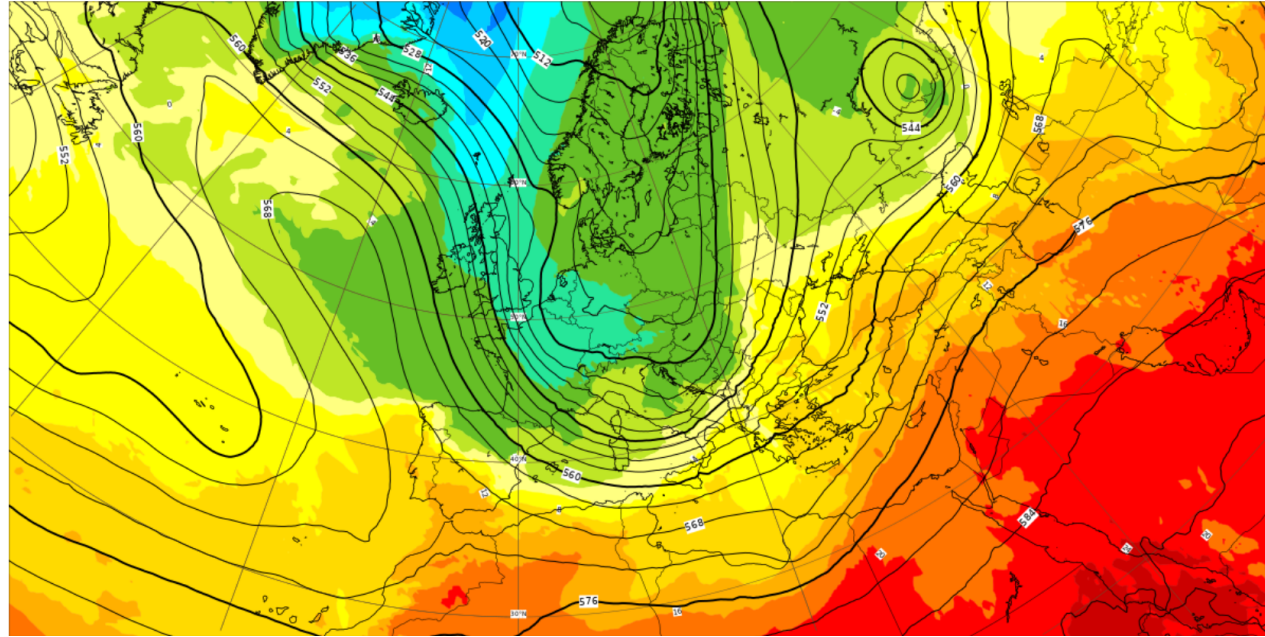# Modélisation Numérique de l'Écoulement Atmosphérique et Assimilation de Données
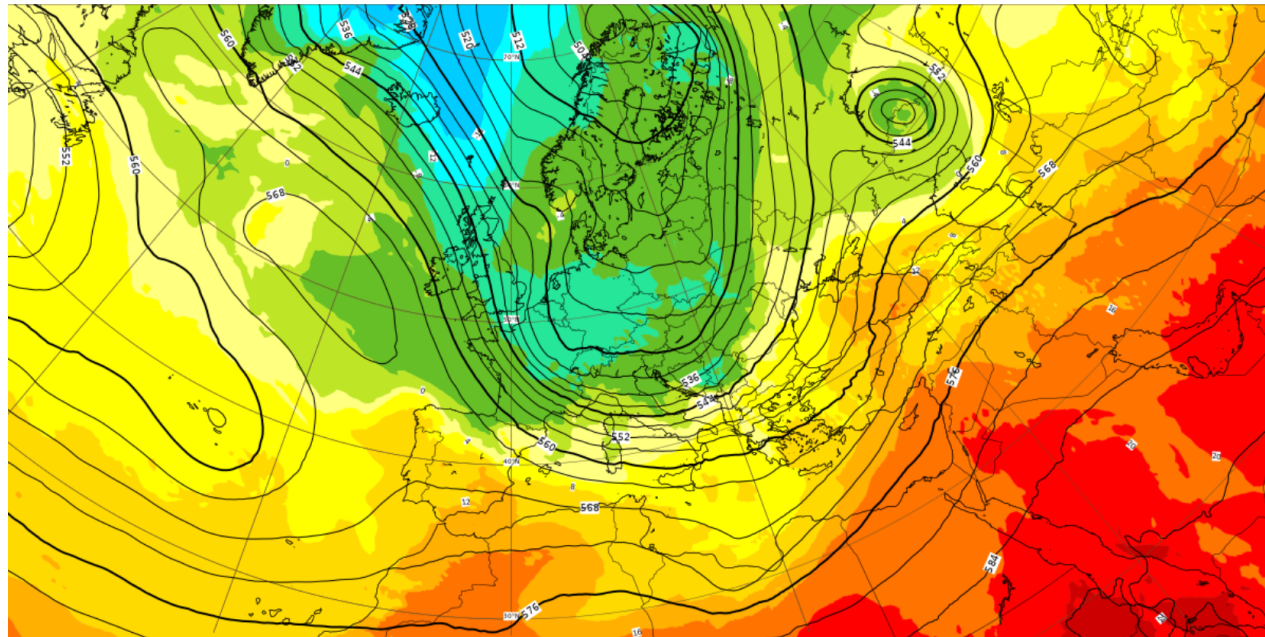
Olivier Talagrand

Cours 3

9 Avril 2021

Base time: Fri 02 Apr 2021 00 UTC, Valid time: Wed 07 Apr 2021 00 UTC, - T+120 h, Area : Europe

Base time: Wed 07 Apr 2021 00 UTC, Valid time: Wed 07 Apr 2021 00 UTC, - T+0 h, Area : Europe

850 hPa temperature (C)

500 hPa geopotential (dm)

2021

- Bayesian estimation. Continuation.

- Reminder on elementary probability theory. Random vectors and covariance matrices, random functions and covariance functions

- *Optimal Interpolation*. Principle, simple examples, basic properties.

Purpose of assimilation : reconstruct as accurately as possible the state of the atmospheric or oceanic flow, using all available appropriate information. The latter essentially consists of

- The observations proper, which vary in nature, resolution and accuracy, and are distributed more or less regularly in space and time.

- The physical laws governing the evolution of the flow, available in practice in the form of a discretized, and necessarily approximate, numerical model.

- 'Asymptotic' properties of the flow, such as, *e. g.*, geostrophic balance of middle latitudes. Although they basically are necessary consequences of the physical laws which govern the flow, these properties can usefully be explicitly introduced in the assimilation process.

Both observations and 'model' are affected with some uncertainty $\Rightarrow$ uncertainty on the estimate.

For some reason, uncertainty is conveniently described by probability distributions (don't know too well why, but it works; see, *e.g.* Jaynes, 2007, *Probability Theory: The Logic of Science,* Cambridge University Press).

Assimilation is a problem in bayesian estimation.

Determine the conditional probability distribution for the state of the system, knowing everything we know (see Tarantola, A., 2005*, Inverse Problem Theory and Methods for Model Parameter Estimation*, SIAM).

Bayesian Estimation

Determine conditional probability distribution of the state of the system, given the probability distribution of the uncertainty on the data

$$z_1 = x + \zeta_1 \qquad \zeta_1 = \mathcal{N}[0, s_1]$$

density function $p_1(\zeta) \propto \exp[-(\zeta^2)/2s_1]$

$$z_2 = x + \zeta_2 \qquad \zeta_2 = \mathcal{N}[0, s_2]$$

density function $p_2(\zeta) \propto \exp[-(\zeta^2)/2s_2]$

- ζ_1 and ζ_2 mutually independent

What is the conditional probability $P(x = \xi \mid z_1, z_2)$ that x be equal to some value ξ ?

$z_1 = x + \zeta_1$ density function $p_1(\zeta) \propto \exp[-(\zeta^2)/2s_1]$

$z_2 = x + \zeta_2$ density function $p_2(\zeta) \propto \exp[-(\zeta^2)/2s_2]$

ζ_1 and ζ_2 mutually independent

$x = \xi \iff \zeta_1 = z_1 - \xi$ and $\zeta_2 = z_2 - \xi$

- $P(x = \xi \mid z_1, z_2) \propto p_1(z_1 - \xi)\, p_2(z_2 - \xi)$

$$\propto \exp[-(\xi - x^a)^2/2p^a]$$

where $1/p^a = 1/s_1 + 1/s_2$, $x^a = p^a\, (z_1/s_1 + z_2/s_2)$

Conditional probability distribution of x, given z_1 and z_2 : $\mathcal{N}[x^a, p^a]$

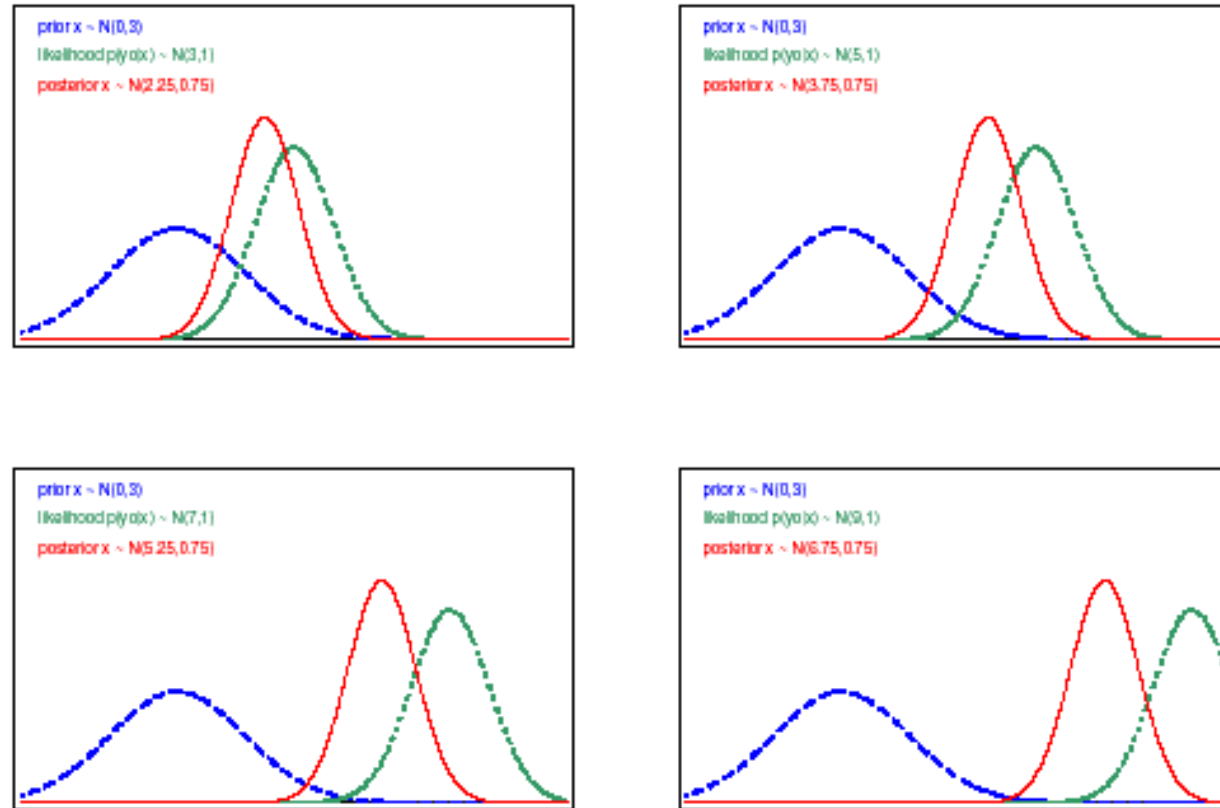$p^a < (s_1, s_2)$ independent of z_1 and z_2

Fig. 1.1: Prior pdf $p(x)$ (dashed line), posterior pdf $p(x|y^o)$ (solid line), and Gaussian likelihood of observation $p(y^o|x)$ (dotted line), plotted against x for various values of y^o. (Adapted from Lorenc and Hammon 1988.)

