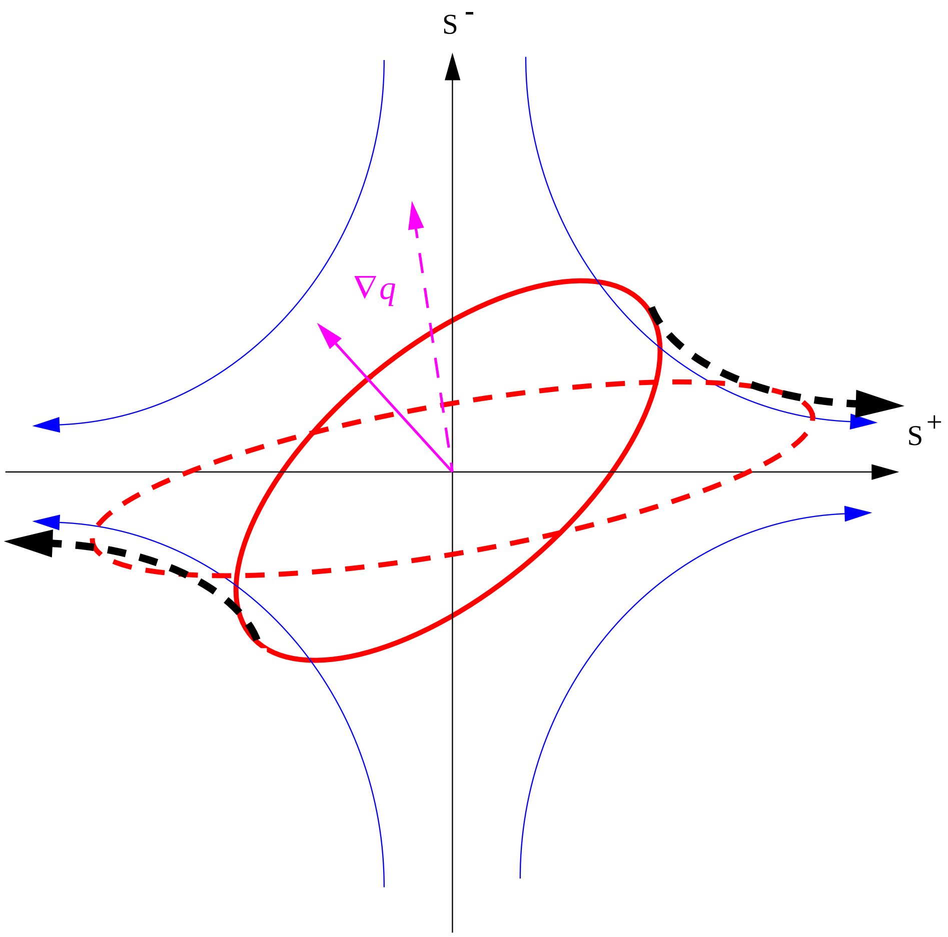
- use information from fields of $\vec{u}$ and $\frac{D\vec{u}}{Dt}$ (velocity AND acceleration)
  $\Rightarrow$ dynamics of tracer gradient
- Mesoscale eddies $\approx$ quasigeostrophic turbulence

Neglect diffusion

Lagrangian reference frame

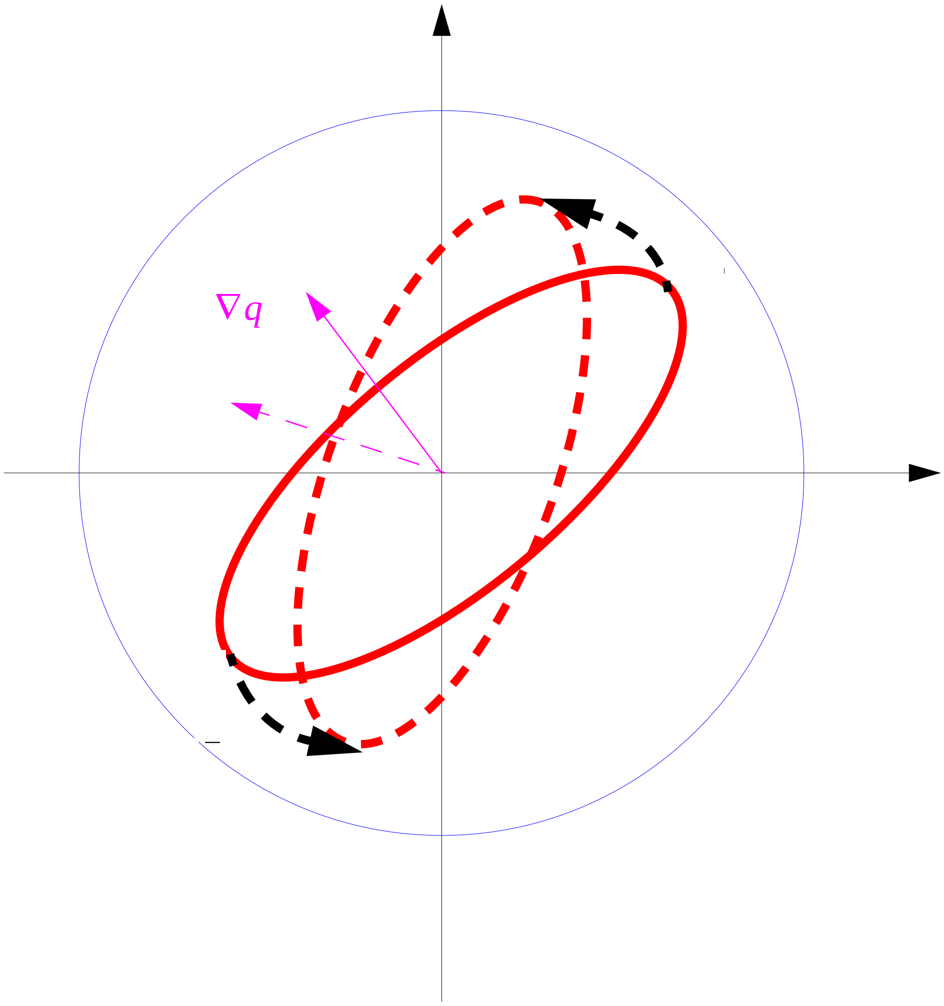
I Simple flows

Pure strain field



- $\psi = \sigma \, xy$
- Alignment of tracer gradient with compressional axis of strain tensor
- Strong growth of tracer gradient norm

Pure vorticity field



- $\psi = \frac{\omega}{2}(x^2 + y^2)$
- Rotation of tracer gradient
- No growth of tracer gradient norm

Results of Okubo and Weiss

passive tracer : $\dfrac{Dq}{Dt} = 0$

$\dfrac{D\nabla q}{Dt} = -[\nabla\vec{u}]^* \; \nabla q$ eigenvalues : $\pm \, \lambda^{1/2}$

