Université Pierre et Marie Curie
Master de Sciences et Technologies
Spécialité Océan, Atmosphère, Climat et Observations Spatiales

Année 2011-2012

Cours B10-2 : *Introduction à l'assimilation de données et modélisation inverse en géophysique*

De la modélisation à l'assimilation de données

Olivier Talagrand
14 Novembre 2011

Fig. 1: Members of day 7 forecast of 500 hPa geopotential height for the ensemble originated from 25 January 1993.

Figure 6 Hurricane Katrina mean-sea-level-pressure (MSLP) analysis for 12 UTC of 29 August 2005 and t+84h high-resolution and EPS forecasts started at 00 UTC of 26 August:

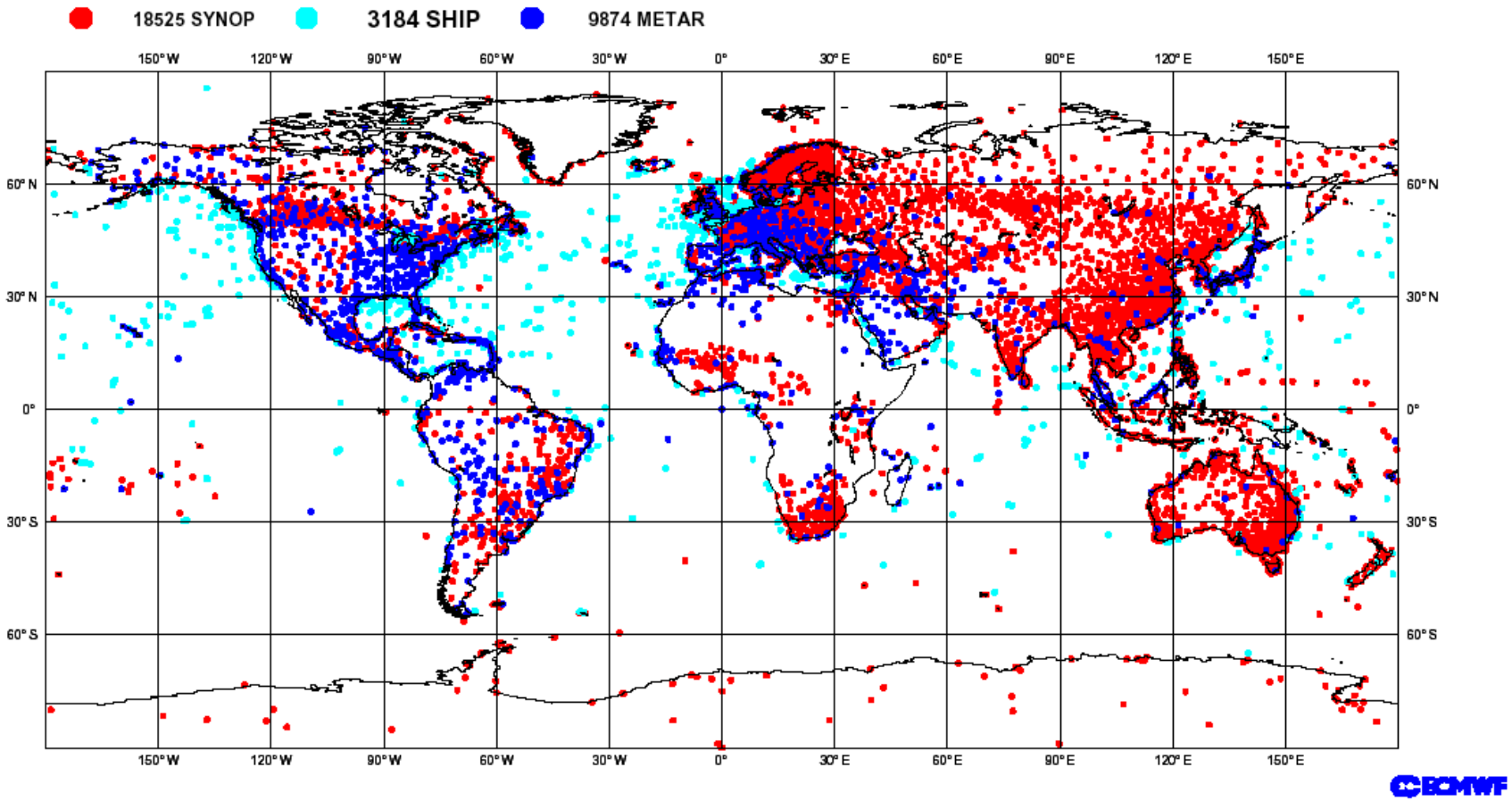
1st row: *1st panel: MSLP analysis for 12 UTC of 29 Aug*
2nd panel: MSLP t+84h $T_L511L60$ forecast started at 00 UTC of 26 Aug
3rd panel: MSLP t+84h EPS-control $T_L255L40$ forecast started at 00 UTC of 26 Aug
Other rows: 50 EPS-perturbed $T_L255L40$ forecast started at 00 UTC of 26 Aug.

The contour interval is 5 hPa, with shading patters for MSLP values lower than 990 hPa.