Conditional expectation x^a minimizes following scalar *objective function*, defined on ξ-space

$$\xi \to \quad \mathcal{J}(\xi) \equiv (1/2)\ [(z_1 - \xi)^2 / s_1 + (z_2 - \xi)^2 / s_2\ ]$$

In addition

$$p^a = 1/\ \mathcal{J}''(x^a)$$

Conditional probability distribution in Gaussian case

$$P(x = \xi\,|\,z_1, z_2) \propto \exp[\,-\,(\xi - x^a)^2/2p^a]$$

$$\mathcal{J}(\xi) + Cst$$

Estimate

$$x^a = p^a \, (z_1/s_1 + z_2/s_2)$$

with error p^a such that

$$1/p^a = 1/s_1 + 1/s_2$$

can also be obtained, independently of any Gaussian hypothesis, as simply corresponding to the linear combination of z_1 and z_2 that minimizes the error $E\left[(x^a - x)^2\right]$

Best Linear Unbiased Estimator (*BLUE*)

$$z_1 = x + \zeta_1$$
$$z_2 = x + \zeta_2$$

Same as before, but ζ_1 and ζ_2 are now distributed according to exponential law with parameter a, $i.e.$

$$p(\zeta) \propto \exp[-|\zeta|/a] \quad ; \quad \mathrm{Var}(\zeta) = 2a^2$$

Conditional probability density function is now uniform over interval $[z_1, z_2]$, exponential with parameter $a/2$ outside that interval

$$E(x \mid z_1, z_2) = (z_1 + z_2)/2$$

$\mathrm{Var}(x \mid z_1, z_2) = a^2 (2\delta^3/3 + \delta^2 + \delta + 1/2) / (1 + 2\delta)$, with $\delta = |z_1 - z_2|/(2a)$
Increases from $a^2/2$ to ∞ as δ increases from 0 to ∞. Can be larger than variance $2a^2$ of original errors (probability 0.08)

Bayesian estimation

State vector $\boldsymbol{x}$, belonging to *state space* $\mathcal{S}$ $(\dim \mathcal{S} = n)$, to be estimated.

Data vector $\boldsymbol{z}$, belonging to *data space* $\mathcal{D}$ $(\dim \mathcal{D} = m)$, available.

$$z = F(\boldsymbol{x}, \zeta) \tag{1}$$

where ζ is a random element representing the uncertainty on the data (or, more precisely, on the link between the data and the unknown state vector).

For example

$$z = \Gamma\boldsymbol{x} + \zeta$$

Bayesian estimation (continued)

Probability that $x = \xi$ for given ξ ?

$$x = \xi \;\;\Rightarrow\;\; z = F(\xi, \zeta)$$

$$P(x = \xi \,|\, z) = P[z = F(\xi, \zeta)] \,/ \int_{\xi'} P[z = F(\xi', \zeta)]$$

Unambiguously defined iff, for any ζ, there is at most one x such that (1) is verified.

$\Leftrightarrow$ data contain information, either directly or indirectly, on any component of x. *Determinacy* condition. Implies $m \geq n$.

Difficulties specific to assimilation of meteorological observations :

- Very large numerical dimensions ($n \approx 10^6$-10^9 parameters to be estimated, $p \approx$ 4-5.10^7 observations per 24-hour period). Difficulty aggravated in Numerical Weather Prediction by the need for the forecast to be ready in time.

- Non-trivial, actually chaotic, underlying dynamics

Bayesian estimation is actually impossible in its general theoretical form in meteorological or oceanographical practice because

- It is impossible to explicitly describe a probability distribution in a space with dimension even as low as $n \approx 10^3$, not to speak of the dimension $n \approx 10^{6\text{-}9}$ of present Numerical Weather Prediction models (the *curse of dimensionality*).

- Probability distribution of errors on data very poorly known (model errors in particular).

One has to restrict oneself to a much more modest goal. Two approaches exist at present

- Obtain some 'central' estimate of the conditional probability distribution (expectation, mode, …), plus some estimate of the corresponding spread (standard deviations and a number of correlations).

- Produce an ensemble of estimates which are meant to sample the conditional probability distribution (dimension $N \approx O(10\text{-}100)$).

Coût des différentes composantes de la chaîne de prévision opérationnelle du CEPMMT (septembre 2015, J.-N. Thépaut) :
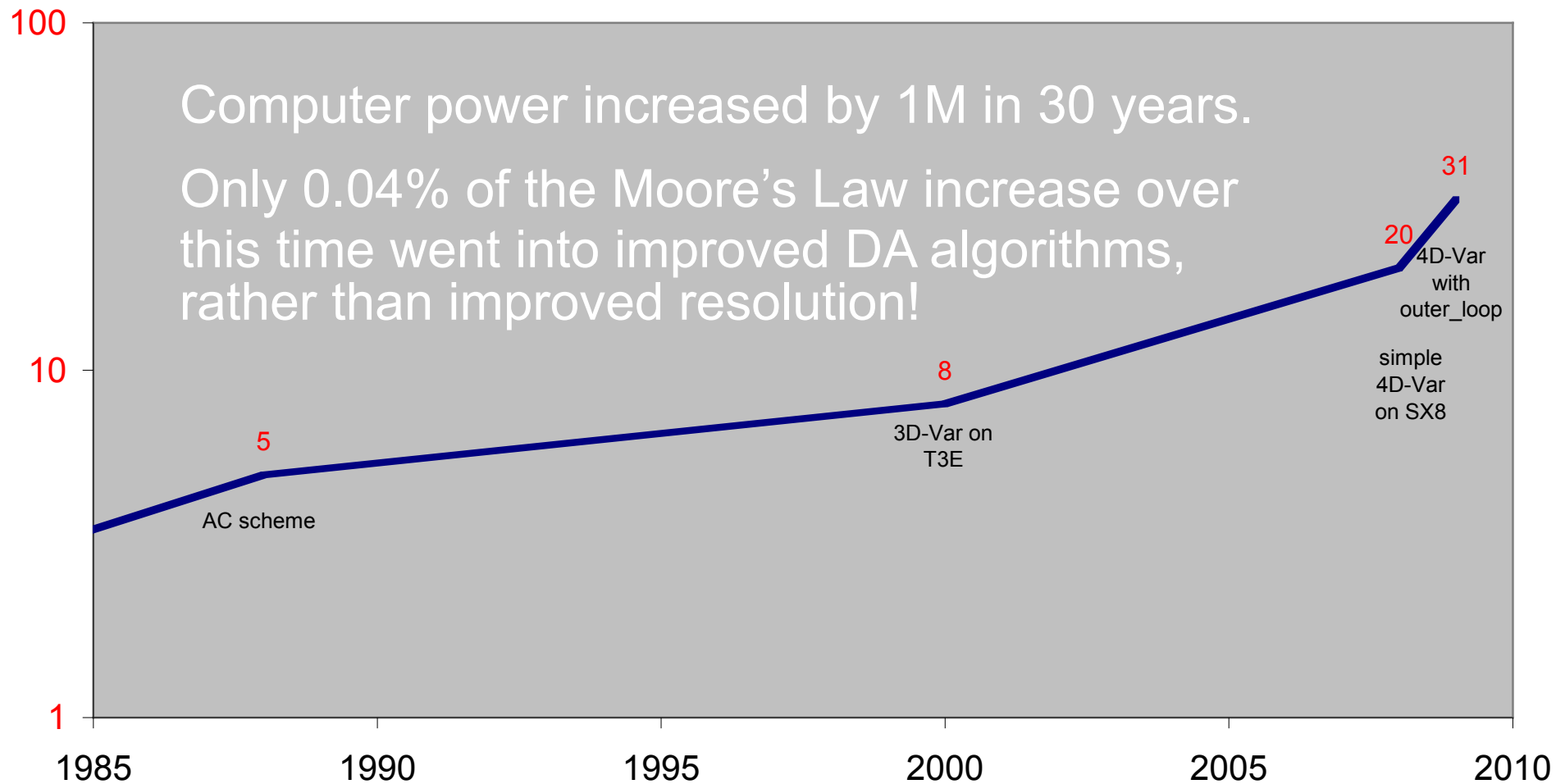
4DVAR: 9.5%

HRES FC: 4.5%

EDA: 30%

ENS: 22%

ENS: hindcasts 14%

Other: 20% of which BC AN: 3.5% BC FC: 4% BC ENS: 9.5%

L'EDA fournit à la fois les variances d'erreur d'ébauche du 4D-Var, et les perturbations initiales (en complément des vecteurs singuliers) de l'EPS.

ratio of supercomputer costs:
1 day's assimilation / 1 day forecast

Computer power increased by 1M in 30 years.

Only 0.04% of the Moore's Law increase over this time went into improved DA algorithms, rather than improved resolution!

- 31 — 4D-Var with outer_loop
- 20 — simple 4D-Var on SX8
- 8 — 3D-Var on T3E
- 5 — AC scheme

Courtesy A. Lorenc

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Scalar random variable x

Observed outcome of 'realizations' of a process that is repeated a large number of times. And also, *a priori* uncertainty on that result.

For any interval $[a, b]$, the probability $P(a < x < b)$ is known (whether inequalities are strict or not may matter).