Okubo (1970) and Weiss (1981) :

$$\nabla q \approx \nabla q_0 \; \exp(\pm\lambda^{1/2} \; t)$$

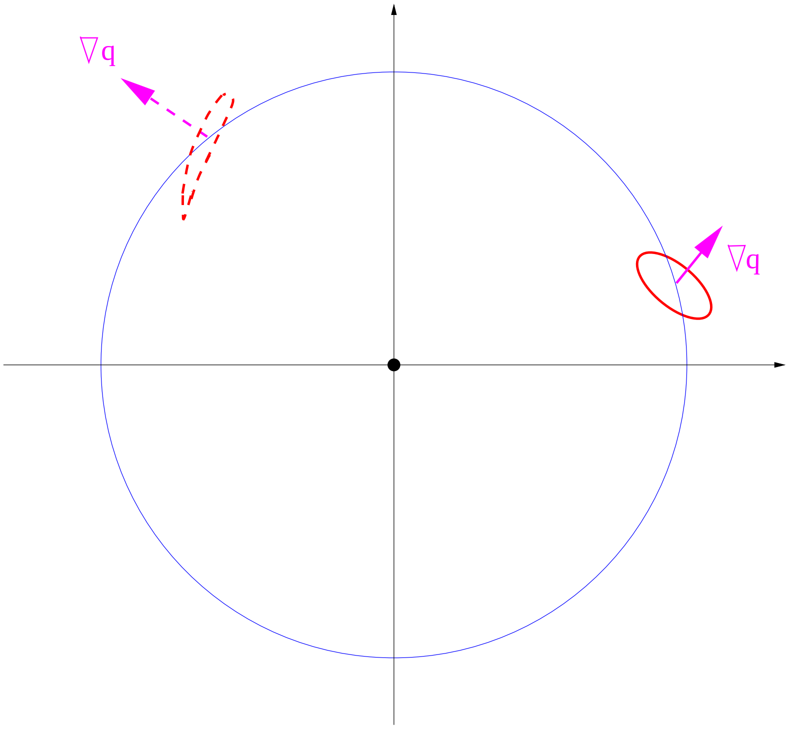
$$\lambda = (\text{strain rate})^2 - (\text{vorticity})^2$$

- $\lambda > 0$ (strain rate dominates): exponential growth
- $\lambda < 0$ (vorticity dominates): rotation of gradients

Assumption :

- $[\nabla\vec{u}]^*$ varies slowly along Lagrangian trajectories

Finite size axisymmetric vortex



Outside axisymmetric vortex,

vorticity $\omega = 0$ and strain rate $\sigma \neq 0$

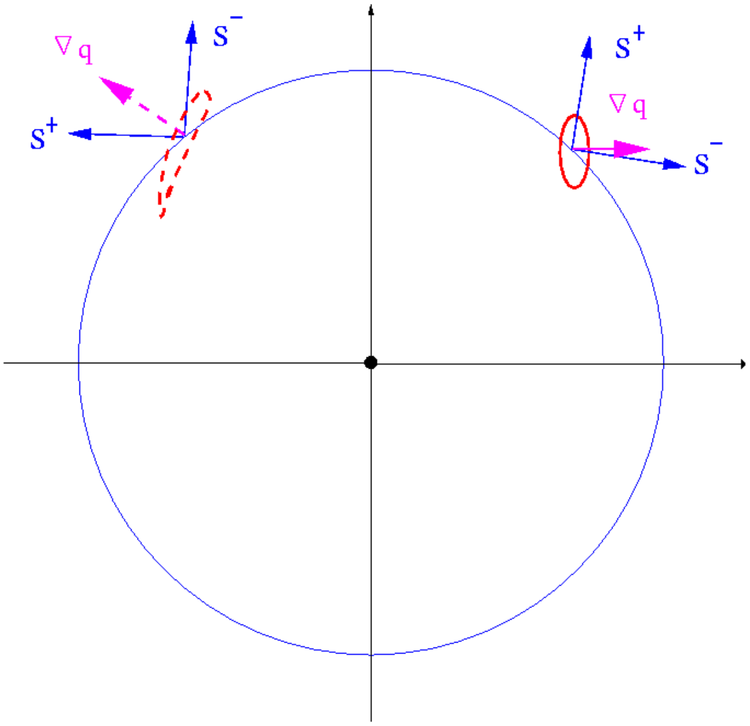
Rotation of gradient and **linear** growth of tracer gradient norm

$\lambda = \sigma^2 - \omega^2 = \sigma^2 > 0$ → **exponential** growth

Okubo-Weiss criterion fails

because the rotation of strain tensor axis $\dfrac{D\nabla\vec{u}}{Dt}$

has not been taken into account



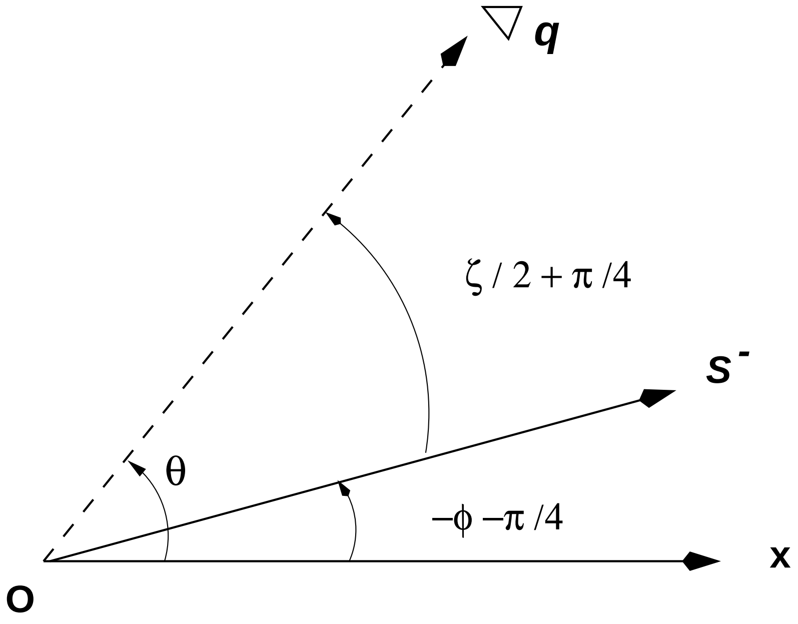
∇q
s⁻
s⁺
s⁺
∇q
s⁻

II Dynamics of tracer gradient

$$\frac{D\nabla q}{Dt} = -[\nabla\vec{u}]^{*}\nabla q = -\frac{1}{2}\begin{pmatrix} \sigma_n & \sigma_s + \omega \\ \sigma_s - \omega & -\sigma_n \end{pmatrix}\nabla q$$

$$\nabla q = |\nabla q| \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\begin{pmatrix} \sigma_n \\ \sigma_s \end{pmatrix} = \sigma \begin{pmatrix} \sin 2\phi \\ \cos 2\phi \end{pmatrix}$$



2 scalar equations for the norm $|\nabla q|$ and the gradient orientation θ

$$\frac{D\log|\nabla q|^2}{Dt} = -\sigma\sin(2(\theta+\phi))$$

$$2\frac{D\theta}{Dt} = \omega - \sigma\cos(2(\theta+\phi))$$

Where does $\frac{D\nabla\vec{u}}{Dt}$ come into play?