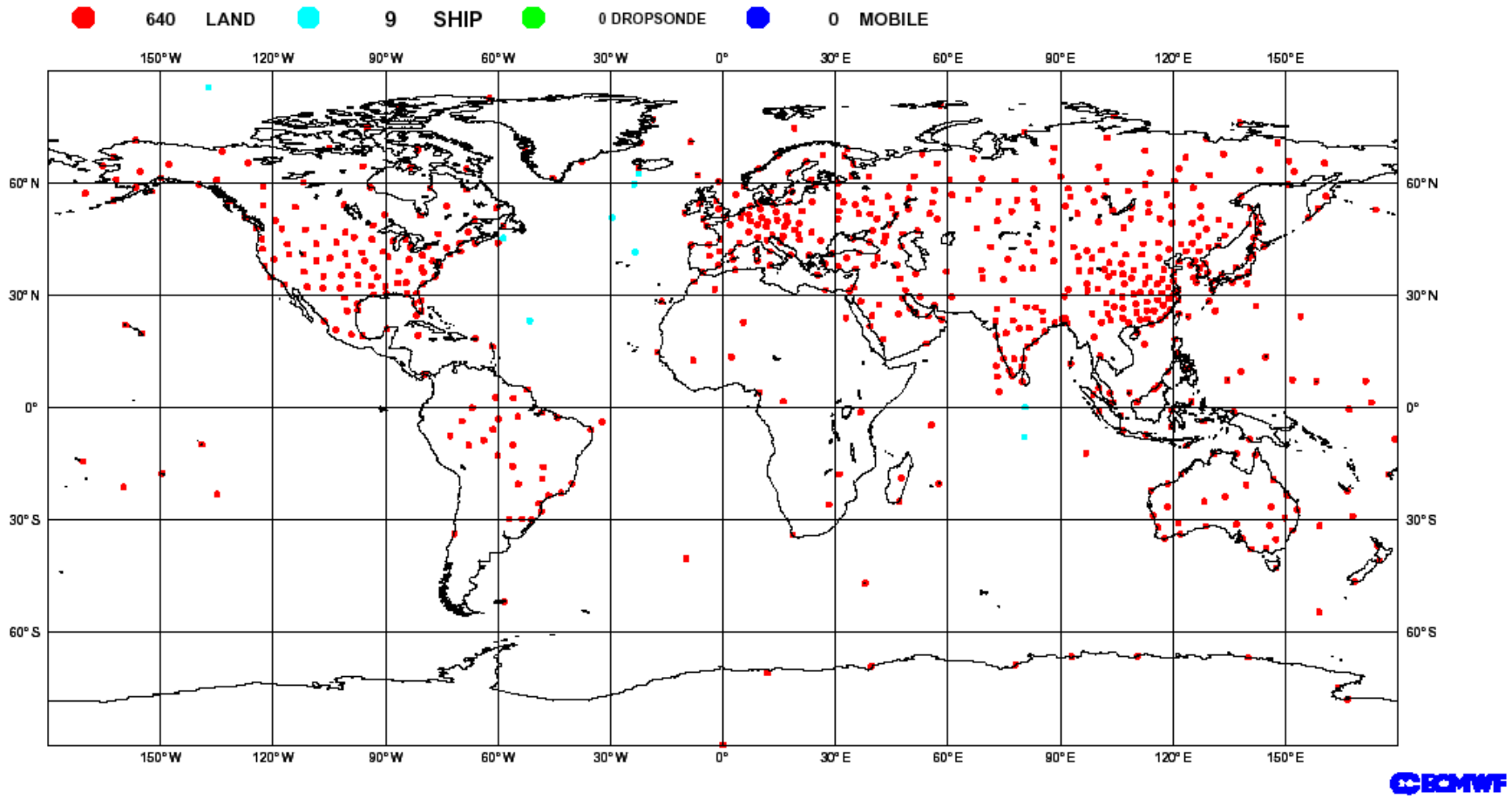
Pourquoi les météorologistes ont-ils tant de peine à prédire le temps avec quelque certitude ? Pourquoi les chutes de pluie, les tempêtes elles-mêmes nous semblent-elles arriver au hasard, de sorte que bien des gens trouvent tout naturel de prier pour avoir la pluie ou le beau temps, alors qu'ils jugeraient ridicule de demander une éclipse par une prière ?[...] un dixième de degré en plus ou en moins en un point quelconque, le cyclone éclate ici et non pas là, et il étend ses ravages sur des contrées qu'il aurait épargnées. Si on avait connu ce dixième de degré, on aurait pu le savoir d'avance, mais les observations n'étaient ni assez serrées, ni assez précises, et c'est pour cela que tout semble dû à l'intervention du hasard.

H. Poincaré, *Science et Méthode*, Paris, 1908

ECMWF Data Coverage (All obs DA) - Synop-Ship-Metar
13/Nov/2011; 00 UTC
Total number of obs = 31583



ECMWF Data Coverage (All obs DA) - Temp
13/Nov/2011; 00 UTC
Total number of obs = 649



ECMWF Data Coverage (All obs DA) - Pilot-Profiler
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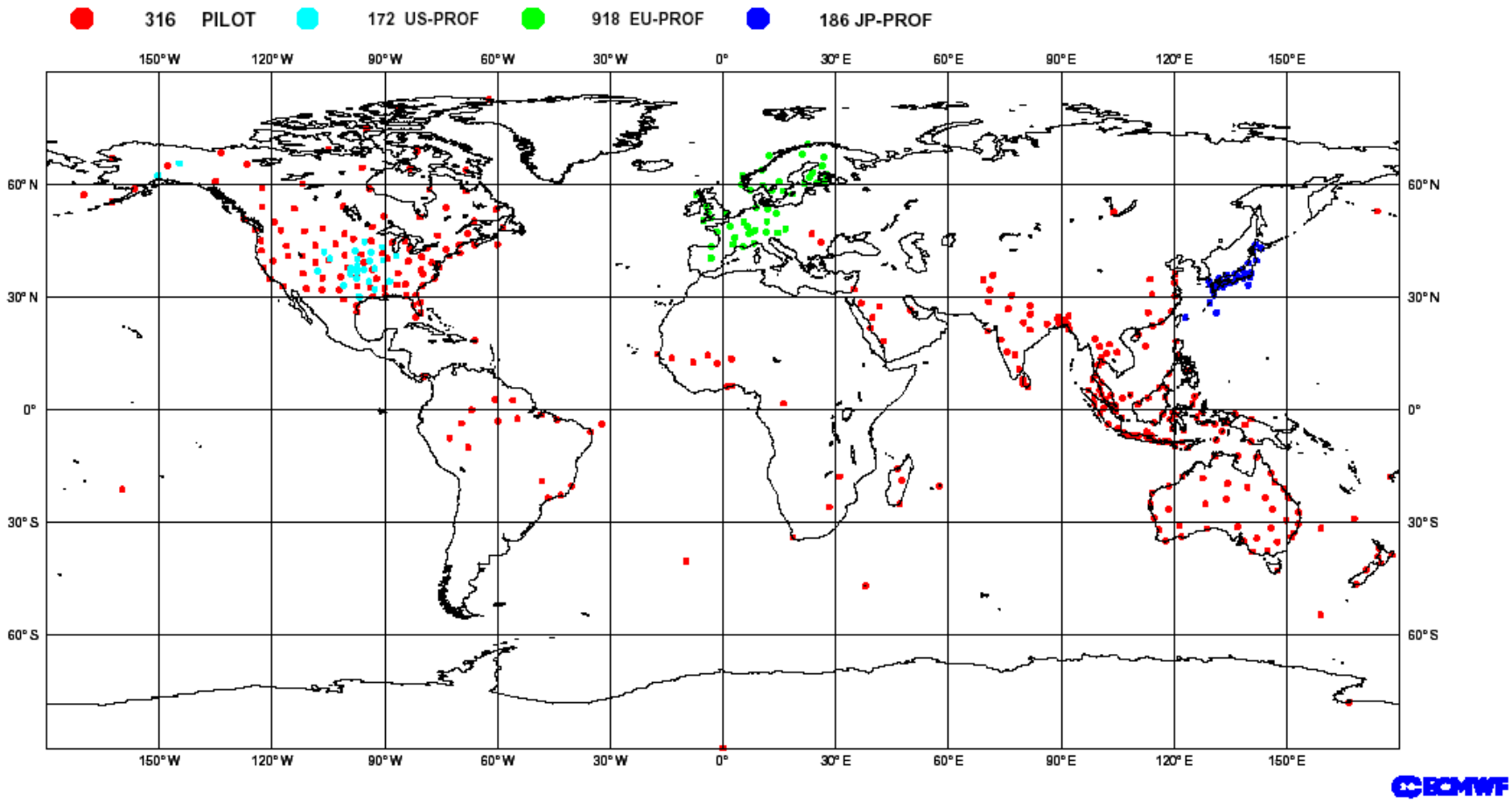