Probability density function (pdf). Function $p(\xi)$ such that, for any interval $[a, b]$

$$P[a < x < b] = \int_a^b p(\xi)d\xi \qquad \int_{-\infty}^{+\infty} p(\xi)d\xi = 1$$

($p(\xi)$ may contain diracs)

Expectation. Mean of a large number of realizations of x

$$E(x) = \int_{-\infty}^{+\infty} \xi p(\xi)d\xi$$

(may not exist)

Scalar random variable x (continued)

Variance

$$Var(x) \equiv E\{[x - E(x)]^2\} = E(x^2) - [E(x)]^2$$

Standard deviation

$$\sigma(x) \equiv \sqrt{Var(x)}$$

Centred variable $x' \equiv x - E(x)$

Couple of random variables $x = (x_1, x_2)^T$

For any intervals $[a_1, b_1]$, $[a_2, b_2]$, probability $P(a_1 < x_1 < b_1$ and $a_2 < x_2 < b_2)$ is known

Extends to any measurable domain $\mathcal{D} \subset R^2$

$$P[(x_1, x_2) \in D] = \int_D p(\xi_1, \xi_2) d\xi_1 \xi_2$$

where $p(\xi_1, \xi_2)$ is probability density function

Expectation

$$E(x_1 + x_2) = E(x_1) + E(x_2)$$

Couple of random variables $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}}$

Covariance

$$Cov(x_1, x_2) \equiv E(x_1{}' \, x_2{}')$$

$$Corr(x_1, x_2) \equiv Cov(x_1, x_2) \, / \, (\sigma(x_1) \, \sigma(x_2)) \; = \; \cos \varphi$$

Covariance is a scalar product, and defines Euclidean geometry (on space of finite-variance random variables on a given trial space)

Modulus = standard deviation σ, angle = $\cos^{-1} (Corr)$, orthogonality = decorrelation

If x_1 and x_2 uncorrelated,

$$Var(x_1 + x_2) = Var(x_1) + Var(x_2) \qquad \text{(Pythagorean theorem)}$$

$$E(x_1 \, x_2) = E(x_1) \, E(x_2)$$

Couple of random variables $x = (x_1, x_2)^T$ (continued)

Independence

x_1 and x_2 independent : knowledge about either one of the variables brings no knowledge about the other one.

For any intervals $[a_1, b_1]$, $[a_2, b_2]$

$$P(a_1 < x_1 < b_1 \text{ and } a_2 < x_2 < b_2) = P(a_1 < x_1 < b_1) \, P(a_2 < x_2 < b_2)$$

Equivalently, pdf's verify

$$p(\xi_1, \xi_2) = p_1(\xi_1) \, p_2 \xi_2)$$

Independence implies decorrelation. Converse is not true
(consider $S = \sin \alpha$, $C = \cos \alpha$, where α is uniformly distributed over $[0, 2\pi]$)

Random vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} = (x_i)$ (*e. g.* pressure, temperature, abundance of given chemical compound at n grid-points of a numerical model)

- Expectation $E(\boldsymbol{x}) \equiv [E(x_i)]$; centred vector $\boldsymbol{x'} \equiv \boldsymbol{x} - E(\boldsymbol{x})$

- Covariance matrix

$$E(\boldsymbol{x'x'}^{\mathrm{T}}) = [E(x_i'x_j')]$$

dimension $n\mathrm{x}n$

Non-random vector $\boldsymbol{\lambda} = (\lambda_i)_{i = 1, \ldots, n}$

$$G \equiv \Sigma_i \lambda_i x_i' \qquad\qquad G^2 = \Sigma_{i,j} \lambda_i \lambda_j x_i' x_j'$$

$$E(G^2) = \Sigma_{i,j} \lambda_i \lambda_j E(x_i'x_j') = \boldsymbol{\lambda}^{\mathrm{T}} E(\boldsymbol{x'x'}^{\mathrm{T}}) \boldsymbol{\lambda} \geq 0$$

Covariance matrix $E(\boldsymbol{x'x'}^{\mathrm{T}})$ is symmetric non negative (strictly definite positive except if linear relationship holds between the x_i''s with probability 1).

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Change

$$x \rightarrow y \equiv Px$$

$$y'y'^{\mathrm{T}} = Px'(Px')^{\mathrm{T}} = P\, x\, x'^{\,\mathrm{T}}\, P^{\mathrm{T}}$$

$$E(y'y'^{\mathrm{T}}) = P\, E(x'x'^{\mathrm{T}})\, P^{\mathrm{T}}$$

In change $x \rightarrow y$, eigenvalues of covariance matrix remain > 0, but can be modified (conserved if $P^{\mathrm{T}} = P^{-1}$, orthogonal matrix).

Eigenvalues can actually take any positive values.

In particular, covariance matrix can be made equal to the unit matrix, for instance in the basis of *principal components*.

- **Two random vectors**

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$$
$$\boldsymbol{z} = (z_1, z_2, \ldots, z_p)^{\mathrm{T}}$$

$$E(\boldsymbol{x'z'}^{\mathrm{T}}) = E(x_i'z_j')$$

dimension nxp

Change

$$\boldsymbol{x} \rightarrow \boldsymbol{u} \equiv A\boldsymbol{x} \qquad \boldsymbol{z} \rightarrow \boldsymbol{v} \equiv B\boldsymbol{z}$$

$$E(\boldsymbol{u'v'}^{\mathrm{T}}) = A\, E(\boldsymbol{x'z'}^{\mathrm{T}})\, B^{\mathrm{T}}$$

Covariance matrices will be denoted

$$C_{xx} \equiv E(x'x'^{\mathrm{T}})$$

$$C_{xy} \equiv E(x'y'^{\mathrm{T}})$$

Random function $\Phi(\xi)$ (field of pressure, temperature, abundance of given chemical compound, … ; ξ is now spatial and/or temporal coordinate) (aka *stochastic process* if function of time)

- Expectation $E[\Phi(\xi)]$; $\qquad \Phi'(\xi) \equiv \Phi(\xi) - E[\Phi(\xi)]$
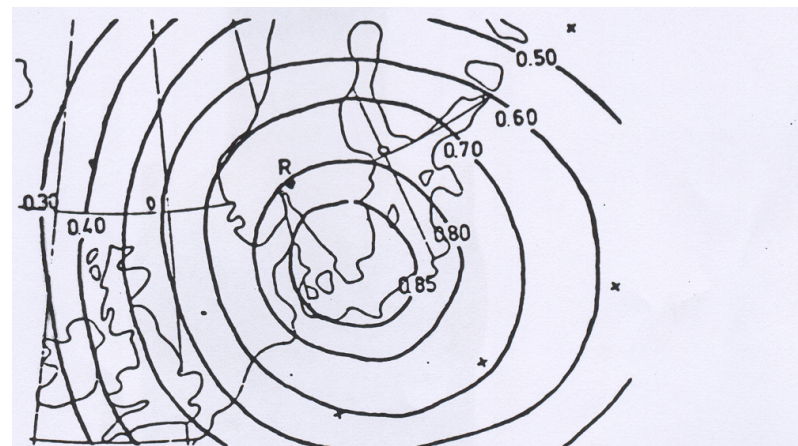- Variance $\quad Var[\Phi(\xi)] = E\{[\Phi'(\xi)]^2\}$

- Covariance function

$$(\xi_1, \xi_2) \rightarrow C_\Phi(\xi_1, \xi_2) \equiv E[\Phi'(\xi_1)\,\Phi'(\xi_2)]$$

- Correlation function

$$Cor_\Phi(\xi_1, \xi_2) \equiv E[\Phi'(\xi_1)\,\Phi'(\xi_2)] / \{Var[\Phi(\xi_1)]\,Var[\Phi(\xi_2)]\}^{1/2}$$
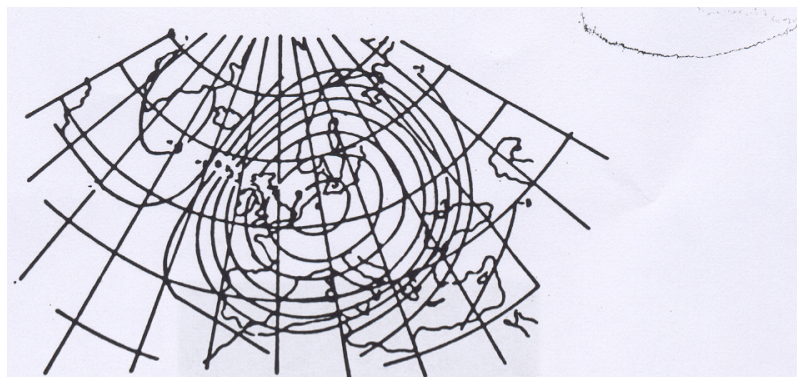
.: Isolines for the auto-correlations of the 500 mb
geopotential between the station in Hannover and
surrounding stations.
From Bertoni and Lund (1963)

: Isolines of the cross-correlation between the 500 mb
geopotential in station 01 384 (R) and the surface
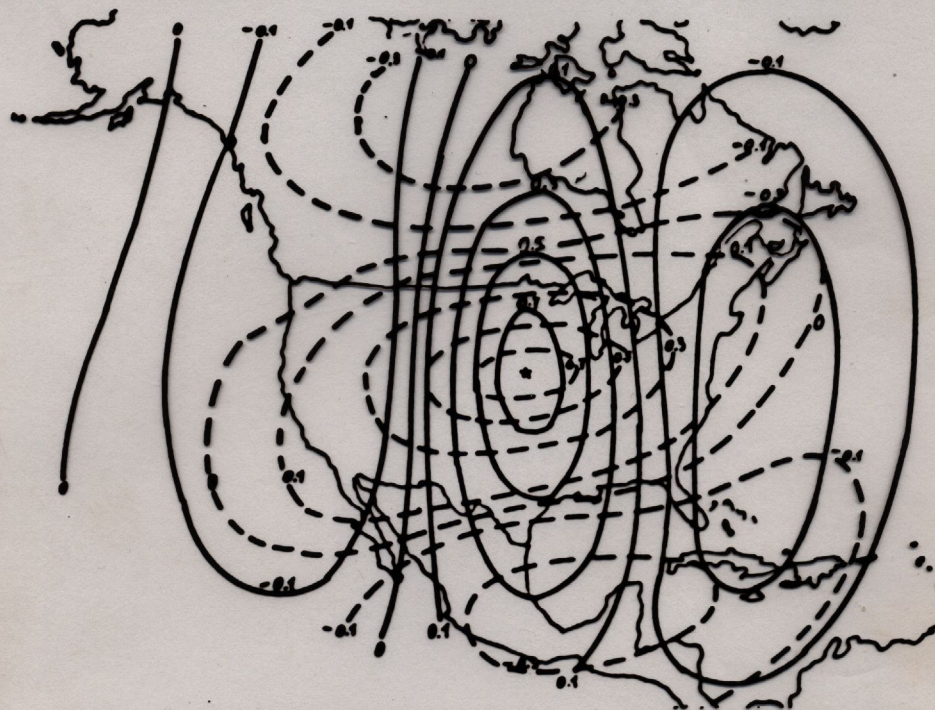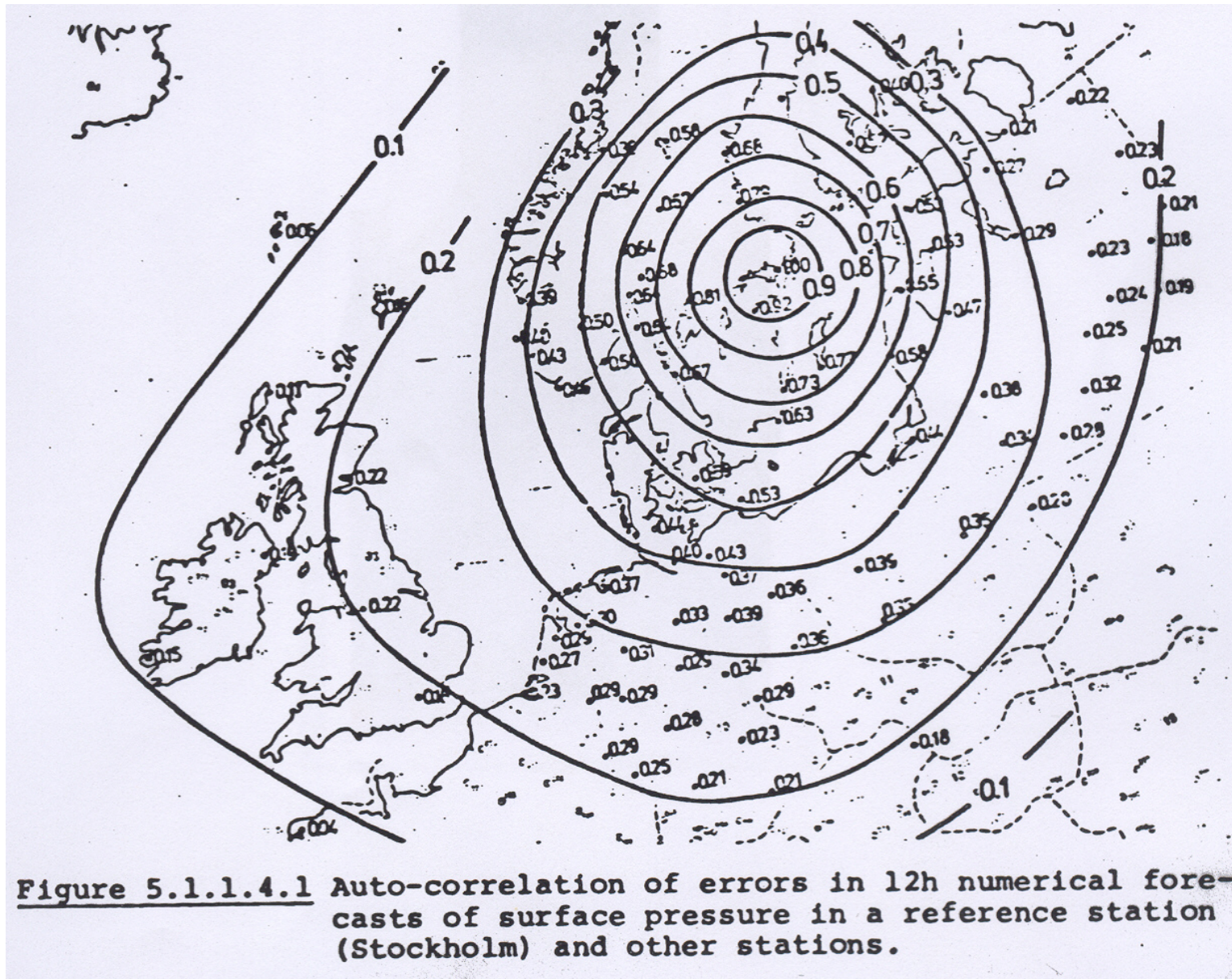pressure in surrounding stations.