Equation for the orientation of tracer gradient

$$2\frac{D\theta}{Dt} = \omega - \sigma\cos(2(\theta+\phi))$$

Relative angle between ∇q

and the strain rate tensor axis

$$\boxed{\zeta = 2(\theta+\phi)}$$

Nondimensional

Lagrangian Time

$$\tau = \int_0^t \sigma(t')dt'$$

$$\boxed{\frac{D\zeta}{D\tau} = r - \cos\zeta}$$

$$r = \frac{\omega + 2D\phi/Dt}{\sigma} = \frac{\text{"effective rotation"}}{\text{strain rate}}$$

$$\boxed{\frac{D\phi}{Dt} \quad \textbf{related to} \quad \frac{D\nabla\vec{u}}{Dt}}$$

Strain-dominated regions

$$\boxed{|\mathbf{r}| < 1}$$

$$\frac{D\zeta}{D\tau} = r - \cos\zeta$$

$$r = \frac{\omega + 2D\phi/Dt}{\sigma}$$

Assumption : r is slowly varying

- Two fixed points $\zeta_{\pm} = \pm \arccos r$
  - stable orientation ζ_{-} $\Rightarrow$ growth of $|\nabla q|$
  - unstable orientation ζ_{+} $\Rightarrow$ decay of $|\nabla q|$
- Alignment of tracer gradient with the stable orientation of ζ_{-}
- Exponential growth of gradients norm

Lapeyre, Klein and Hua, 1999, Physics of Fluids

“Effective rotation”-dominated regions

$$\boxed{|\mathbf{r}| > 1}$$

$$\frac{D\zeta}{D\tau} = r - \cos\zeta$$

$$r = \frac{\omega + 2D\phi/Dt}{\sigma}$$

- non uniform rotation of gradient ($\frac{D\zeta}{Dt}$ is variable)
- the gradient spends most of its time near the direction with minimal rotation rate ($D^2\zeta/Dt^2 = 0$)
- The most probable orientation of this direction is α such that

$$\alpha = \arctan\left(\frac{s}{r}\right) + (1 - sign(r))\frac{\pi}{2})$$

$$s = -\frac{D(\sigma^{-1})}{Dt}$$

σ^{-1} : stirring time scale

Klein, Hua and Lapeyre, 2000, Physica D

Summary of Dynamics of tracer gradient

$$r = \frac{\omega + 2D\phi/Dt}{\sigma}$$

$$s = -\frac{D\left(\sigma^{-1}\right)}{Dt}$$

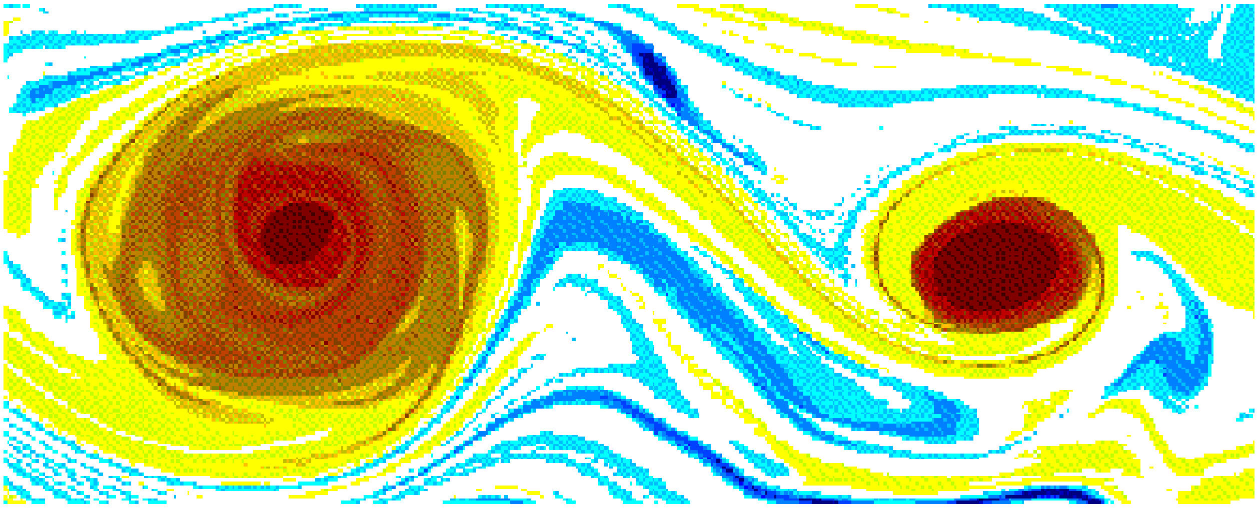
- $|r| \leq 1$: strain rate dominates
  - Alignment with the orientation ζ_- (r)
  - Exponential growth of $|\nabla q|$ for $|r| < 1$
  - Linear growth of $|\nabla q|$ for $|r| = 1$

- $|r| > 1$: effective rotation dominates
  - non uniform rotation
  - statistical alignment with the orientation α (r and s)
  - weak growth or decay