ECMWF Data Coverage (All obs DA) - Aircraft
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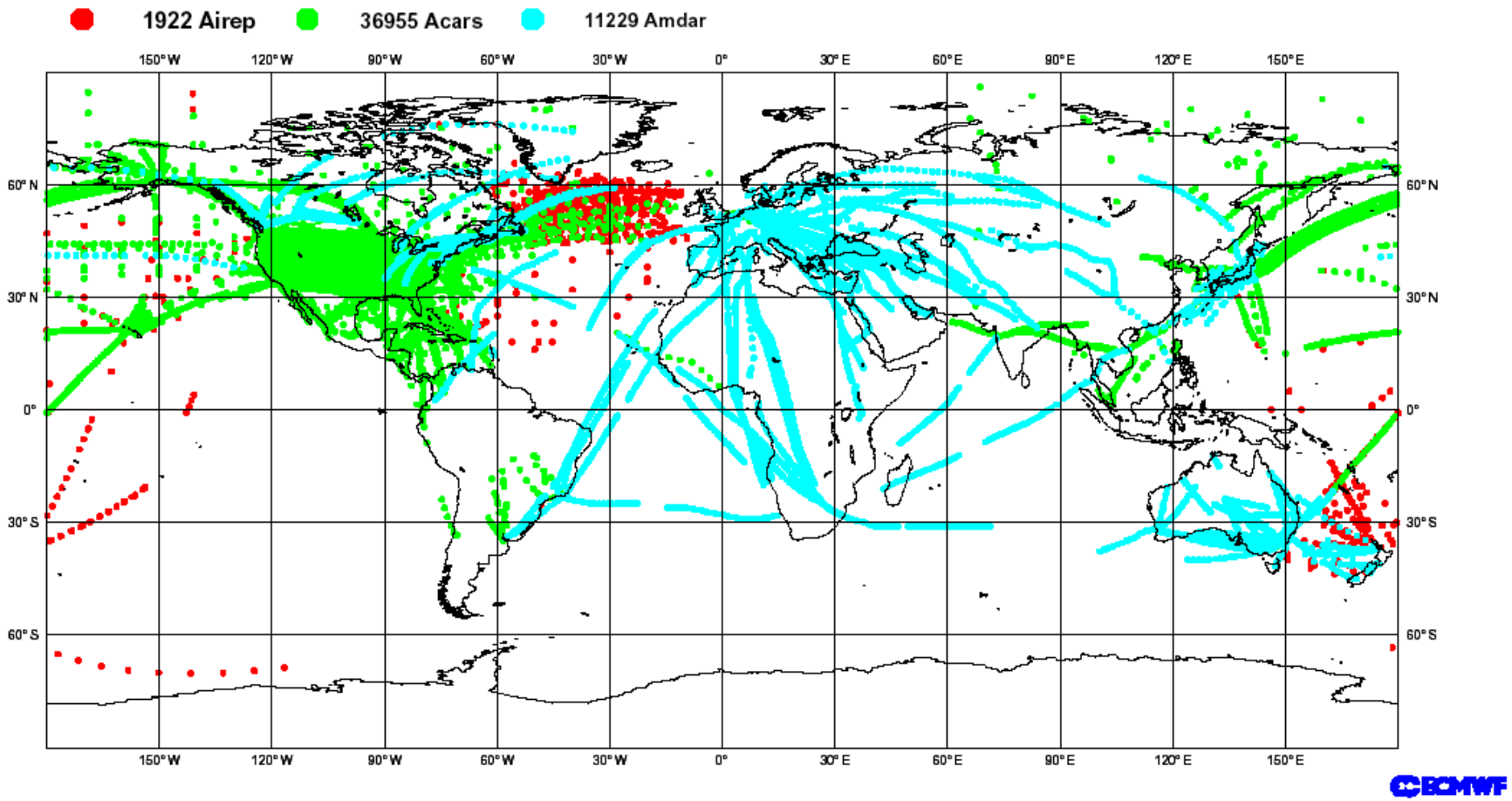
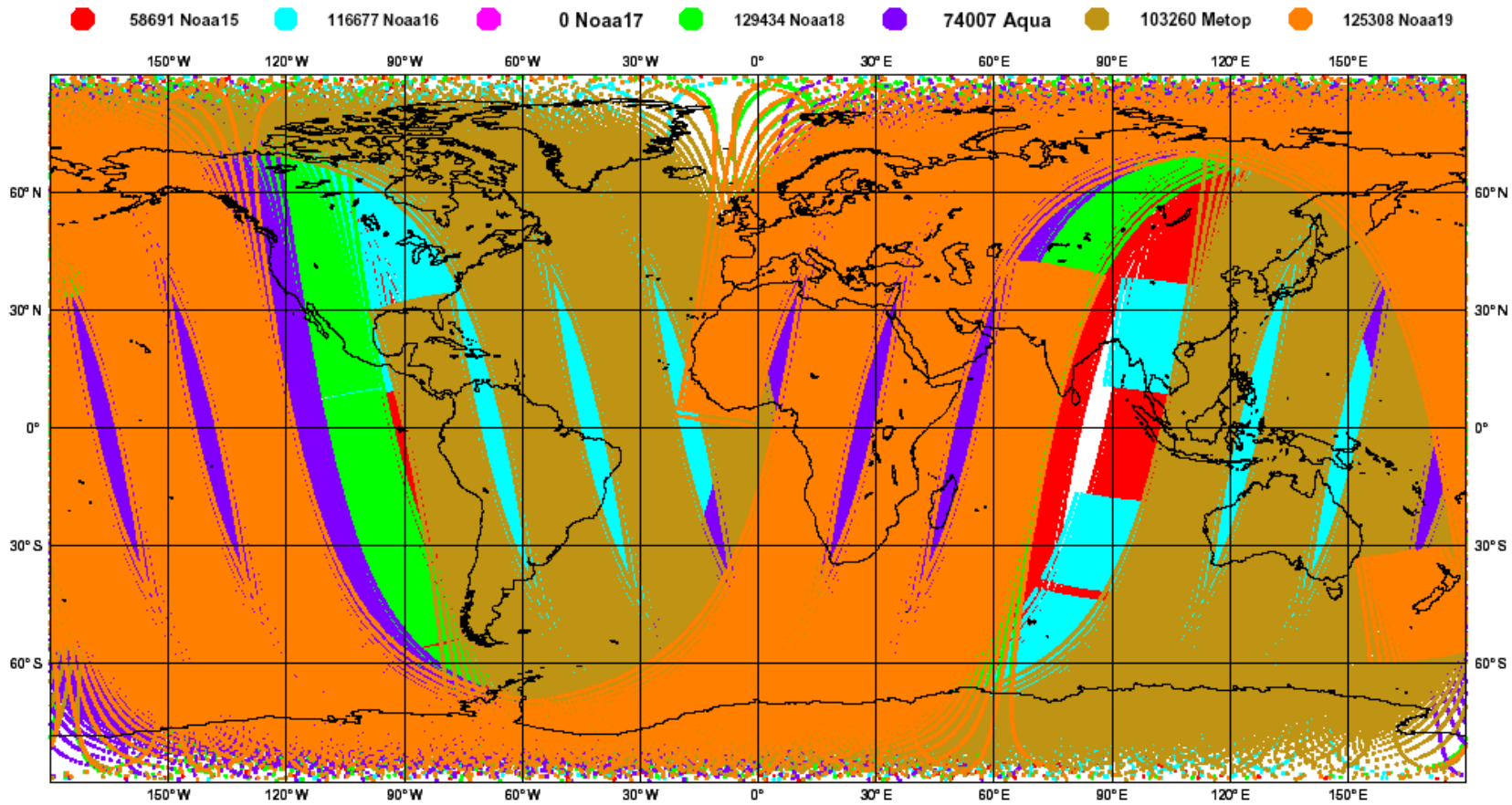