After N. Gustafsson

Figure 4.2.4.3: Isolines for the auto-correlation of the 500 mb u-wind component (dashed line) and the auto-correlation of the 500 mb v-wind component (full line). The "star" indicates the position of the reference station. (From Buel (1972).

After N. Gustafsson

Figure 5.1.1.4.1 Auto-correlation of errors in 12h numerical forecasts of surface pressure in a reference station (Stockholm) and other stations.

After N. Gustafsson

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Covariance function can be

homogeneous $\qquad C_{\Phi}(\xi_1, \xi_2) = H(\xi_1 - \xi_2)$

or *isotropic* $\qquad C_{\Phi}(\xi_1, \xi_2) = K(|\xi_1 - \xi_2|)$

(on the sphere, no difference)

N points $\xi_1, \xi_2, \ldots, \xi_N$ in state space

N non-random coefficients $\lambda_1, \lambda_2, \ldots, \lambda_N$

$$G \equiv \Sigma_i \, \lambda_i \, \Phi'(\xi_i)$$

$$E(G^2) = \Sigma_{i,j} \, \lambda_i \, \lambda_j \, C_{\Phi}(\xi_i, \xi_j) \geq 0$$

$$E(G^2) = \Sigma_{i,j}\, \lambda_i\, \lambda_j\, C_{\Phi}(\xi_i, \xi_j) \geq 0$$

covariance functions are of *positive type* (or *definite positive*). Conversely, a function of positive type can be shown to be the covariance function of a random function.

Examples

On a circle, function $C(\xi_1, \xi_2) = \cos(\xi_1 - \xi_2)$ is covariance function of random function $\Phi(\xi) = 2\cos(\xi + \alpha)$, where α is uniformly distributed over $[0, 2\pi]$.

In R^n, squared exponential

$$C(\xi_1, \xi_2) = \exp[-\,(\xi_1 - \xi_2)^T B^{-1} (\xi_1 - \xi_2)\,] \quad B > 0$$

Bochner-Khintchin theorem. Homogeneous function C $(\xi_1, \xi_2) = H(\xi_1 - \xi_2)$ over R^n of positive type $\Leftrightarrow$ Fourier Transform of H is real ≥ 0.

Gaussian variables

Unidimensional

$$\mathcal{N}[m, a] \sim (2\pi a)^{-1/2} \exp\left[-(1/2a)(\xi - m)^2\right]$$

Dimension n

$$\mathcal{N}[\boldsymbol{m}, \boldsymbol{A}] \sim$$
$$\left[(2\pi)^n \det\boldsymbol{A}\right]^{-1/2} \exp\left[-(1/2)(\boldsymbol{\xi} - \boldsymbol{m})^{\mathrm{T}} \boldsymbol{A}^{-1}(\boldsymbol{\xi} - \boldsymbol{m})\right]$$

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Gaussian variables

Gaussian couple $z = (x^{\mathrm{T}}, y^{\mathrm{T}})^{\mathrm{T}}$ with distribution $\mathcal{N}[0, C]$

pdf $\sim$ exp $[-(1/2)\, z^{\mathrm{T}} C^{-1} z]$
$$C \equiv \begin{pmatrix} C_{xx} & C_{xy} \\ C_{yx} & C_{yy} \end{pmatrix}$$

x and y uncorrelated $C_{xy} = 0, C_{yx} = 0$
$$C^{-1} = \begin{pmatrix} C_{xx}^{-1} & 0 \\ 0 & C_{yy}^{-1} \end{pmatrix}$$

$$z^{\mathrm{T}} C^{-1} z = x^{\mathrm{T}} C_{xx}^{-1} x + y^{\mathrm{T}} C_{yy}^{-1} y$$

Gaussian variables

$$z^T C^{-1} z = x^T C_{xx}^{-1} x + y^T C_{yy}^{-1} y$$

$$\exp[-(1/2)\, z^T C^{-1} z] =$$
$$\exp[-(1/2)\, x^T C_{xx}^{-1} x]\, \exp[-(1/2)\, y^T C_{yy}^{-1} y]$$

$$p(z) = p(x)\, p(y)$$

For globally Gaussian variables, decorrelation implies independence

- 'Optimal Interpolation'. Basic theory and basic properties. A simple example.

Optimal Interpolation

x $\boldsymbol{\xi}_1$

x $\boldsymbol{\xi}_3$

X $\boldsymbol{\xi}$

x $\boldsymbol{\xi}_2$ x $\boldsymbol{\xi}_5$

x $\boldsymbol{\xi}_4$
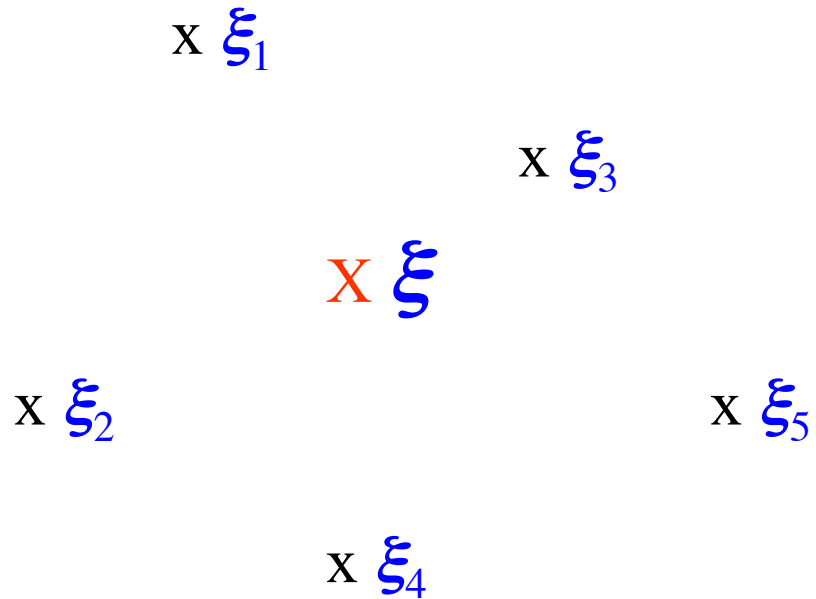
Observations $y_j = \Phi(\boldsymbol{\xi}_j) + \varepsilon_j$ at points $\boldsymbol{\xi}_j$

Value $x = \Phi(\boldsymbol{\xi})$ at point $\boldsymbol{\xi}$?

Optimal Interpolation

Random field $\Phi(\xi)$

Observation network $\xi_1, \xi_2, \ldots, \xi_p$

For one particular realization of the field, observations

$$y_j = \Phi(\xi_j) + \varepsilon_j \quad, \ j = 1, \ldots, p \quad , \qquad \text{making up vector } y = (y_j)$$

Estimate $x = \Phi(\xi)$ at given point ξ, in the form

$$x^a = \alpha + \Sigma_j \, \beta_j \, y_j = \alpha + \beta^{\mathrm{T}} y \ , \qquad \text{where } \beta = (\beta_j)$$

α and the β_j's being determined so as to minimize the expected quadratic estimation error $E[(x - x^a)^2]$

Optimal Interpolation (continued 1)

$$E[(x\text{-}x^a)^2] \text{ minimum} \Rightarrow E(x\text{-}x^a) = 0 \quad \text{Estimate } x^a \text{ is unbiased.}$$

$$x^a = \alpha + \Sigma_j \, \beta_j \, y_j$$

$$E(x^a) = \alpha + \Sigma_j \, \beta_j \, E(y_j)$$

$$x^a - E(x) = \Sigma_j \, \beta_j \, [y_j - E(y_j)]$$

Computations are to be made on centred variables

$x'^a \equiv x^a - E(x)$ is the linear combination of the $y_j' = y_j - E(y_j)$ that minimizes the distance to $x' = x - E(x)$. It is the orthogonal projection, in the sense of covariance, of x' onto the space spanned by the y_j''s.