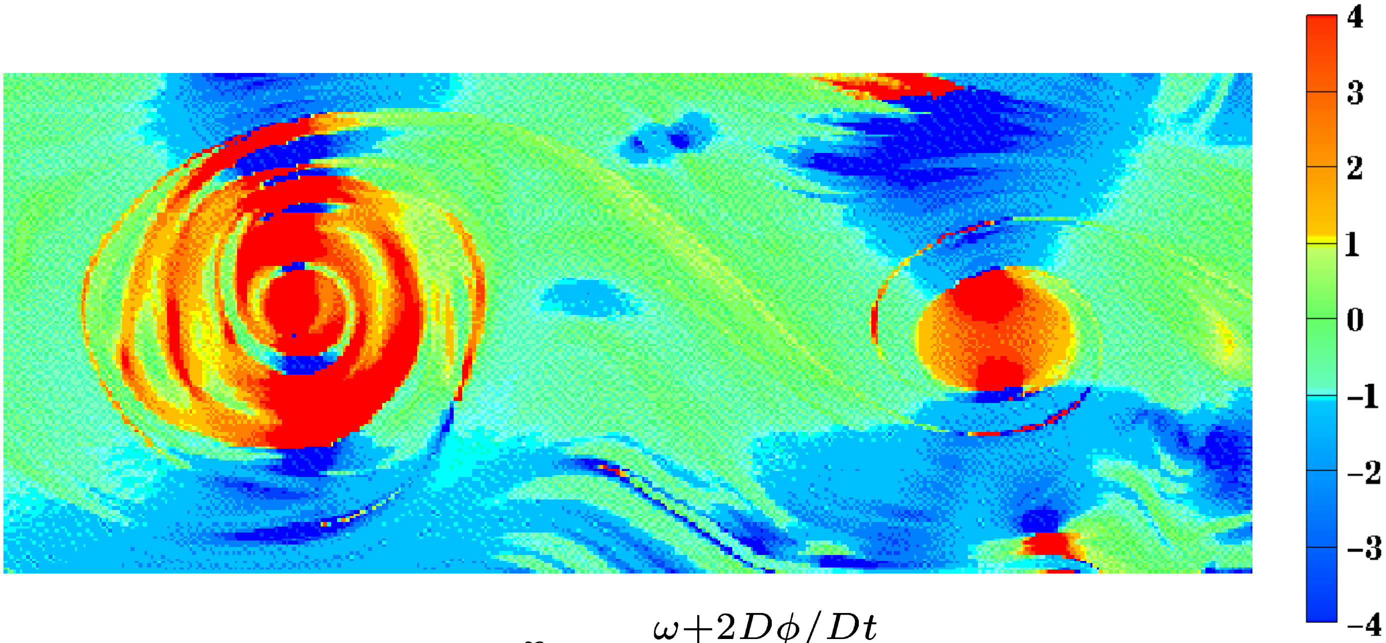
- r and s depend on $\nabla\vec{u}$ and $\frac{D\nabla\vec{u}}{Dt}$
  - saddle point : $r = 0$ and $s = 0$
  - axisymmetric vortex : $|r| = 1$ and $s = 0$
  - strong rotation : $|r| >> 1$

Both r and s are independent of the reference frame

III Numerical Simulations (1024^2)



vortictiy ω



$$r = \frac{\omega + 2D\phi/Dt}{\sigma}$$

- green: strain dominates
- blue and red: effective rotation dominates

  red ↔ ω blue ↔ $2D\phi/Dt$
- yellow ↔ $|r| = 1$

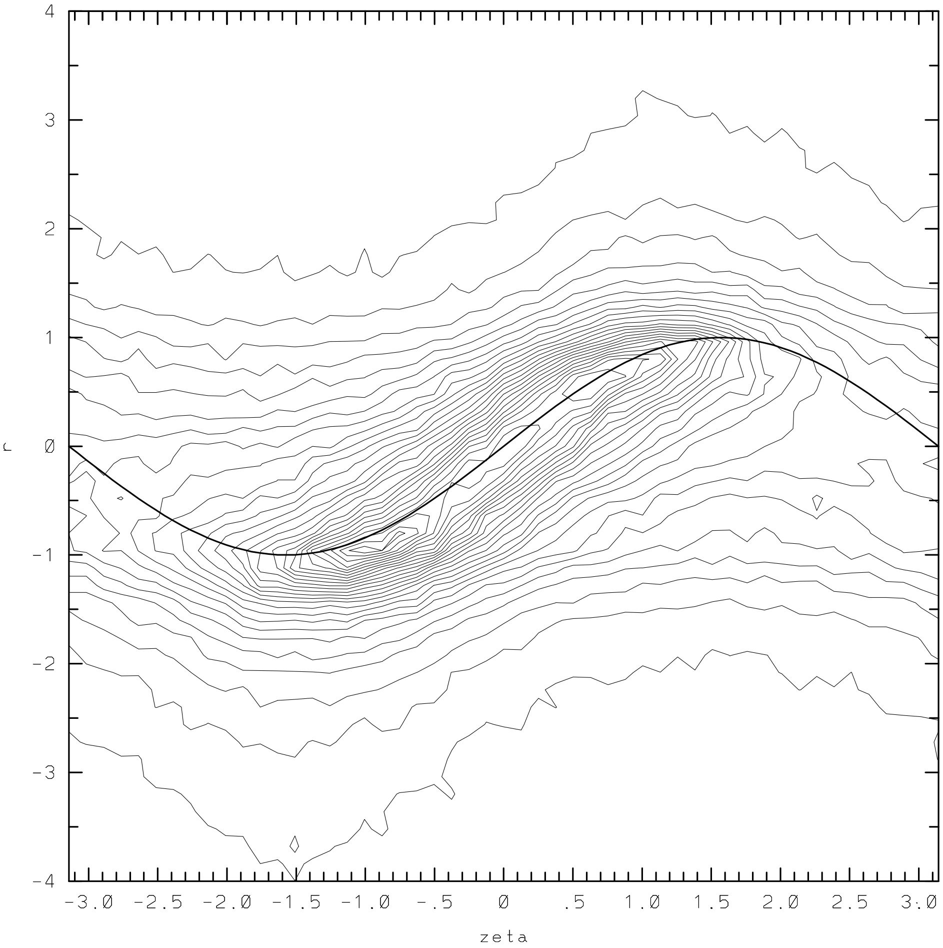
Active tracer (vorticity ω) and passive tracer (C) rapidly show the same orientations in their gradients at specific locations of the flow