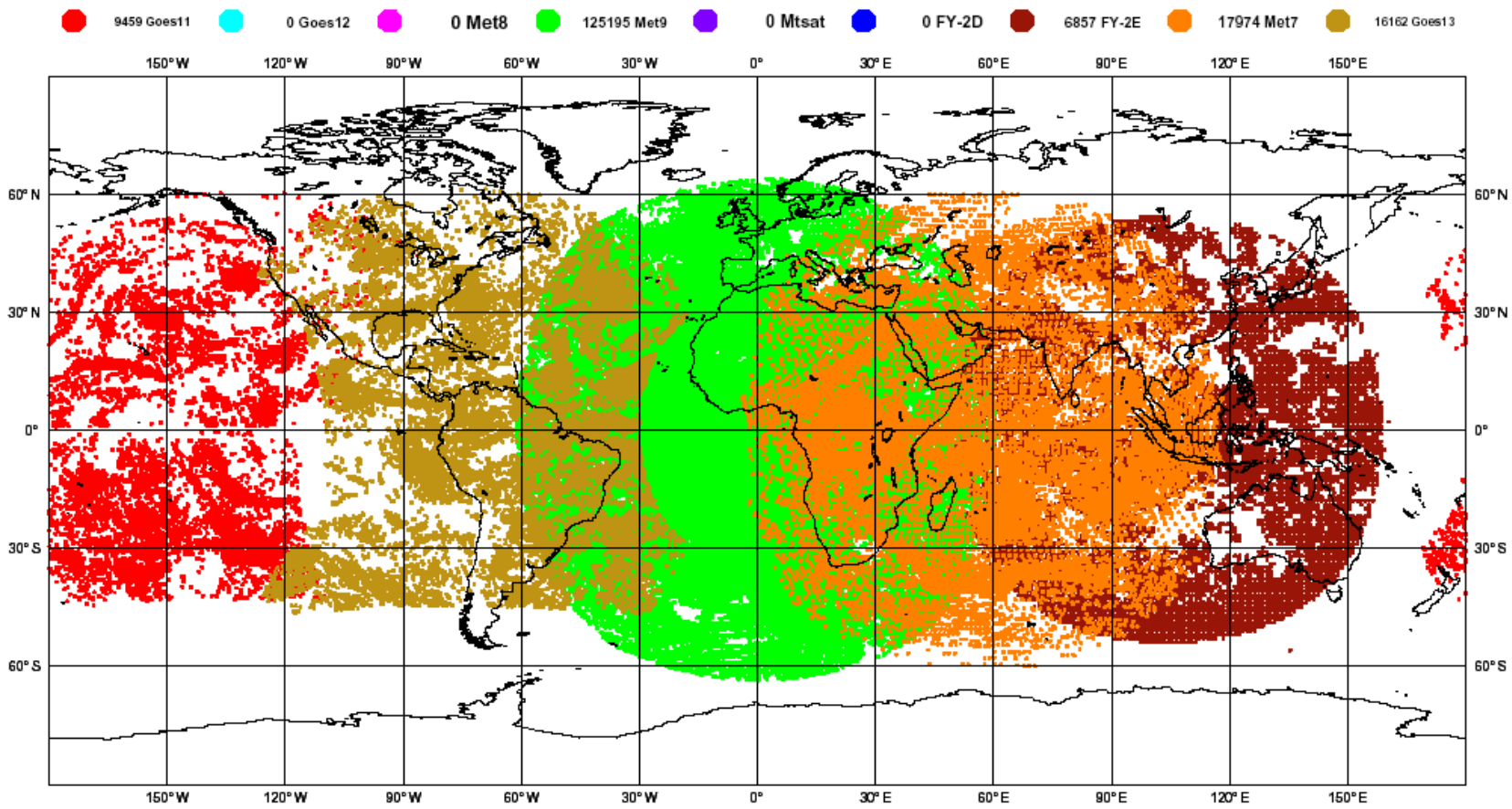
ECMWF

ECMWF Data Coverage (All obs DA) - AMSU-A
13/Nov/2011; 00 UTC
Total number of obs = 607377



