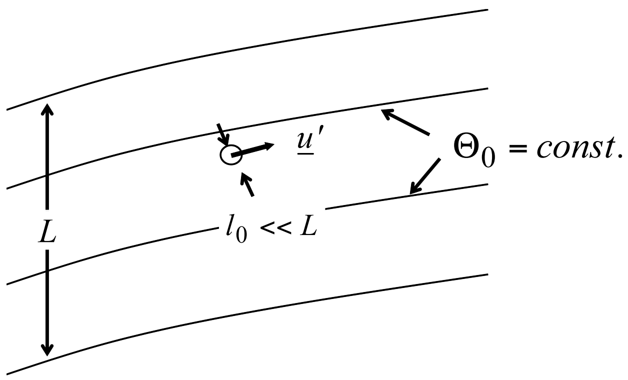
Exercise 1 – Turbulent mixing of a passive scalar

Let consider a passive scalar field $\theta(\underline{x},t)$ such as temperature or particle concentration (aerosols, spray) in a turbulent flow **without mean flow**. We suppose this is a passive scalar field, which means that it depends on the velocity field without any feedback on it.



The equation for $\theta(\underline{x},t)$ in a turbulent flow of incompressible fluid without mean velocity reads in this case:

$$\frac{d\theta}{dt} = \frac{\partial\theta}{\partial t} + \underline{u}' \,.\, \underline{grad}\,\theta = \chi\,\Delta\theta \tag{1}$$

where $\underline{u}'$ is the velocity fluctuation (here $\underline{u}' = \underline{u} - \cancel{\langle \underline{u} \rangle}$) and where χ denotes the diffusivity coefficient of $\theta(\underline{x},t)$.

Question 1 - Let suppose that initialy $\theta(\underline{x},t) = \Theta_0(\underline{x})$ is a deterministic field characterized by variation amplitudes $\delta\Theta_0/L = \| \underline{grad}\,\Theta_0 \|$ on the characteristic length scale L. Let l_0 be the characteristic lengthscale of the energetic turbulent eddies, such that $l_0 << L$. Introducing the Reynolds decomposition $\theta(\underline{x},t) = \langle\, \theta(\underline{x},t) \rangle + \theta'(\underline{x},t)$ show that the $\langle \theta(\underline{x},t) \rangle$ - equation reads:

$$\frac{d\langle\theta\rangle}{dt} = div\left(\chi\, \underline{grad}\langle\theta\rangle - \langle \theta' \underline{u}' \rangle \right) \tag{2}$$

with $-\langle \theta' \underline{u}' \rangle$ the turbulent flux of $\langle \theta \rangle$.

Solution - Putting $\theta(\underline{x},t) = \langle \theta(\underline{x},t) \rangle + \theta'(\underline{x},t)$ in (1) yields:

$$\frac{\partial \langle \theta \rangle}{\partial t} + \frac{\partial \theta'}{\partial t} + \underline{u}'.\underline{grad}\langle \theta \rangle + \underline{u}'.\underline{grad}\,\theta' = \chi\, \Delta\langle \theta \rangle + \chi\, \Delta\theta' \,.$$

An ensemble average of the above equation gives:

$$\frac{\partial \langle \theta \rangle}{\partial t} + \langle\, \underline{u}'.\underline{grad}\,\theta' \rangle = \chi\, \Delta\langle \theta \rangle$$

the other terms being functions of averaged fluctuations, so being nil. Writing:

$$\langle \underline{u}'.\underline{grad}\,\theta' \rangle = div\langle\, \underline{u}'\, \theta' \rangle + \theta'\, \cancel{div\,\underline{u}}$$

one gets:

$$\frac{\partial \langle \theta \rangle}{\partial t} = \chi\, \Delta\langle \theta \rangle - div\langle \theta' \underline{u}' \rangle$$

That is :

$$\frac{\partial \langle \theta \rangle}{\partial t} = div\Big(\underbrace{\chi\, \underline{grad}\langle \theta \rangle}_{\text{molecular flux of}\langle\theta\rangle} - \underbrace{\langle \theta' \underline{u}' \rangle}_{\text{turbulent flux of}\langle\theta\rangle} \Big)$$

Question 2 - Let's introduce an eddy diffusivity χ_ε which transforms (2) into:

$$\frac{d\langle \theta \rangle}{dt} = (\chi + \chi_\varepsilon)\, \Delta \langle \theta \rangle \tag{3}$$

Our objective is to show that in the present problem, (3) is not an approximation, but an exact equation (!).