Optimal Interpolation (continued 2)

x' - x'^a uncorrelated with y_j'

$$E[(x' - x'^a)\, y_j'] = 0$$
$$x'^a = \Sigma_k\, \beta_k\, y_k'$$

$$\Rightarrow \qquad \Sigma_k\, \beta_k\, E(y_k'\, y_j') = E(x'\, y_j')$$

in matrix form $\qquad C_{yy}\, \boldsymbol{\beta} = C_{yx}$

Optimal Interpolation (continued 3)

Solution

$$x^a = E(x) + E(x'\mathbf{y'}^{\mathrm{T}})\,[E(\mathbf{y'}\mathbf{y'}^{\mathrm{T}})]^{-1}\,[\mathbf{y} - E(\mathbf{y})]$$

$$= E(x) + \mathbf{C}_{xy}\,[\mathbf{C}_{yy}]^{-1}\,[\mathbf{y} - E(\mathbf{y})]$$

$i.\,e.,$ $\qquad \boldsymbol{\beta}^{\mathrm{T}} = \mathbf{C}_{xy}\,[\mathbf{C}_{yy}]^{-1}$

$$\alpha = E(x) - \boldsymbol{\beta}^{\mathrm{T}} E(\mathbf{y})$$

Estimate is unbiased $\qquad E(x\text{-}x^a) = 0$

Minimized quadratic estimation error

$$E[(x\text{-}x^a)^2] = E(x'^2) - E[(x'^a)^2])$$

$$= \mathbf{C}_{xx} - \mathbf{C}_{xy}\,[\mathbf{C}_{yy}]^{-1}\,\mathbf{C}_{yx}$$

Estimation made in terms of deviations x' and $\mathbf{y'}$ from expectations $E(x)$ and $E(\mathbf{y})$.

Optimal Interpolation (continued 4)

$$x^a = E(x) + E(x'\boldsymbol{y}'^{\mathrm{T}}) \, [E(\boldsymbol{y}'\boldsymbol{y}'^{\mathrm{T}})]^{-1} \, [\boldsymbol{y} - E(\boldsymbol{y})]$$

$$y_j = \Phi(\boldsymbol{\xi}_j) + \varepsilon_j$$

$$E(y_j'y_k') = E\left\{[\Phi'(\boldsymbol{\xi}_j) + \varepsilon_j'][\Phi'(\boldsymbol{\xi}_k) + \varepsilon_k']\right\}$$

If observation errors ε_j are mutually uncorrelated, have common variance r, and are uncorrelated with field Φ, then

$$E(y_j'y_k') = C_\Phi(\boldsymbol{\xi}_{j,}\,\boldsymbol{\xi}_k) + r\delta_{jk}$$

and

$$E(x'y_j') = C_\Phi(\boldsymbol{\xi}_,\,\boldsymbol{\xi}_j)$$

Optimal Interpolation (continued 5)

Unique observation ($p=1$) $\qquad y_1 = \Phi(\xi_1) + \varepsilon_1$

Value $x = \Phi(\xi)$ at some point ξ to be estimated

(all values assumed to be centred)

$$C_{yy}\,\beta = C_{yx}$$

$$C_{yy} = E(y_1^2) = C_{\Phi}(\xi_1, \xi_1) + r \qquad\qquad C_{yx} = C_{\Phi}(\xi, \xi_1)$$

$$x^a = \Phi^a(\xi) = \frac{C_{\Phi}(\xi, \xi_1)}{C_{\Phi}(\xi_1, \xi_1) + r}\, y_1$$

Optimal Interpolation (continued 6)

$$x^a = \Phi^a(\xi) = \frac{C_\Phi(\xi, \xi_1)}{C_\Phi(\xi_1, \xi_1) + r} y_1$$

Figure 5.1.1.4.1 Auto-correlation of errors in 12h numerical fore-casts of surface pressure in a reference station (Stockholm) and other stations.
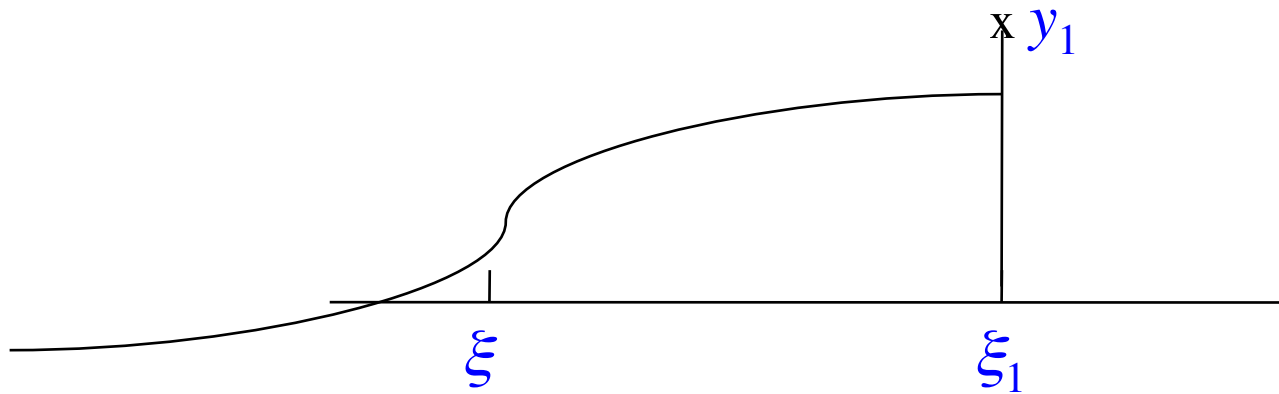
After N. Gustafsson

Optimal Interpolation (continued 7)

Two mutually close observations ($p=2$) $y_j = \Phi(\boldsymbol{\xi}_j) + \varepsilon_j$, $j = 1,2$
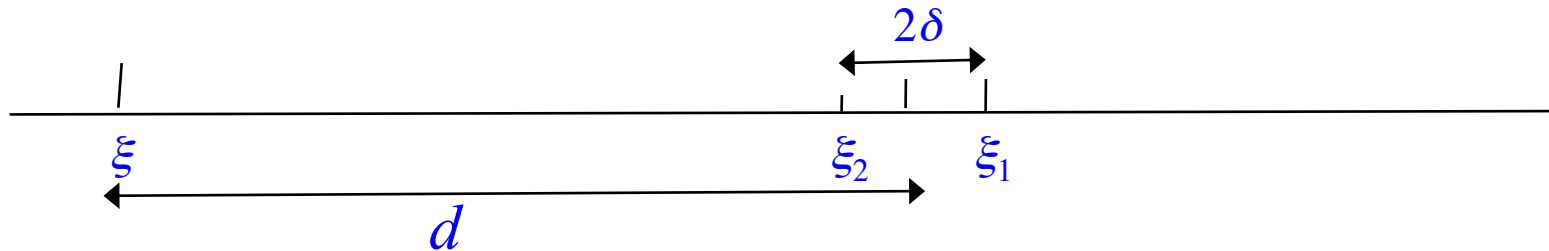
Homogeneous covariance function $C_\Phi(\chi_1, \chi_2) = \Gamma(\chi_1 - \chi_2)$

Linear system for weights β_j's

$$\begin{pmatrix} \Gamma(0) + r & \Gamma(2\delta) \\ \Gamma(2\delta) & \Gamma(0) + r \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \Gamma(d + \delta) \\ \Gamma(d - \delta) \end{pmatrix}$$

Optimal Interpolation (continued 8)