Lapeyre, Hua and Klein, Physics of Fluids, 2001

(a) **vorticity gradient**

(c) **tracer gradient**

Alignment in strain-dominated regions

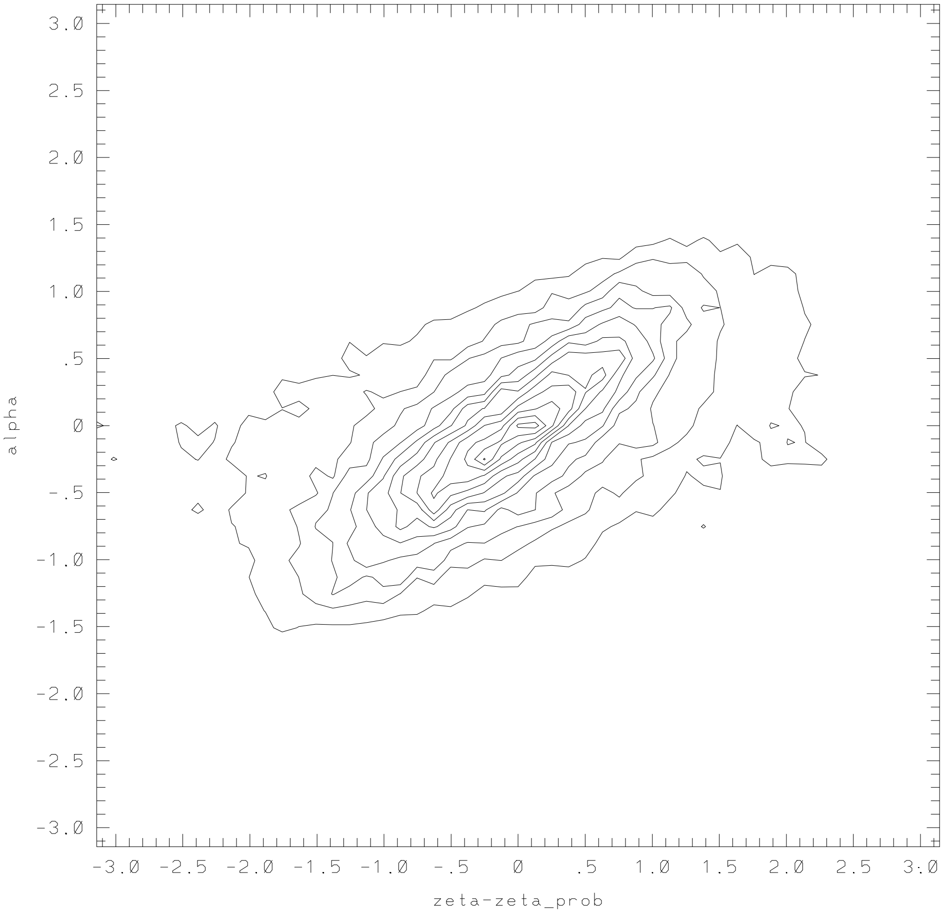


Joint PDF of $\zeta + \pi/2$ **and** $r = \frac{\omega + 2\frac{D\phi}{Dt}}{\sigma}$