2.1- Prove at first that the equation of the scalar fluctuation $\theta'(\underline{x},t)$ reads:

$$\frac{\partial \theta'}{\partial t} = -\underline{u}'.\underline{grad}\langle \theta \rangle - div\left[\, \theta' \underline{u}' - \langle \theta' \underline{u}' \rangle \,\right] + \chi\, \Delta\theta' \tag{4}$$

Solution – Let's substract equation (2)

$$\frac{d\langle\theta\rangle}{dt} = \frac{\partial\langle\theta\rangle}{\partial t} = div\left(\chi\,\underline{grad}\langle\theta\rangle - \langle\theta'\underline{u}'\rangle\right)$$

to the full equation (1) :

$$\frac{d\theta}{dt} = \frac{\partial\theta}{\partial t} + \underline{u}'.\underline{grad}\theta = div\left(\chi\,\underline{grad}\theta\right)$$

which also reads after decomposition of θ:

$$\frac{d\theta}{dt} = \frac{\partial\langle\theta\rangle}{\partial t} + \frac{\partial\theta'}{\partial t} + \underline{u}'.\underline{grad}\langle\theta\rangle + \underbrace{\underline{u}'.\underline{grad}\theta'}_{\substack{=div\,(\theta'\underline{u}')\\ -\theta'\cancel{div\underline{u}'}}} = \chi\Delta\langle\theta\rangle + \chi\Delta\theta'$$

This yields :

$$\frac{\partial\theta'}{\partial t} = \underbrace{-\underline{u}'.\underline{grad}\langle\theta\rangle}_{\text{forcing}} + \chi\Delta\theta' - div\left[\theta'\underline{u}' - \langle\theta'\underline{u}'\rangle\right]$$

2.2 – In the case where the turbulent energetic scales are such that $l_0 << L$, one may consider the mean gradient $\underline{grad}\langle\theta\rangle$ as being nearly constant. In that case, $\underline{u}'$ being independant of θ' (passive scalar), the θ'- equation (4) becomes **a linear equation** of the variable θ' forced by $\underline{u}'.\underline{grad}\langle\theta\rangle$. Let $\boldsymbol{L}$ denoting the linear kernel of this equation, one can write formally:

$$\begin{cases} \boldsymbol{L}(\theta') = -\underline{u}'.\underline{grad}\langle\theta\rangle \\ \theta' = -\boldsymbol{L}^{-1}\left(\underline{u}'.\underline{grad}\langle\theta\rangle\right) \end{cases} \tag{6}$$

This leads to a linear relation between the turbulent flux $-\langle\theta'\underline{u}'\rangle$ and $\underline{grad}\langle\theta\rangle$:

$$\langle\theta'\underline{u}'\rangle = -\langle\underline{u}'\boldsymbol{L}^{-1}\left(\underline{u}'.\underline{grad}\langle\theta\rangle\right)\rangle \tag{7}$$

This relation being linear with respect to $\underline{grad}\langle\theta\rangle$, it can be also be written formally as

$$\langle\theta'\underline{u}'\rangle = -\langle\,\underline{\underline{\chi_\varepsilon}}\,\rangle.\,\underline{grad}\langle\theta\rangle \tag{8}$$

where $\langle\,\underline{\underline{\chi_\varepsilon}}\,\rangle$ is a second order diffusivity tensor.

2.3 – Under what condition do we obtain $-\langle\theta'\underline{u}'\rangle = \chi_\varepsilon\,\underline{grad}\langle\theta\rangle$?

Solution – The condition required is isotropy of the turbulence. In that case:

$$\begin{cases} \underline{\underline{\chi_\varepsilon}} = \chi_\varepsilon \, \underline{\underline{1}} \\ \chi_\varepsilon = \frac{1}{3} trace \left\{ \underline{\underline{\chi_\varepsilon}} \right\} \end{cases}$$

Using indices:

$$\begin{cases} \chi_{\varepsilon ij} = \chi_\varepsilon \, \delta_{ij} \\ \chi_\varepsilon = \frac{1}{3} \chi_{\varepsilon ii} \end{cases}$$

2.4 – Compare the physical diffusivity χ to the eddy diffusivity χ_ε using the mean field scales $\delta\Theta_0$, L and the turbulence scale u_0, l_0 ($l_0 << L$).