Two mutually close observations (p=2) $y_j = \Phi(\xi_j) + \varepsilon_j$, $j = 1,2$
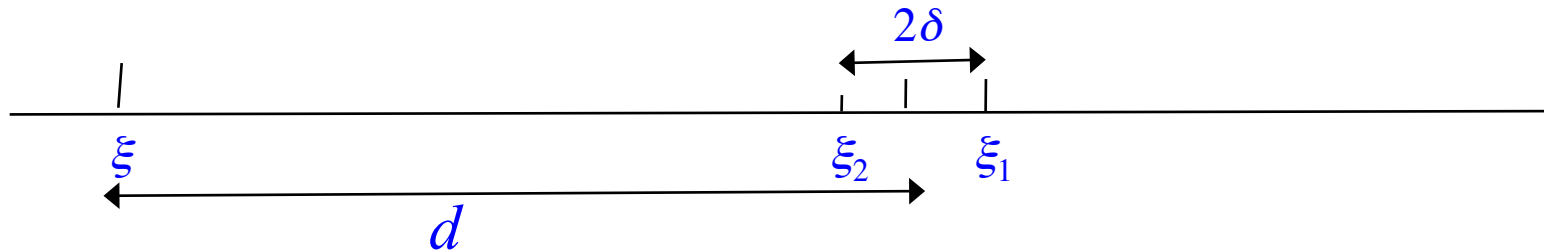
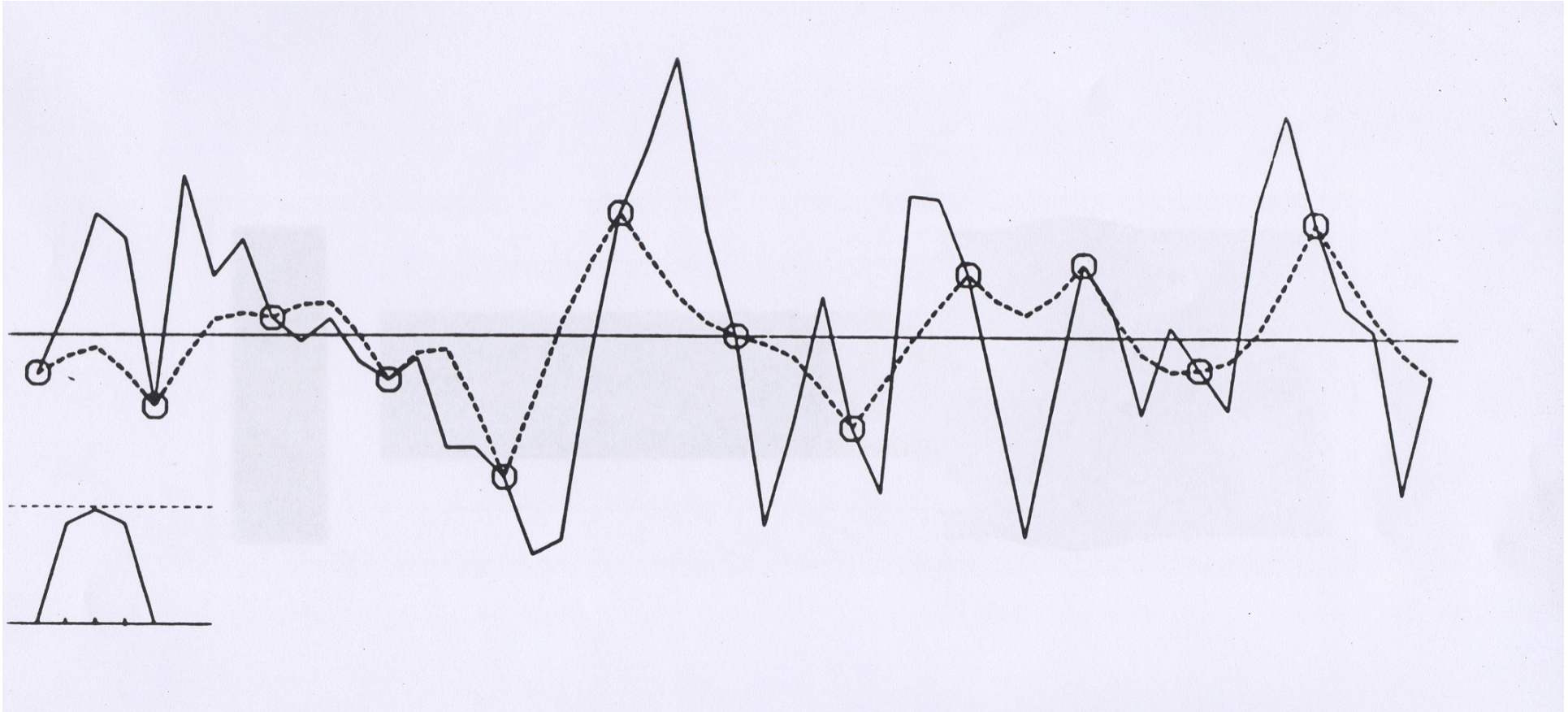$$\beta_1 + \beta_2 = \frac{\Gamma(d+\delta) + \Gamma(d-\delta)}{\Gamma(0) + \Gamma(2\delta) + r}$$
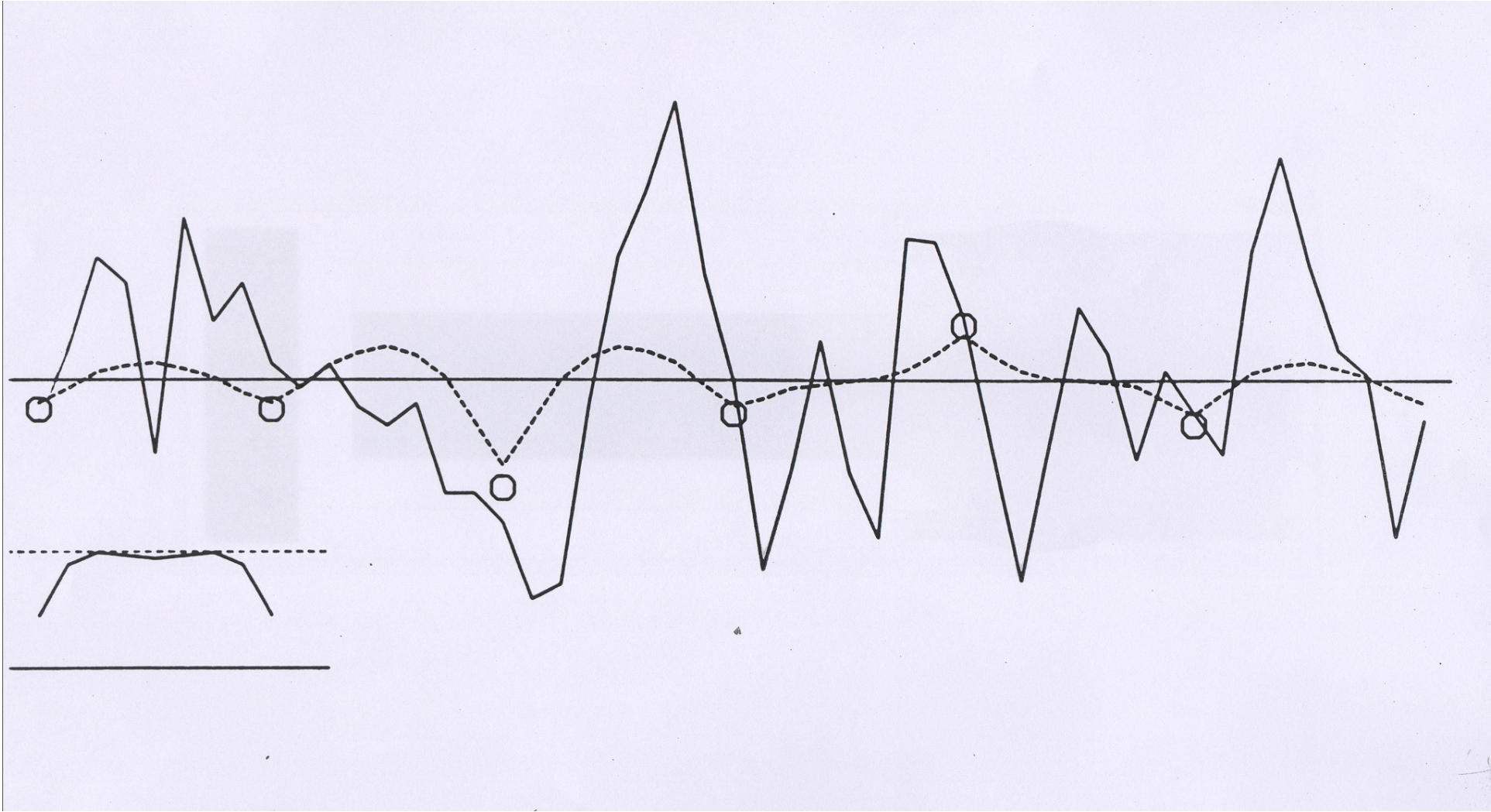
For small δ,

$$\beta_1 + \beta_2 = \frac{\Gamma(d)}{\Gamma(0) + r/2}$$

Sum equals weight that would be given to a unique observation located at position d, with error $r/2$
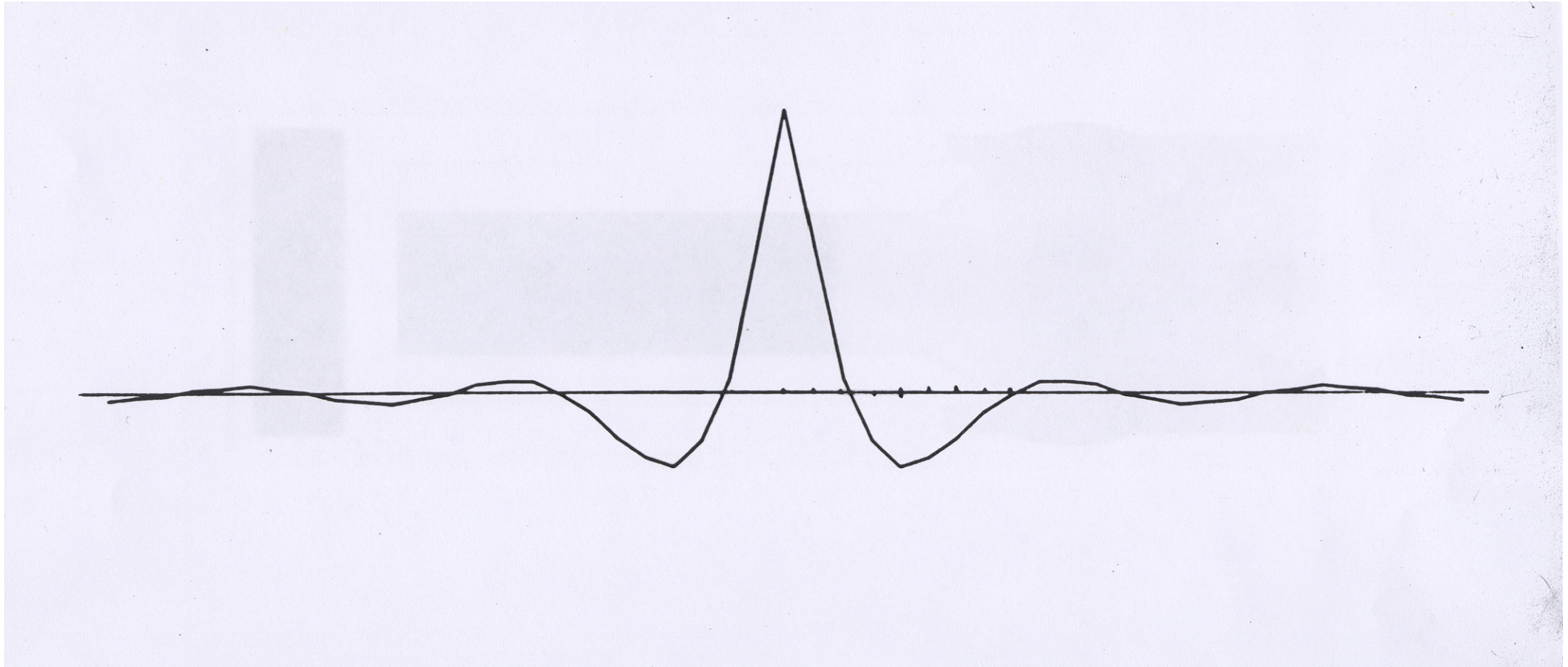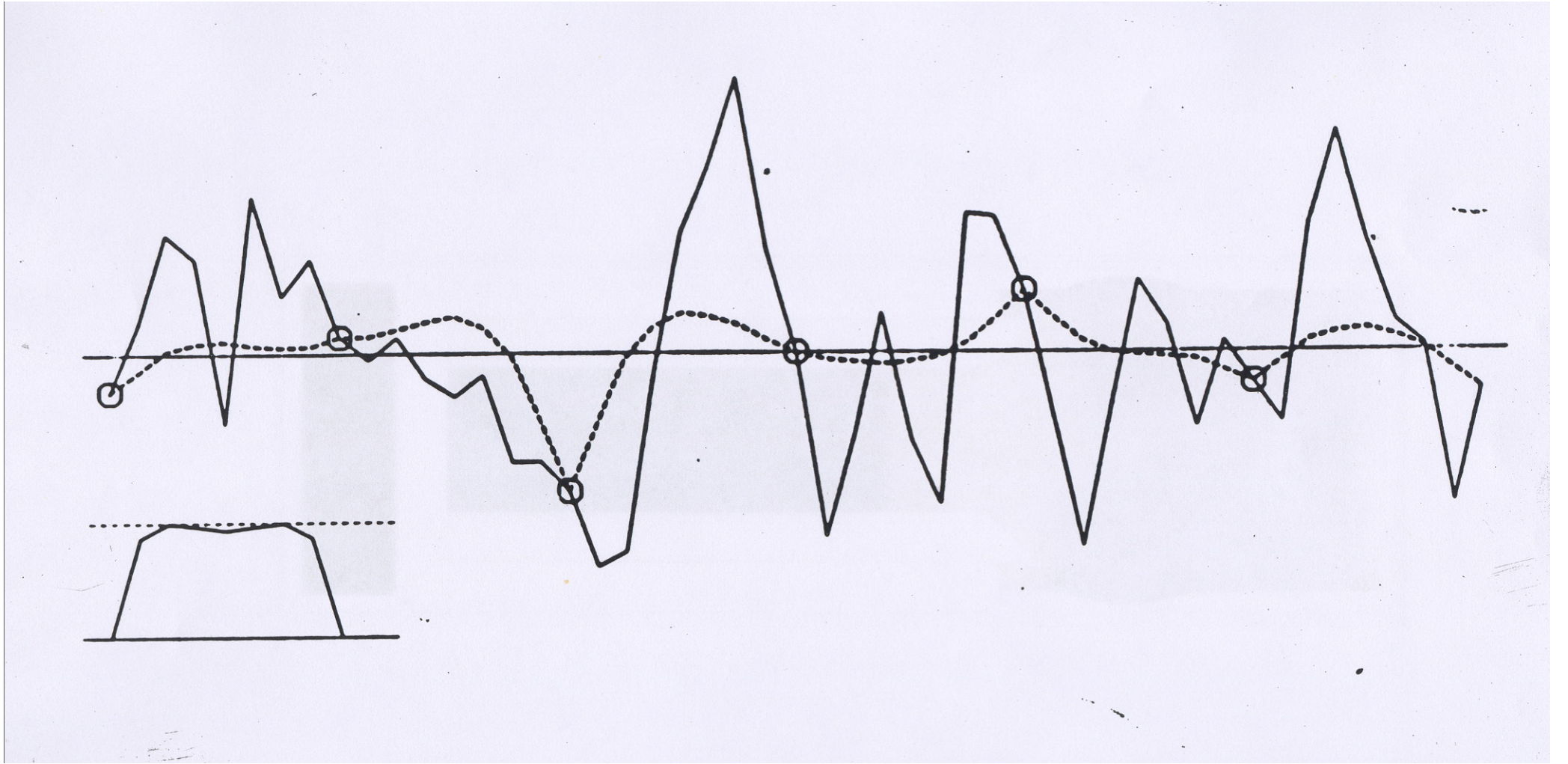
Optimal Interpolation (continued 10)