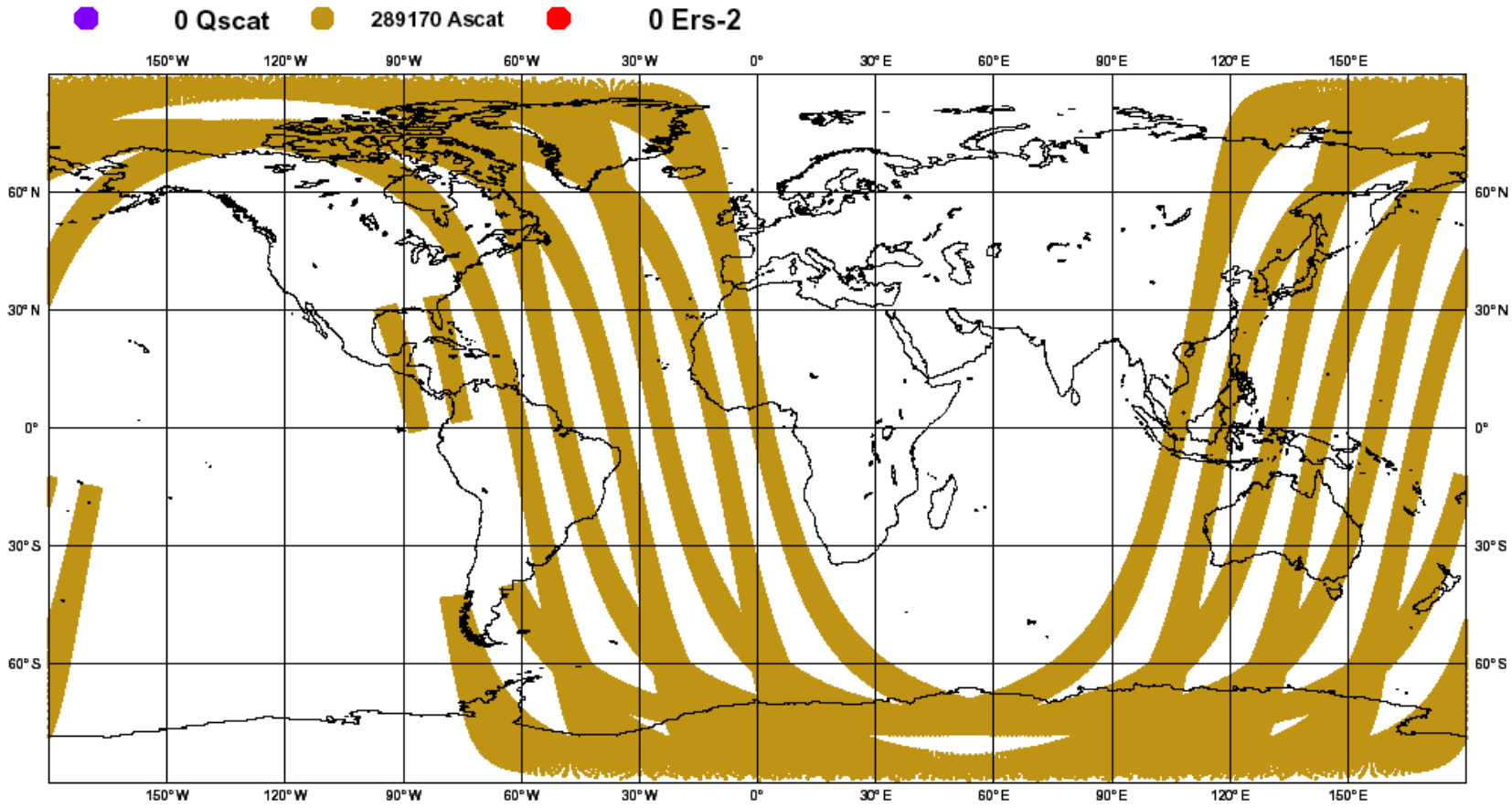
ECMWF

ECMWF Data Coverage (All obs DA) - AMV WV
13/Nov/2011; 00 UTC
Total number of obs = 175647