Solution – One has

$$\frac{\chi_\varepsilon}{\chi} = \frac{\left\| \langle \underline{u}'\theta' \rangle \right\|}{\chi \left\| \underline{grad} \langle \theta \rangle \right\|} \sim \frac{u_0 \delta\Theta_0}{\chi \, \delta\Theta_0 / L} = \frac{u_0 L}{\chi} = \underbrace{\frac{u_0 l_0}{\nu}}_{Re_0 >> 1} \underbrace{\frac{L}{l_0}}_{>>1} \underbrace{\frac{\nu}{\chi}}_{\substack{\text{Prandtl} > 1 \\ \text{Schmidt} >> 1}} >> 1$$

Exercise 2 – Temporal decay of turbulence

Question 1 - Show that the temporal decay of a homogeneous - isotropic turbulence using a k - ε model formulation reads:

$$\begin{cases} \dfrac{dk}{dt} = -\varepsilon \\ \dfrac{d\varepsilon}{dt} = -C_{\varepsilon 2} \dfrac{\varepsilon^2}{k} \end{cases} \tag{1.1}$$

Solution – The general expression of the $k-\varepsilon$ model given in the lecture is:

$$\begin{cases} \dfrac{Dk}{Dt} = P + div\left(\dfrac{\nu_\varepsilon}{\sigma_k} \underline{grad\, k} \right) - \varepsilon \\ \dfrac{D\varepsilon}{Dt} = C_{\varepsilon 1} \dfrac{\varepsilon}{k} P + div\left(\dfrac{\nu_\varepsilon}{\sigma_\varepsilon} \underline{grad \varepsilon} \right) - C_{\varepsilon 2} \dfrac{\varepsilon}{k} \varepsilon \end{cases}$$

Statistic homogeneity eliminates the diffusion terms (div(…)) and the absence of turbulence production (P = 0) then leads to (1).

Question 2 - Show that the solution of (1) reads

$$k(t) = k(0)\left[1 + \frac{t}{n\,\tau(0)} \right]^{-n} \tag{1.2}$$

with:

$$n = \frac{1}{C_{\varepsilon 2} - 1} \tag{1.3}$$

Solution - The equation of ε becomes:

$$\frac{1}{\varepsilon}\frac{d\varepsilon}{dt} = C_{\varepsilon 2}\frac{1}{k}\frac{dk}{dt}$$

So:

$$\frac{\varepsilon(t)}{\varepsilon(0)} = \left(\frac{k(t)}{k(0)} \right)^{C_{\varepsilon 2}}$$

For the equation of k this yields:

$$\frac{dk}{dt} = -\varepsilon = -Ak^{C_{\varepsilon 2}} = 0,\ A = \frac{\varepsilon(0)}{k^{C_{\varepsilon 2}}(0)}$$

This integrates as follows:

$$k^{-C_{\varepsilon 2}} \frac{dk}{dt} = -A$$

$$\frac{1}{1 - C_{\varepsilon 2}} \frac{dk^{(1-C_{\varepsilon 2})}}{dt} = -A.$$

$$\frac{dk^{(1-C_{\varepsilon 2})}}{dt} = -A\left(1 - C_{\varepsilon 2}\right)$$

$$k^{(1-C_{\varepsilon 2})} = B - A\left(1 - C_{\varepsilon 2}\right)t$$

Namely, by introducing $n = 1/\left(C_{\varepsilon 2} - 1\right)$:

$$k(t) = \left[\ B + \frac{A}{n}t\ \right]^{-n}$$

The initial condition $t = 0$ stipulates:

$$k(0) = B^{-n} \quad \Rightarrow \quad B = k(0)^{-1/n}$$

Therefore:

$$k(t) = k(0)\left[\ 1 + \frac{A}{n}k(0)^{1/n}\ t\ \right]^{-n}$$

By introducing $A = \dfrac{\varepsilon(0)}{k^{C_{\varepsilon 2}}(0)}$, $\tau_0 = k(0)/\varepsilon(0) = k(0)/\varepsilon(0)$ and $C_{\varepsilon 2} = 1 + \dfrac{1}{n}$:

$$k(t) = k(0)\left[\ 1 + \frac{1}{n}\frac{t}{\tau_0}\ \right]^{-n}$$

Question 3 - Recall the consensus for n, so for $C_{\varepsilon 2}$, as provided by the experiments and by the DNS of the isotropic turbulence decay

Solution - Experiments and DNS provide:

$$n \approx 1.3$$

which leads to:

$$C_{\varepsilon 2} = 1.77$$