$$x^a = E(x) + C_{xy} [C_{yy}]^{-1} [y - E(y)]$$

Vector

$$\mu = (\mu_j) \equiv [C_{yy}]^{-1} [y - E(y)]$$

is independent of variable to be estimated

$$x^a = E(x) + \Sigma_j \, \mu_j \, E(x'y_j')$$

Optimal Interpolation (continued 11)

$$x^a = E(x) + \Sigma_j \, \mu_j \, E(x'y_j')$$

$$\Phi^a(\xi) = E[\Phi(\xi)] + \Sigma_j \, \mu_j \, E[\Phi'(\xi) \, y_j']$$

Under hypotheses made above, $E[\Phi'(\xi) \, y_j'] = C_\Phi(\xi, \xi_j)$

$$\Phi^a(\xi) = E[\Phi(\xi)] + \Sigma_j \, \mu_j \, C_\Phi(\xi, \xi_j)$$

Correction made on background expectation is a linear combination of the p functions $C_\Phi(\xi, \xi_j)$

$C_\Phi(\xi, \xi_j)$, considered as a function of estimation position ξ, is the *representer* associated with observation y_j.

Optimal Interpolation (continued 12)

Univariate interpolation. Each physical field (*e. g.* temperature) determined from observations of that field only.

Multivariate interpolation. Observations of different physical fields are used simultaneously. Requires specification of cross-covariances between various fields.

Cross-covariances between mass and velocity fields can simply be modelled on the basis of geostrophic balance.

Cross-covariances between humidity and temperature (and other) fields still a problem.
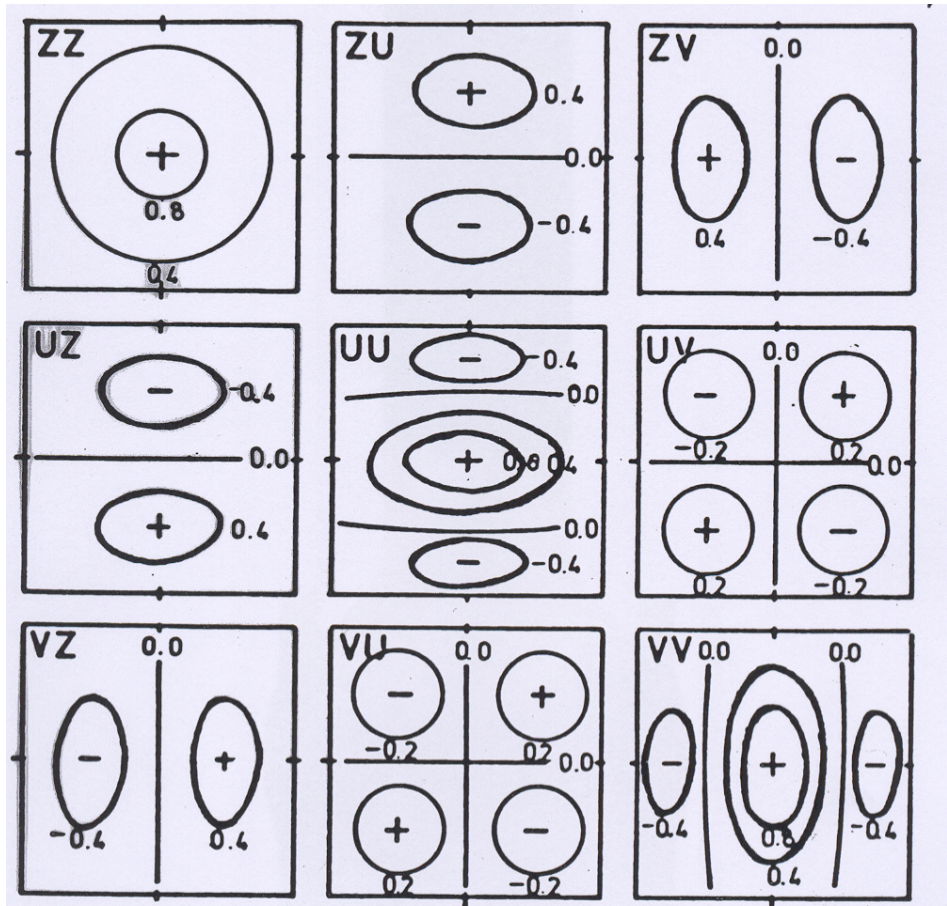
4.: Schematic illustration of correlation functions and cross-correlation functions for multi-variate analysis derived by the geostrophic assumption.
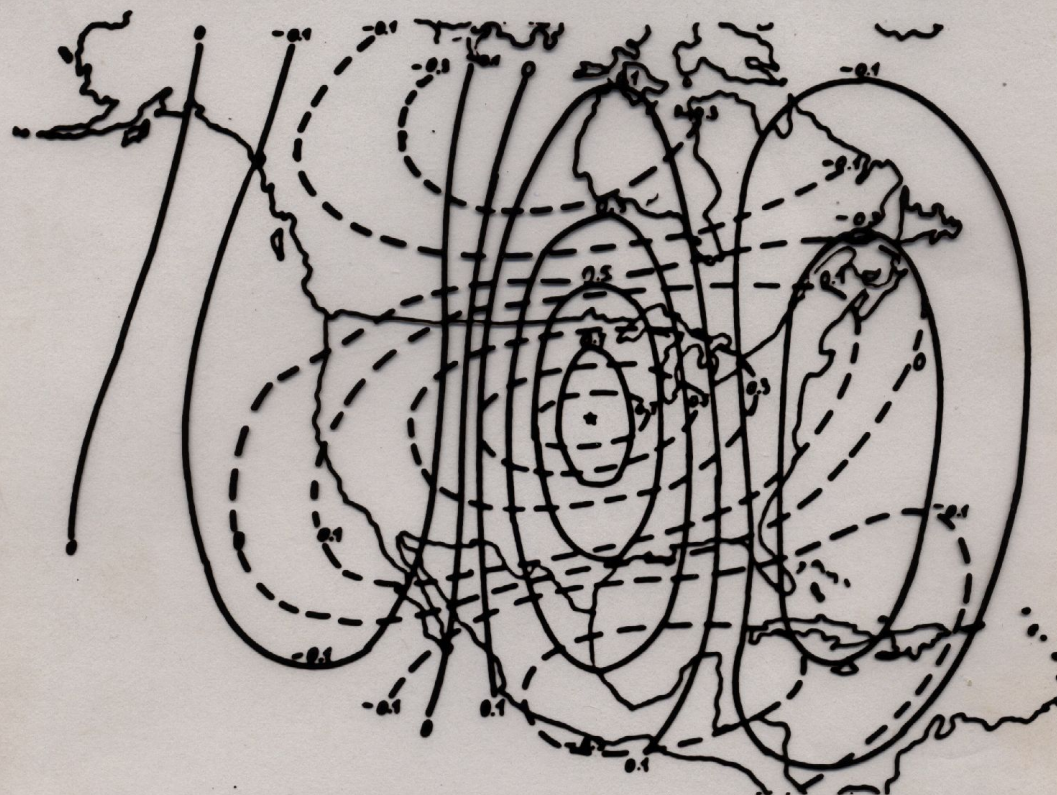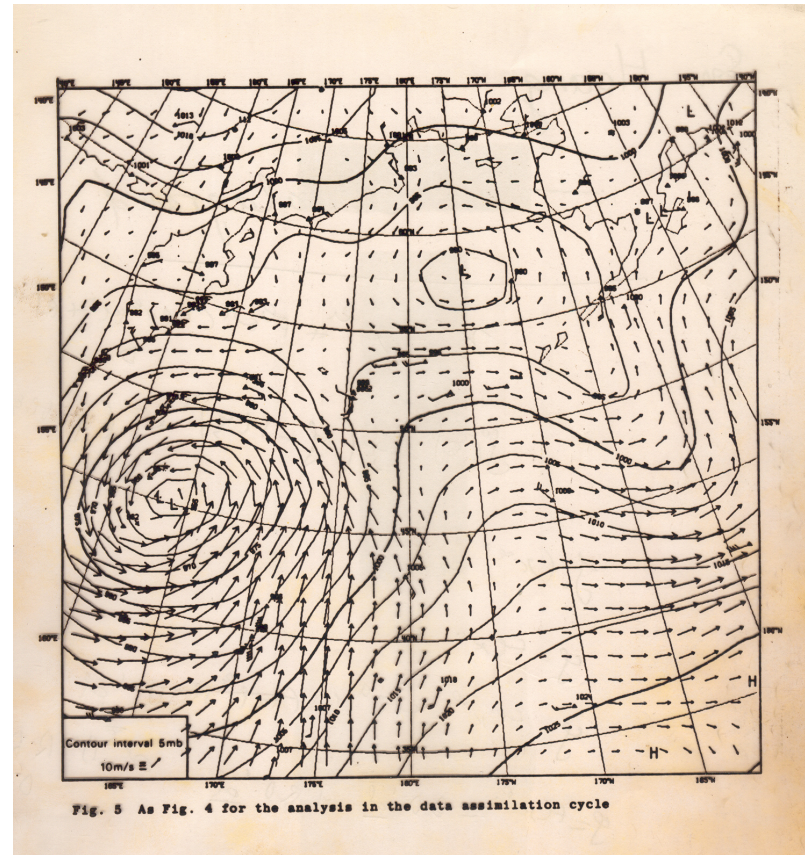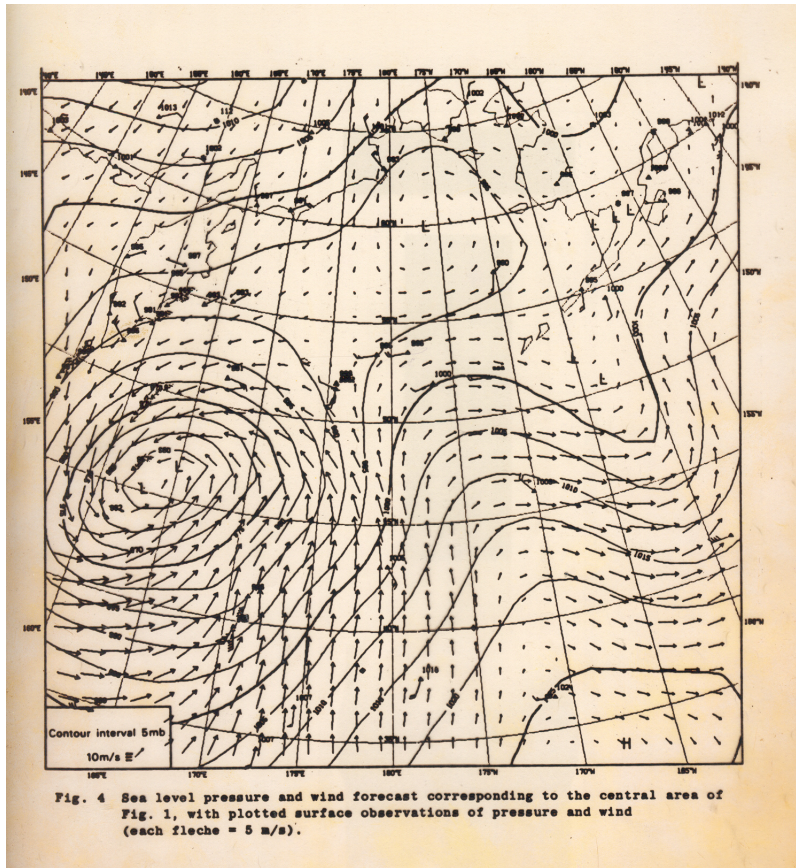
After N. Gustafsson