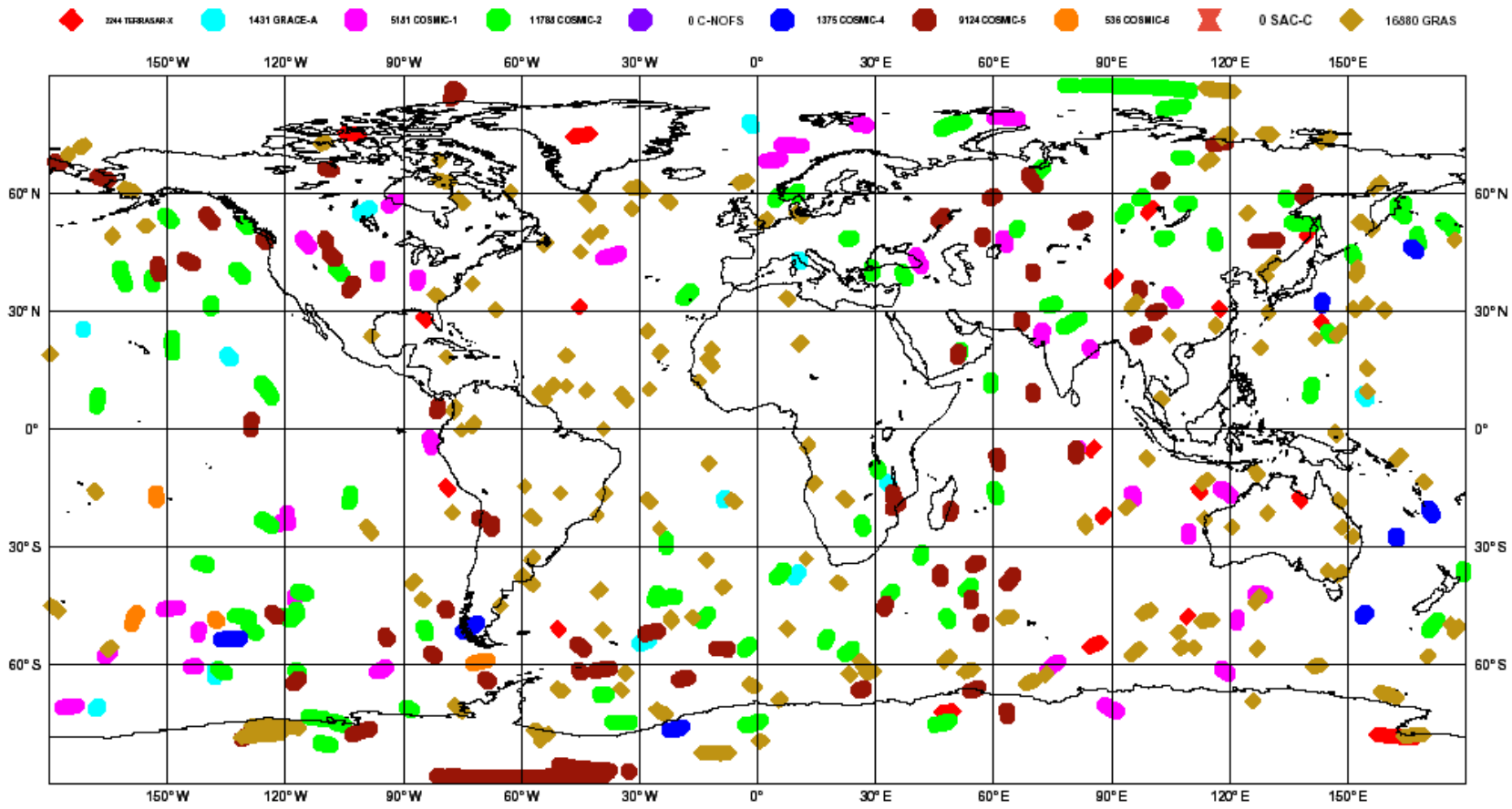


ECMWF

ECMWF Data Coverage (All obs DA) - SCAT
13/Nov/2011; 00 UTC
Total number of obs = 289170

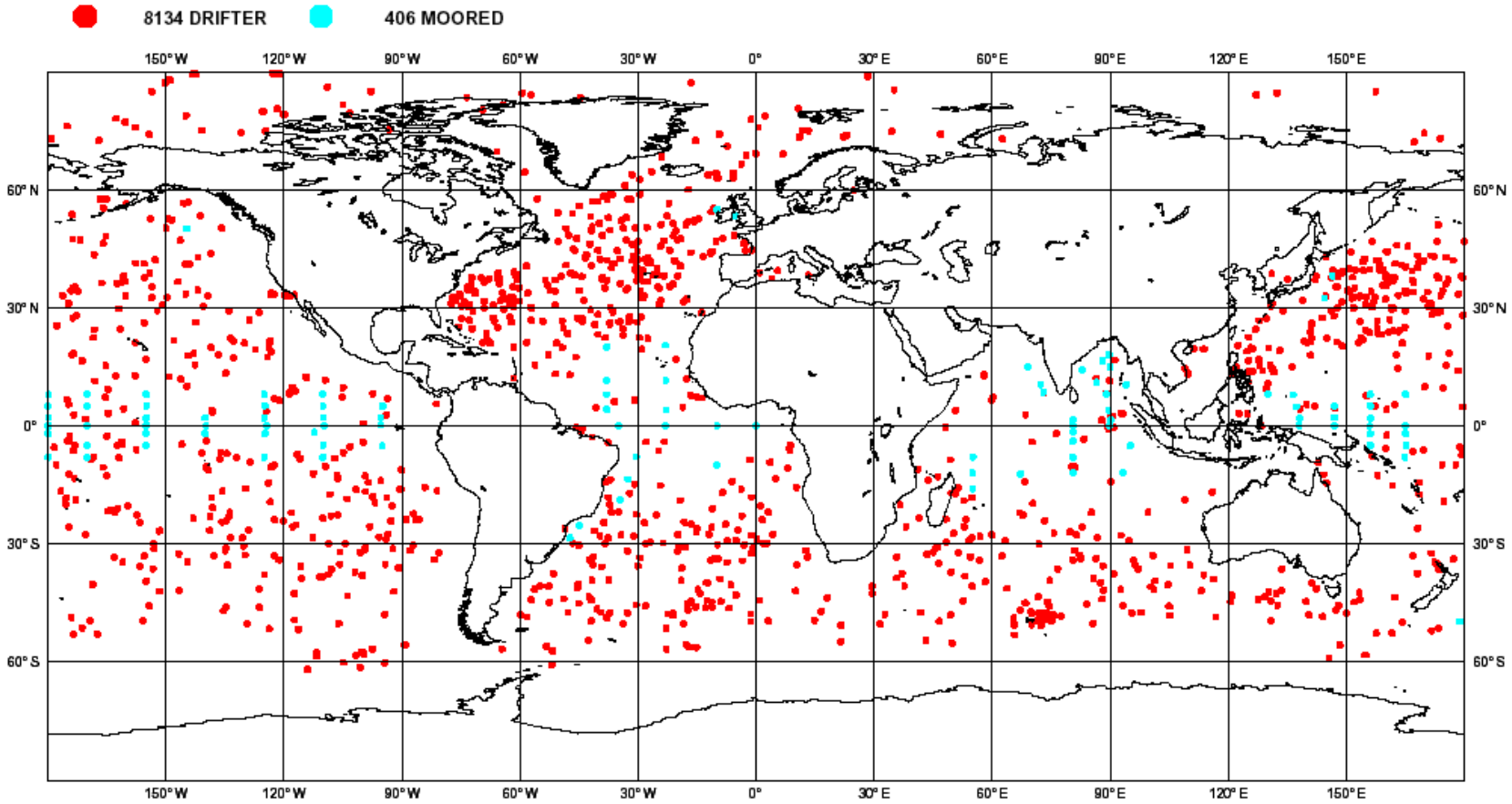


ECMWF Data Coverage (All obs DA) - GPSRO
13/Nov/2011; 00 UTC
Total number of obs = 48559