$$\frac{D\zeta}{D\tau} = r - \cos\zeta$$

$\zeta \approx \zeta_{-} \Rightarrow$ **predicted dynamical alignment**

Alignment in “effective rotation”-dominated regions



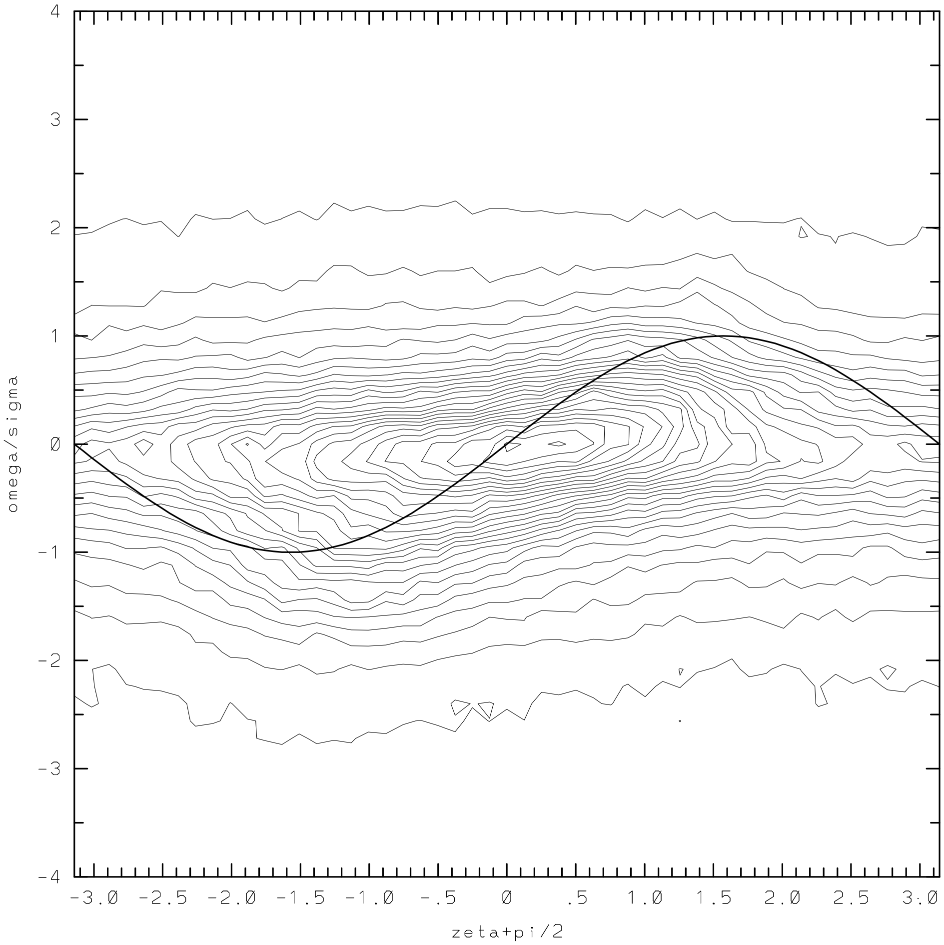
Joint PDF of ζ and α

$$\alpha = \arctan(\frac{s}{r}) + (1 - sign(r))\frac{\pi}{2})$$

$\zeta \approx \alpha \Rightarrow$ **predicted dynamical alignment**

Alignment dans strain-dominated regions

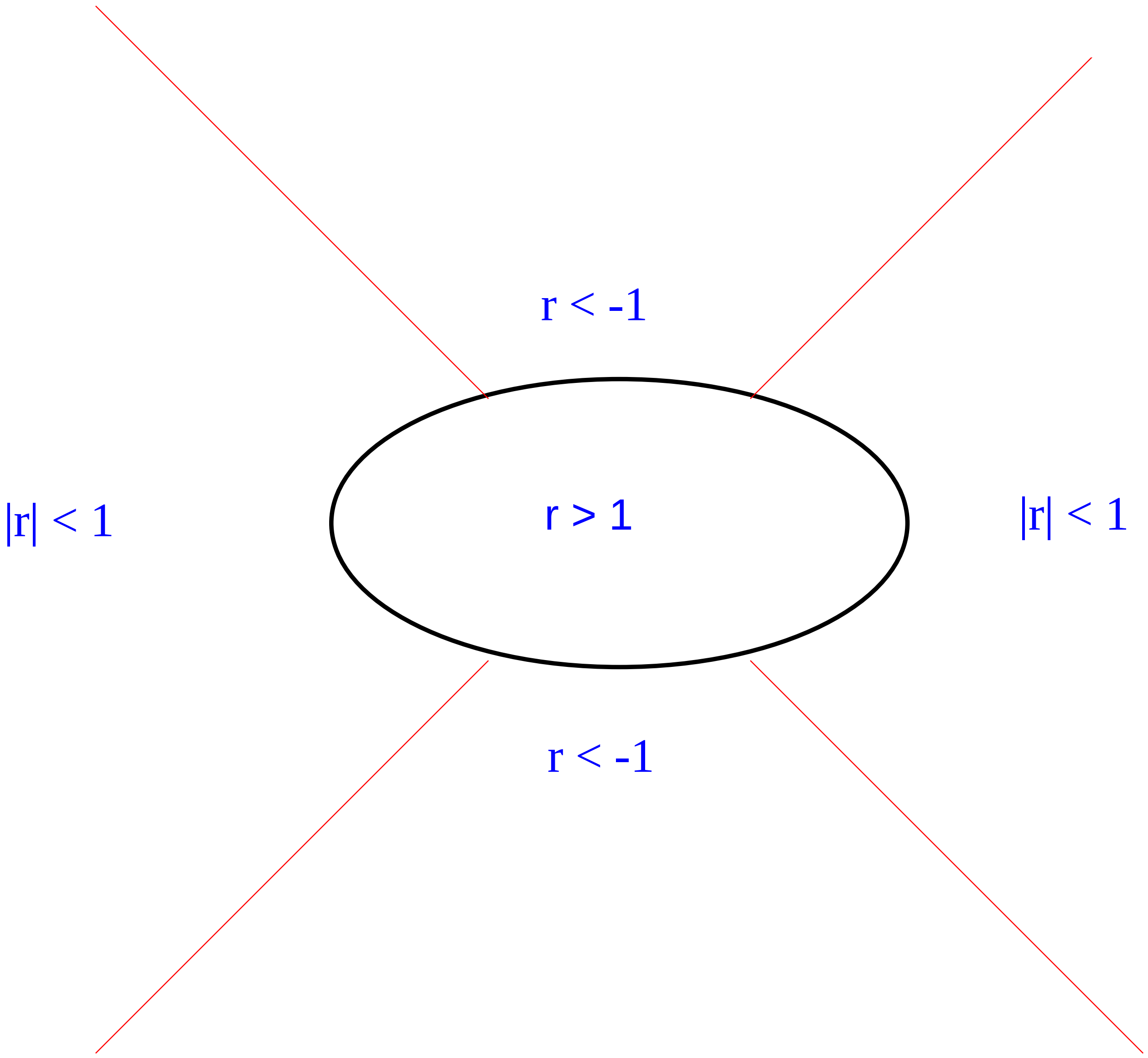
for Okubo-Weiss prediction ($\frac{D\phi}{Dt} = 0$)



Joint PDF of $\zeta + \pi/2$ **and** ω/σ

No alignment occurs for Okubo-Weiss criterion

Distribution of strain-dominated and effective-rotation dominated regions for a uniform elliptic vortex (positive)



Palinstrophy production

- Gradient enhancement of vorticity ω, $P_s \equiv \frac{D\,|\nabla\omega|^2}{Dt}$ (palinstrophy production) is such that its domain-averaged value measures the strength of the cascade of enstrophy ($|\omega|^2$) to ever smaller scales.
- Spatial patterns of palinstrophy production/destruction are quadrupoles for *non-uniform* elliptic vortex (Kimura and Herring, 2001, JFM)

