Homework on the instability of a cloud layer due to entrainment of its basis or its top

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(First part)

In this first part, we derive a new form of potential temperature called liquid potential temperature θ_ℓ.

The notations are the same as in the course

1) Using the expression

$$s = s_d + r_t s_v + r_\ell(s_\ell - s_v^S) + r_\ell(s_v^S - s_v)$$

show that one can define a potential temperature θ_ℓ, conserved for all adiabatic and frictionless transforms such that

$$\theta_\ell = T\left(\frac{p_0}{p_d}\right)^\chi \left(\frac{r_t}{r_v}\right)^\beta H^\alpha \exp - \left(\frac{r_\ell L}{(C_p + r_t C_{pv})T}\right).$$

Provide the expressions of α, χ et β as a function of known thermodynamic quantities.

2) Show that one can generalize θ_ℓ as $\theta_{i\ell}$ including also the presence of ice. Use r_i for the ice mass mixing ratio, L_i for the sublimation latent heat and H_i for relative humidity with respect to ice.

The remaining part of the problem aims at showing how the potential equivalent temperature and the liquid potential temperature can be used to understand cloud dynamics and how one has to be preferred over the other according to the case.

(Second part)

We consider a stratiform cloud with profiles of potential temperature θ, water vapour mixing ratio r and liquid water content ℓ as indicated on figure 1. Air is assumed to be at saturation level with respect to water inside the cloud. We do not take ice into account.

We use simplified linearized versions of the potential equivalent temperature θ_e and the liquid potential temperature θ_ℓ

$$\begin{aligned} \theta_e &= \theta + \gamma r \\ \theta_\ell &= \theta_e - \gamma r_t = \theta - \gamma r_\ell \end{aligned}$$

where r is the mass mixing ratio of water vapour, r_ℓ is the mass mixing ratio of liquid water and $r_t = r + r_\ell$ is the total water mas mixing ratio.

a) What approximations are made to obtain these simplified expressions ? Give the value of γ.

We assume that the buoyancy of a parcel with a potential temperature θ immersed within an environment of potential temperature θ_0 is

$$B = g\frac{\theta - \theta_0}{\theta_0}.$$

Notice that this expression always holds whether the environment is clear air or cloudy air.

We consider 4 scenarios

1. A parcel of clear air is entrained across the cloud basis and is mixed with the cloudy air located above.
2. A parcel of cloudy air is entrained across the cloud top and is mixed with clear air located above.
3. A parcel of cloudy air is entrained across the basis of the cloud and is mixed with clear air located below.
4. A parcel of clear air is entrained across the cloud top and is mixed with the cloudy air located below.

In these four cases there is partial or total evaporation of the condensed water contained in the cloudy part of the mixture.

b) Explain without calculation why the first two cases are stable and why there is a possibility of instability in the last two cases. [Take into account the stratification shown on figure 1 and the effect of partial transformation between liquid and vapour]

We consider now the third case and we are looking for the conditions under which the downward displaced parcel can be negative buoyant and thus amplifies its descent. This process is a theory for the generation of mamatus clouds from convective anvils (see figure 2).

We therefore consider a mixture of a part σ of cloud air and $1 - \sigma$ of clear air. We assume that the equivalent potential temperature and the liquid potential temperature of the mixture are a combination of the original same temperatures under these proportions. The total water is conserved but some water may evaporate or condensate during the process.

We denote as r_s the value of saturation mixing ratio. This quantity depends on temperature but as the variations of temperature are small during the processes considered in this exercise, we assume that there is a single value of r_s which is constant over the motion of the parcel and the mixing event.

c) Give the value of the conserved thermodynamic properties of the mixture (total water, θ_ℓ and θ_e) as a function of the initial quantities. Use index c for the cloudy air, index a for clear air and index m for the mixing. For instance, the total water mixing ratio of the mixture is r_{tm} and the equivalent potential temperature of the clear air is θ_{ea}. [Notice that $\theta_{\ell a} \equiv \theta_a$ but that $\theta_{ea} \neq \theta_a$, and use corresponding properties of the mixing ratios]

d) Give the value of $r_{\ell m}$ and r_m [as a function r_{tm} and r_s] by distinguishing the two cases where the mixture is saturated and non saturated.

e) Give the value σ^* of σ when the mixture is in the limit state between saturation and non saturation as a function of r_s, r_a et r_{lc}.