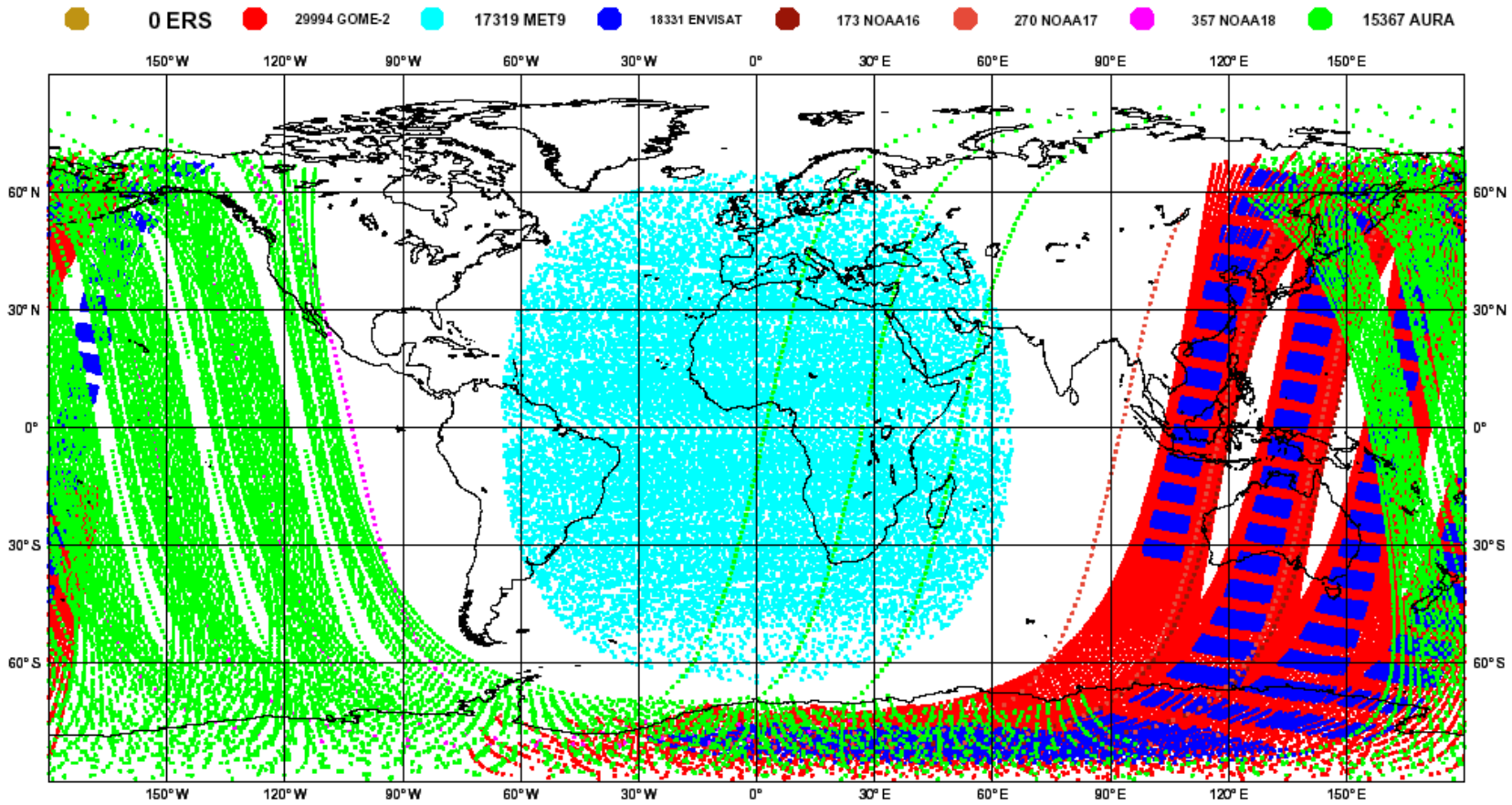
ECMWF

ECMWF Data Coverage (All obs DA) - Buoy
13/Nov/2011; 00 UTC
Total number of obs = 8540



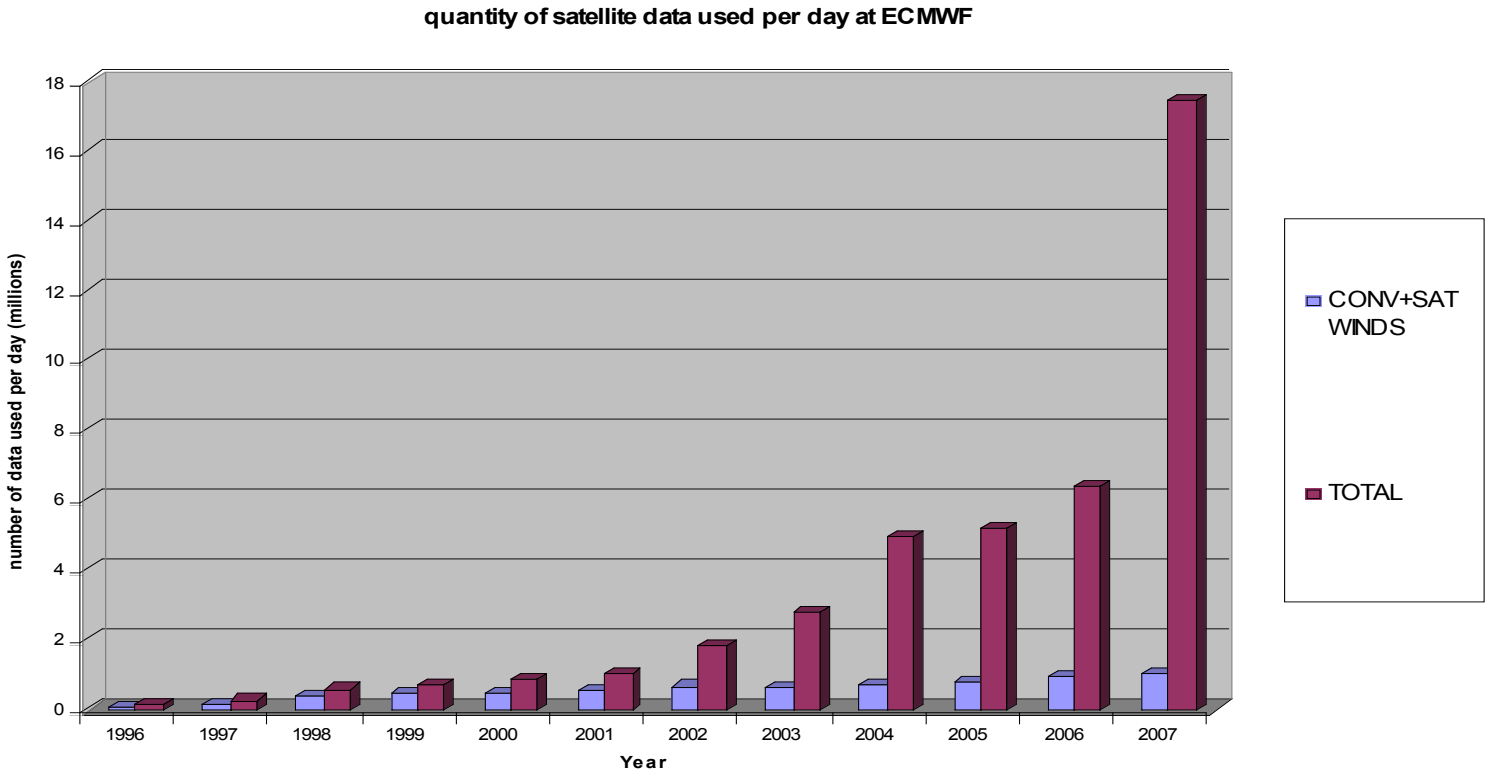
ECMWF

ECMWF Data Coverage (All obs DA) - OZONE
13/Nov/2011; 00 UTC
Total number of obs = 81811



ECMWF

December 2007: Satellite data volumes used: around 18 millions per day



Value as of March 2010 : 25 millions per day

- Observations *synoptiques* (observations au sol, radiosondages), effectuées simultanément, par convention internationale, dans toutes les stations météorologiques du globe (00:00, 06:00, 12:00, 18:00 TU)