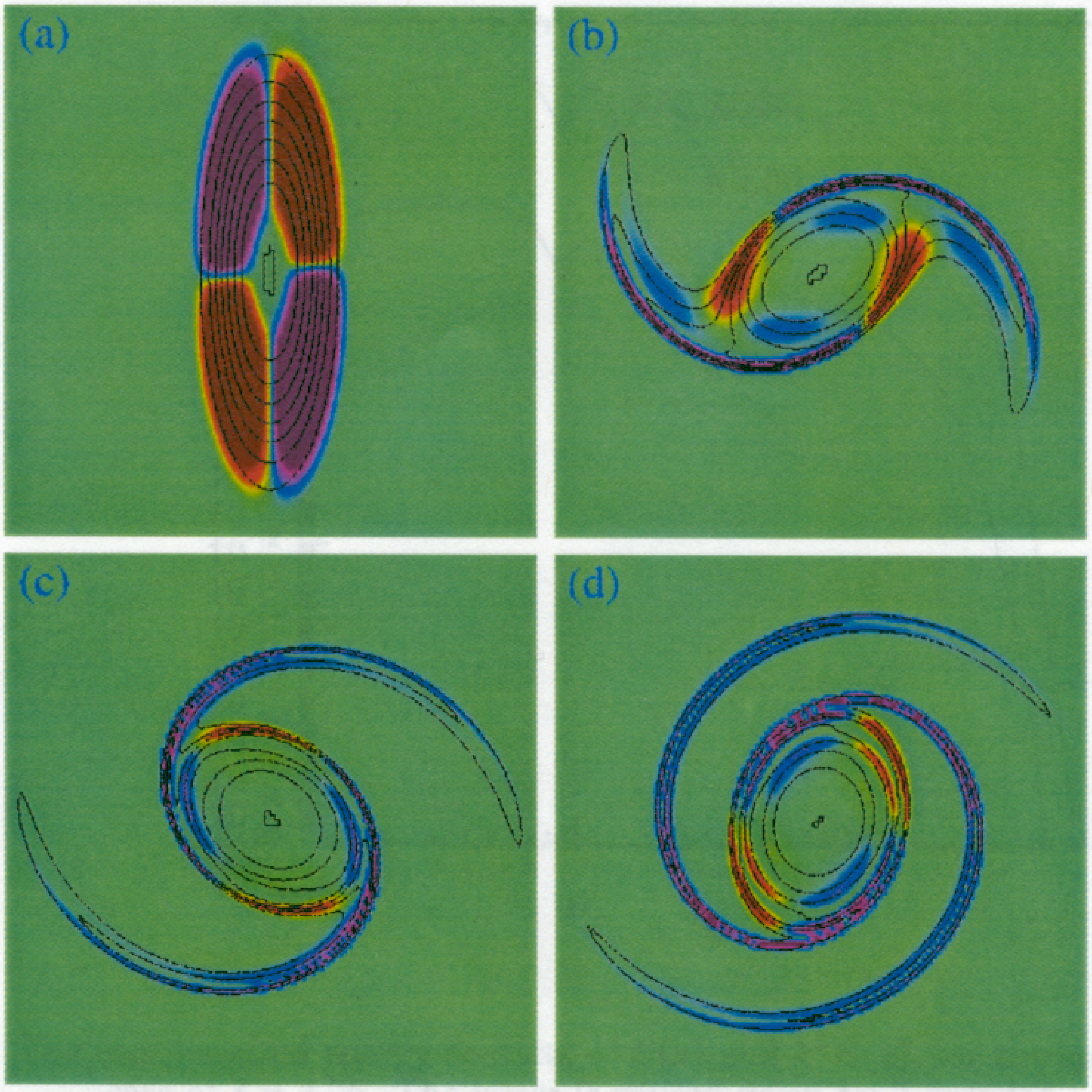
  elliptic streamfunction

  $\Psi(x,y) = F(ax^2 + by^2)$ $\quad$ F differentiable function

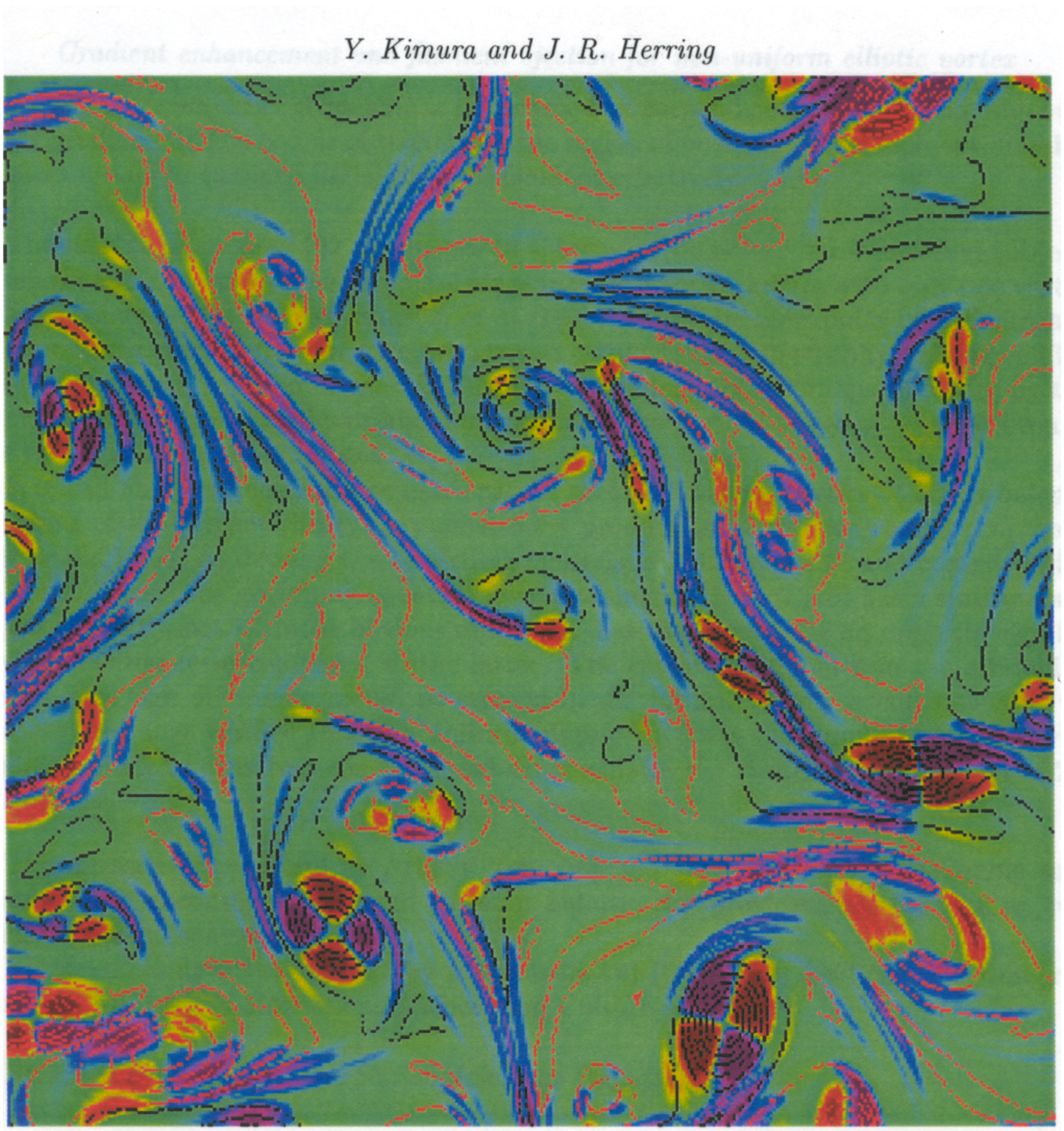
  $\Longrightarrow \quad P_s = ab(a-b)\underset{\substack{| \\ xy}}{G}(x^2, y^2, a, b)$, G functional of F.

  - $a = b$, $P_s = 0$ no palinstrophy production for circular patterns
  - x, y axes separate $P_s > 0$ and $P_s < 0$: quadrupole pattern

Y. Kimura and J. R. Herring



Palinstrophy production P_s (colored) and contours of vorticity



Free-decay turbulence with random initial elliptic vortex conditions

Generically seen that regions of strong palinstrophy production ($P_s > 0$): purple-blue) correspond to spiral filamentary extrusions from regions of near elliptically organized vortices.

Estimation of palinstrophy production

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\boldsymbol{\nabla} q|^2 &= -\boldsymbol{\nabla} q^* \boldsymbol{S} \boldsymbol{\nabla} q \\ \frac{1}{2}\frac{d^2}{dt^2}|\boldsymbol{\nabla} q|^2 &= \boldsymbol{\nabla} q^* \boldsymbol{N} \boldsymbol{\nabla} q\end{aligned}$$

the expression of $\boldsymbol{S}$ and $\boldsymbol{N}$ *in the strain coordinates* become :

$$\boldsymbol{S_{strain}} \equiv \frac{\sigma}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad , \quad \boldsymbol{N_{strain}} \equiv \frac{\sigma^2}{2}\begin{bmatrix} 1-s & r \\ r & 1+s \end{bmatrix}$$

In terms of the gradient norm ρ :

$$\begin{aligned}\frac{1}{\rho^2}\frac{d}{dt}\rho^2 &= -\sigma\sin\zeta \\ \frac{1}{\rho^2}\frac{d^2}{dt^2}\rho^2 &= \sigma^2(1-\chi\cos(\zeta-\alpha))\end{aligned}$$

with χ and α defined by :

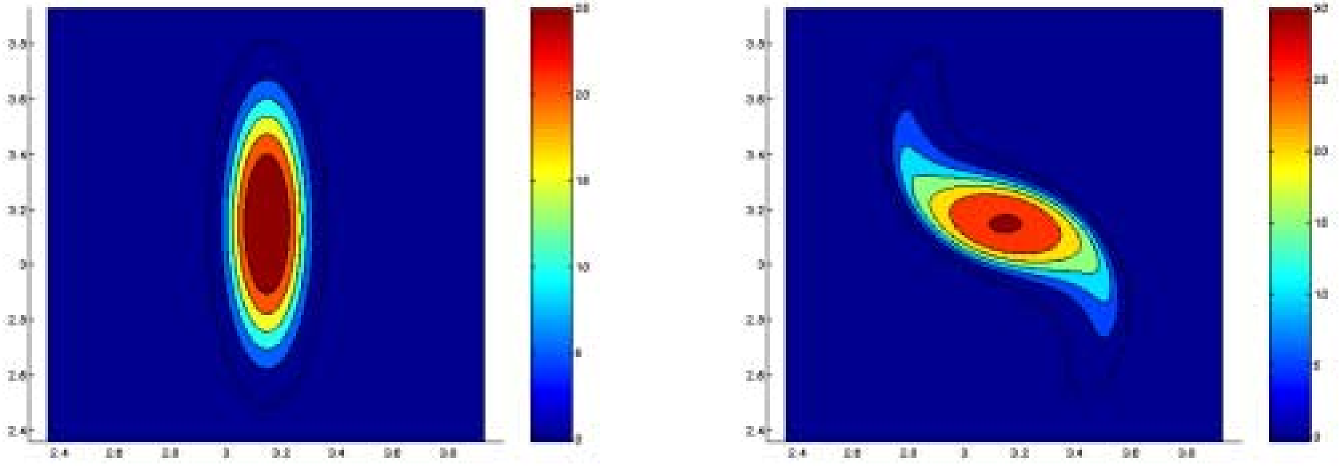
$$\chi = \sqrt{r^2+s^2} \qquad (\sin\alpha, \cos\alpha) = \left(\frac{s}{\chi}, \frac{r}{\chi}\right).$$

- $\zeta = \alpha$ alignment of $\boldsymbol{\nabla}$q with eigenvectors of $\boldsymbol{N}$.