f) Give the buoyancy of the mixture in the non saturated case as a function of the liquid potential temperatures of its two components. [Show that it is proportional to $\theta_a - \theta_{\ell c}$]. What is the criterion for the buoyancy to be negative in terms of the liquid potential temperature profile at the bottom of the cloud ?

g) Give the buoyancy of the mixture in the saturated case. [Let appear σ^* by eliminating r_s and obtain an expression that adds a new term to the result obtained for question f). If you are in trouble here, check what you have done for question e) above.]

h) Under the hypothesis that the buoyancy is negative, show that it is minimal when the mixture is at the limit between saturation and non saturation. The demonstration needs to calculate the derivative of B with respect to σ in the saturated and non saturated cases. The sign of the derivative in the saturated case is found by combining the inequalities of potential temperatures and equivalent potential temperatures at the bottom of the cloud.

i) Plot the variation of B between $\sigma = 0$ and $\sigma = 1$.

We consider now the fourth scenario at the cloud top. This process is relevant for the instability at the top of the boundary layer clouds.

j) Give the buoyancy as a function of the potential temperatures of the mixture and of the surrounding cloudy air.

k) Give the buoyancy in the saturated case as a function of the equivalent potential temperatures of the two components. Give the instability condition.

l) Give the buoyancy in the non saturated case. [Let appear again σ^*]

m) Show that in the unstable case, the buoyancy is again minimal under the limit condition between saturation and non saturation.

n) Plot the variation of B between $\sigma = 0$ and $\sigma = 1$.

o) If the density effect of parcel load in liquid water is accounted in the fourth scenario, is it be stabilizing or destabilizing ?

p) Did we take into account the irreversibility of mixing ?

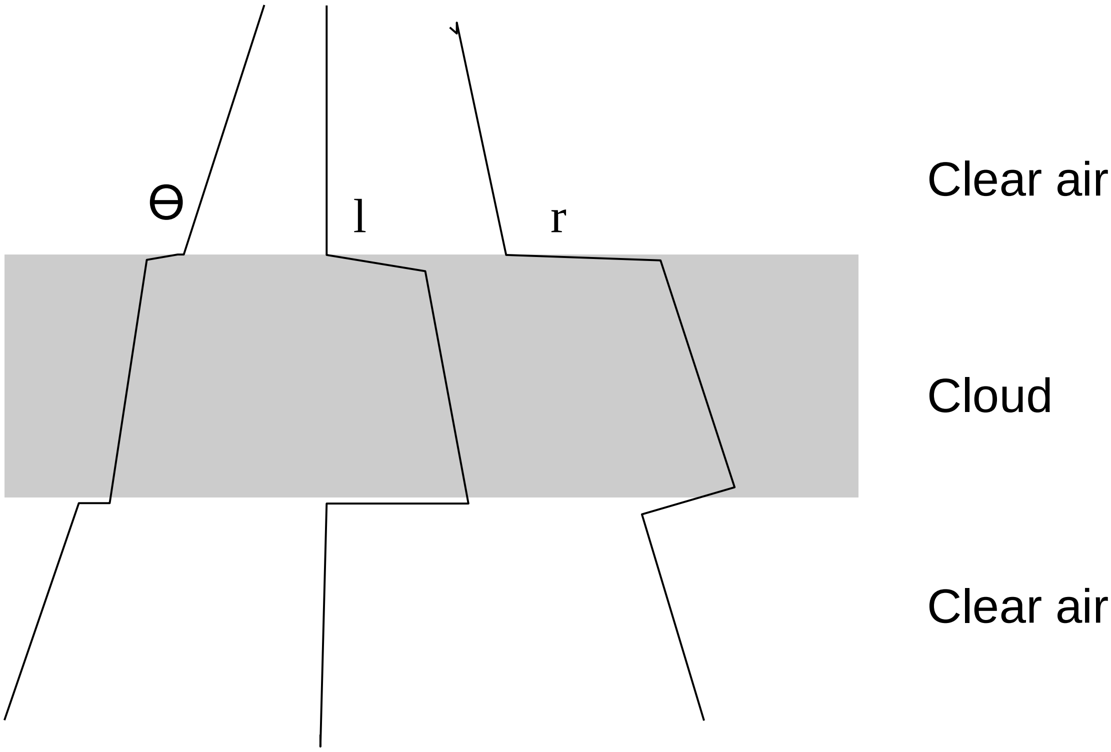

Figure Sketch of the cloud layer and the profiles of potential θ, liquid water l and water vapour mixing ratio r. The liquid water is 0 outside the cloud.

Figure 2 : Example of mamatus