Figure 4.2.4.3: Isolines for the auto-correlation of the 500 mb u-wind component (dashed line) and the auto-correlation of the 500 mb v-wind component (full line). The "star" indicates the position of the reference station. (From Buel (1972).

After N. Gustafsson

Fig. 4 Sea level pressure and wind forecast corresponding to the central area of Fig. 1, with plotted surface observations of pressure and wind (each fleche = 5 m/s).

Fig. 5 As Fig. 4 for the analysis in the data assimilation cycle
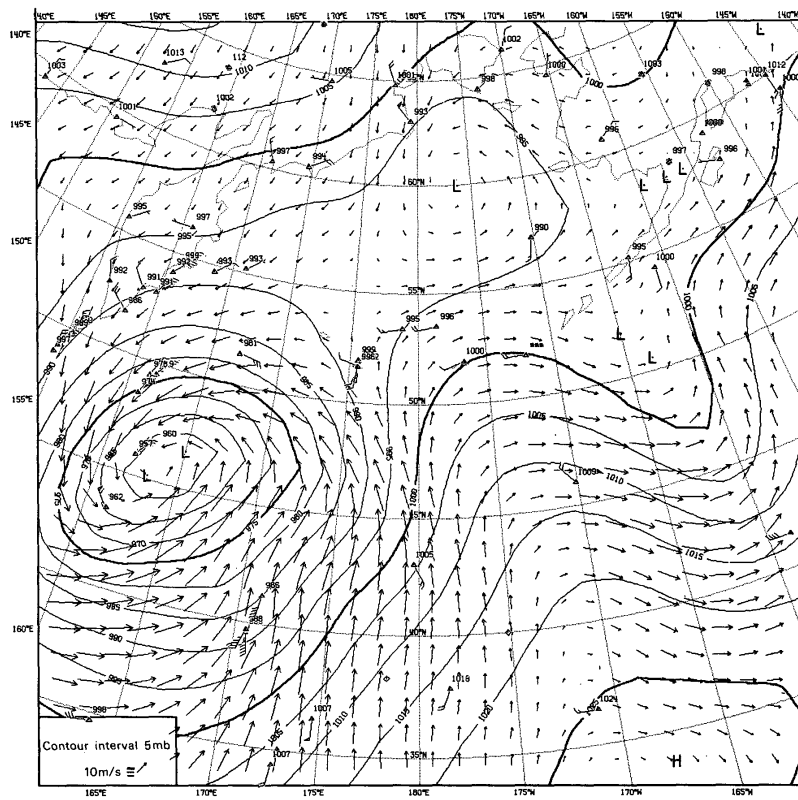
After A. Lorenc, MWR, 1981

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FIG. 14. Sea level pressure and wind forecast corresponding to the central area of Fig. 11, with plotted surface observations of sea level pressure and wind (each barb = 5 m s⁻¹).
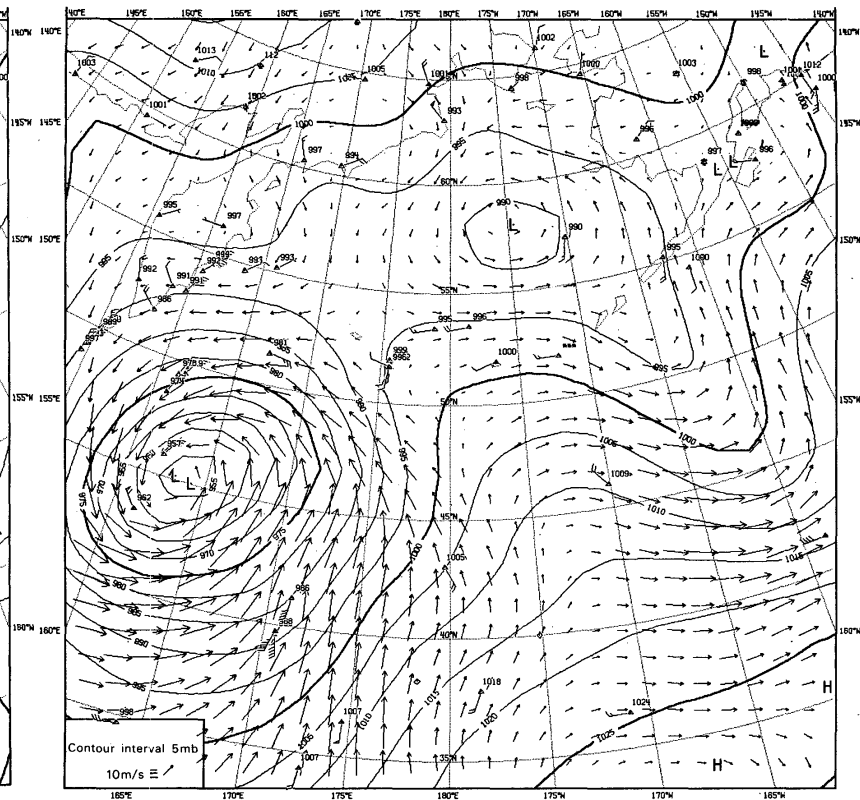
FIG. 15. As in Fig. 14 for the analysis in the data-assimilation cycle.

After A. Lorenc, MWR, 1981

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Optimal Interpolation (continued 13)

Observation vector $\boldsymbol{y}$

Estimation of a scalar x

$$x^a = E(x) + \boldsymbol{C_{xy}}\,[\boldsymbol{C_{yy}}]^{-1}\,[\boldsymbol{y} - E(\boldsymbol{y})]$$

$$p^a \equiv E[(x-x^a)^2] = E(x'^2) - E[(x'^a)^2])$$
$$= C_{xx} - \boldsymbol{C_{xy}}\,[\boldsymbol{C_{yy}}]^{-1}\,\boldsymbol{C_{yx}}$$

Estimation of a vector $\boldsymbol{x}$

$$\boldsymbol{x}^a = E(\boldsymbol{x}) + \boldsymbol{C_{xy}}\,[\boldsymbol{C_{yy}}]^{-1}\,[\boldsymbol{y} - E(\boldsymbol{y})]$$

$$\boldsymbol{P}^a \equiv E[(\boldsymbol{x}-\boldsymbol{x}^a)\,(\boldsymbol{x}-\boldsymbol{x}^a)^{\mathrm{T}}] = E(\boldsymbol{x}'\boldsymbol{x}'^{\mathrm{T}}) - E(\boldsymbol{x}'^a\,\boldsymbol{x}'^{a\mathrm{T}})$$
$$= C_{xx} - \boldsymbol{C_{xy}}\,[\boldsymbol{C_{yy}}]^{-1}\,\boldsymbol{C_{yx}}$$

Optimal Interpolation (continued 14)

$$x^a = E(x) + C_{xy} [C_{yy}]^{-1} [y - E(y)]$$

$$P^a = C_{xx} - C_{xy} [C_{yy}]^{-1} C_{yx}$$

If probability distribution for couple (x, y) is Gaussian (with, in particular, covariance matrix

$$C \equiv \begin{pmatrix} C_{xx} & C_{xy} \\ C_{yx} & C_{yy} \end{pmatrix}$$

then Optimal Interpolation achieves Bayesian estimation, in the sense that

$$P(x \mid y) = \mathcal{N}[x^a, P^a]$$

Convective Instability

In dry atmosphere in hydrostatic balance, adiabatic lapse rate (vertical gradient of temperature)

$(dT/dz)_{ad} = -g/C_p$

$g \approx 10\ m\ s^{-2}$, $C_p \approx 10^3$ SI, $-g/C_p \approx -10\ °C/km$

Water vapour is present in the atmosphere, and will usually condense, and emit heat, in an ascending motion. In practice, dT/dz is observed to have value about $-6\ °C/km$, which is close to *its adiabatic wet* value.

Reminder

Potential temperature

$\theta \equiv T(p_0/p)^{\kappa}$ with $\kappa \equiv r/C_p$ (≈ 0.285 for dry air)

Potential temperature is conserved in adiabatic transformation

Stratified atmosphere at rest with temperature gradient dT/dz and associated gradient of potential temperature $d\theta/dz$.

Particle displaced adiabatically upward from its equilibrium position. Expands taking pressure of background stratification.

- if background temperature larger than temperature of displaced particle, *i. e.* $dT/dz > (dT/dz)_{ad}$ (potential temperature increases with altitude), buoyancy force will pull particle back to its original position. Stratification is said to be *convectively stable*. Particle will oscillate with *Brunt-Väisälä frequency N*

$$N^2 \equiv (g/\theta)\,(d\theta/dz)$$

In the atmosphere, the corresponding period has typical value of a few minutes.

- if background temperature lower than temperature of displaced particle, *i. e.* $dT/dz < (dT/dz)_{ad}$ (potential temperature decreases with altitude), particle will move farther away from its original position $\Rightarrow$ *convective instability*

Convective instability is at the origin of intense *convective cells* (cumulus clouds, thunderstorms), with core of intense ascending motion surrounded by slower subsiding motion. Convective instability is the main process through which energy is carried from the lower surface (continents, oceans) into the atmosphere. It also carries water and momentum.

Convection occurs also in the ocean, when the upper surface is cooled by radiation.

A similar phenomenon occurs in the ocean (but with no thermodynamical effects involved) when dense water (whose salinity has been increased by evaporation) is transported (for instance by wind) above less dense water. This phenomenon is a component of the *thermohaline* circulation.

Cours à venir

~~Vendredi 26 mars~~

~~Vendredi 2 avril~~

~~Vendredi 9 avril~~

Vendredi 16 avril

Vendredi 7 mai

Vendredi 14 mai

Vendredi 21 mai

Vendredi 28 mai