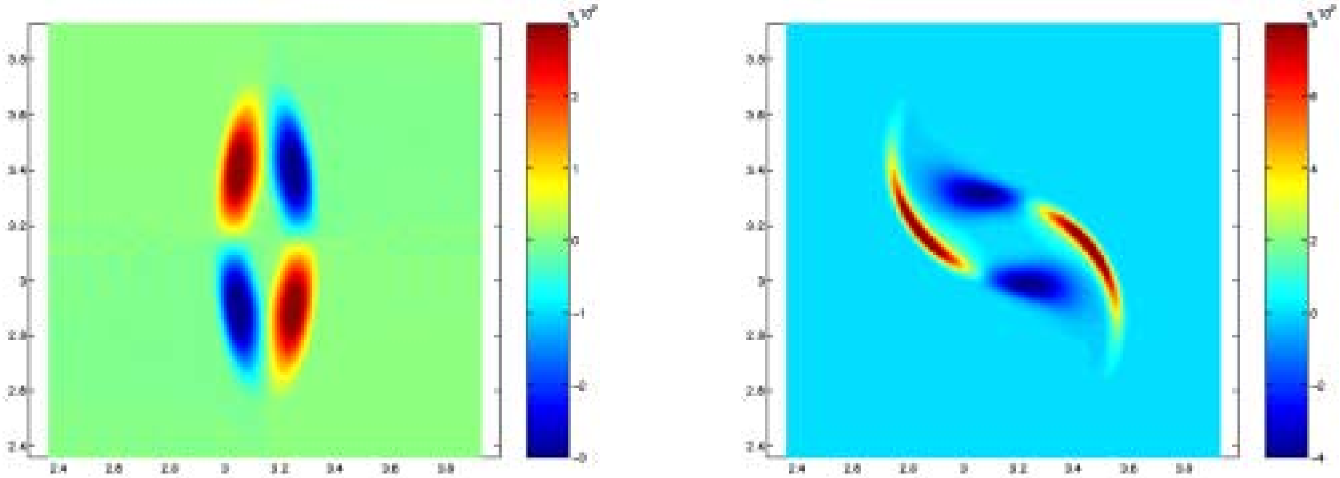
Palinstrophy production

$$
\begin{aligned}
-\sigma \sin\zeta &= \quad \sigma \sin(\arccos r) = \quad \sigma\sqrt{1-r^2} \quad \text{when} \quad |r| \leq 1 \\
-\sigma \sin\zeta &= \quad -\sigma \sin(\alpha) = \quad -\sigma \frac{s}{\chi} \quad \text{when} \quad |r| > 1
\end{aligned}
$$

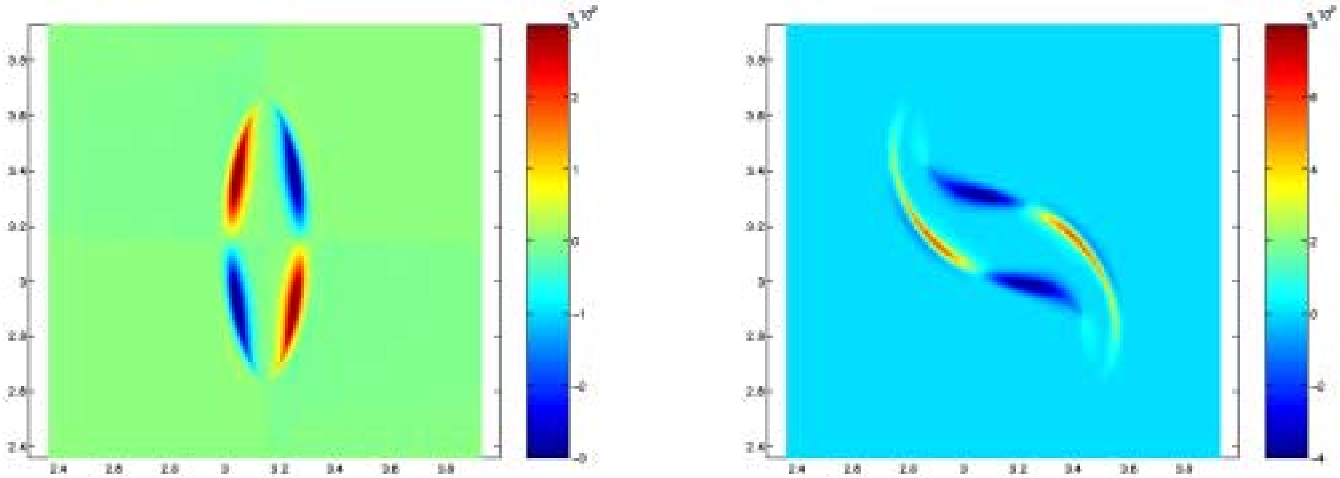
(a) **vorticity t=0, 0.3**



(b) **palenstrophy production**



(c) **analytical prediction of palenstrophy production**



Summary

Dynamics of tracer gradients as a function of flow topology

- mechanism 1 : r Competition between strain rate and effective rotation

$$r = \frac{\omega + 2\frac{D\phi}{Dt}}{\sigma}$$

- mechanism 2 : s Variation of stirring time scale (σ^{-1})

$$s = \frac{\frac{D\sigma}{Dt}}{\sigma^2}$$

- preferred orientations for tracer gradient that depend on
$\nabla \vec{u}$ local properties of velocity field and
$\nabla \frac{D\vec{u}}{Dt}$ acceleration gradient tensor: long-range influence of mesoscale eddies in physical space(Hua and Klein, 1998);

- both $\nabla \vec{u}$ and $\nabla \frac{D\vec{u}}{Dt}$ are entirely diagnostic for Quasi-Geostrophic dynamics (Hua, McWilliams & Klein, 1998)

$\Rightarrow$ better identification in physical space of tracer cascade toward small spatial scales

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