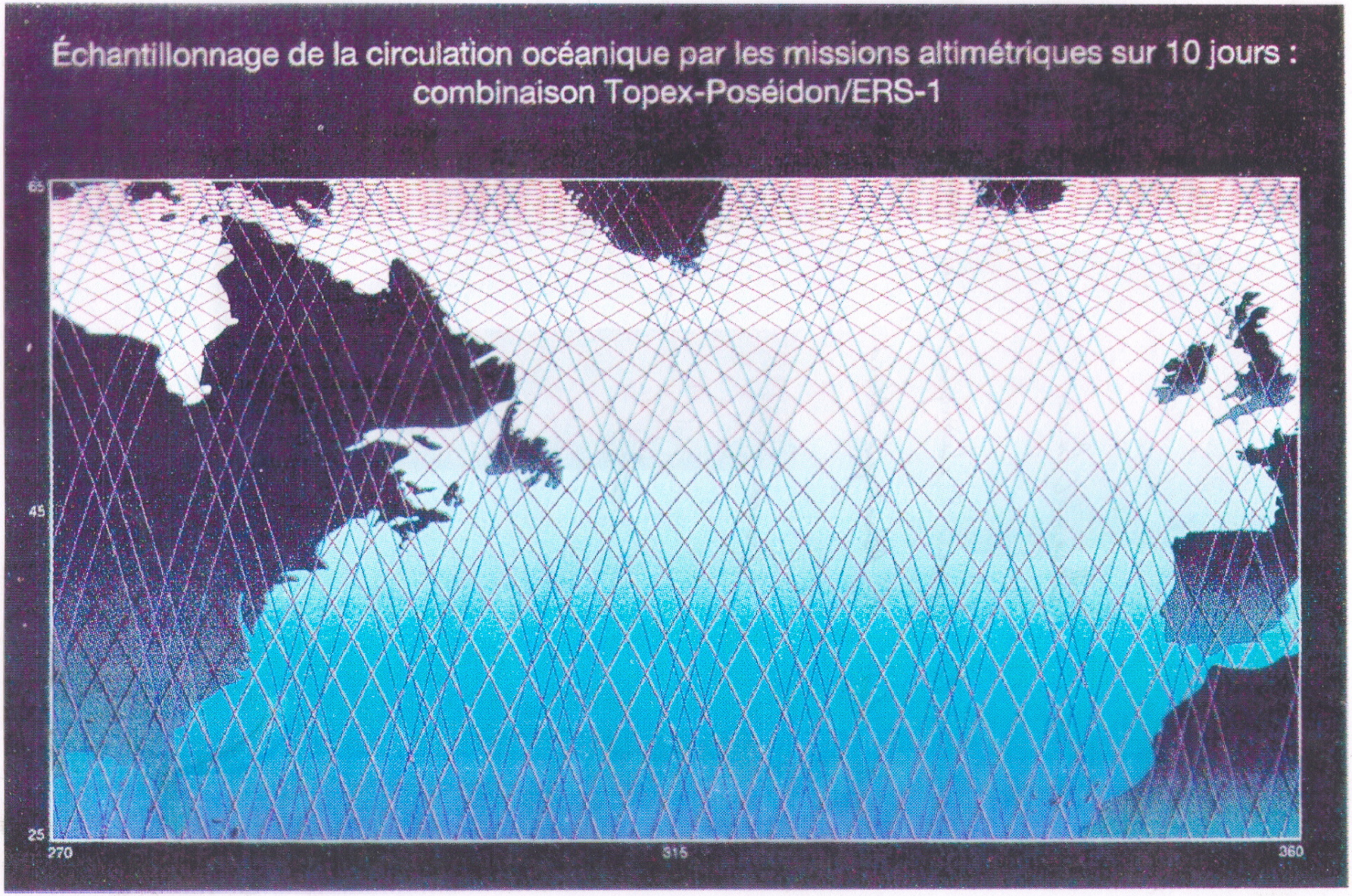
- Observations *asynoptiques* (satellites, avions), effectuées plus ou moins continûment dans le temps.

- Observations *directes* (température, pression, composantes du vent, humidité), portant sur les variables utilisées pour décrire l'état de l'écoulement dans les modèles numériques

- Observations *indirectes* (observations radiométriques, …), portant sur une combinaison plus ou moins complexe (le plus souvent, une intégrale d'espace unidimensionnelle) des variables utilisées pour décrire l'état de l'écoulement

$$y = H(\boldsymbol{x})$$

H : *opérateur d'observation* (par exemple, équation de transfert radiatif)



S. Louvel, Doctoral Dissertation, 1999

Modèles numériques de prévision météorologique

Construits sur les lois physiques qui gouvernent l'évolution de l'écoulement atmosphérique (conservation de la masse, de l'énergie et de la quantité de mouvement), discrétisées de façon appropriée dans l'espace et le temps.

Lois physiques régissant l'écoulement

- Conservation de la masse

  $D\rho/Dt + \rho \, \mathrm{div}\underline{U} = 0$

- Bilan d'énergie interne

  $De/Dt - (p/\rho^2) \, D\rho/Dt = Q$

- Bilan de quantité de mouvement

  $D\underline{U}/Dt + (1/\rho) \, \underline{\mathrm{grad}}p - \underline{g} + 2 \, \underline{\Omega} \wedge \underline{U} = \underline{F}$

- Equation d'état thermodynamique

  $f(p, \rho, e) = 0$ ($p/\rho = rT$, $e = C_v T$ pour un gaz parfait)

- Bilan de masse pour les composants secondaires (eau pour l'atmosphère, sel pour l'océan, …)

  $Dq/Dt + q \, \mathrm{div}\underline{U} = S$

Vocabulaire du métier :

- Processus adiabatiques et inviscides, et donc thermodynamiquement réversibles (tout sauf Q, $\underline{F}$ et S) : *'dynamique'*

- Processus décrits par les termes Q, $\underline{F}$ et S : *'physique'*

Plusieurs hypothèses simplificatrices sont faites pour les besoins de la modélisation du climat et la prévision météorologique de grande échelle

- Dans la direction verticale, approximation *hydrostatique* :

$$\partial p/\partial z + \rho g = 0$$

Élimine l'équation du mouvement pour la direction verticale; en outre, l'écoulement est incompressible dans les coordonnées (x, y, p) $\Rightarrow$ nombre d'équations diminué de deux unités.

Approximation hydrostatique valide dans l'atmosphère pour échelles horizontales $>$ 20-30 km

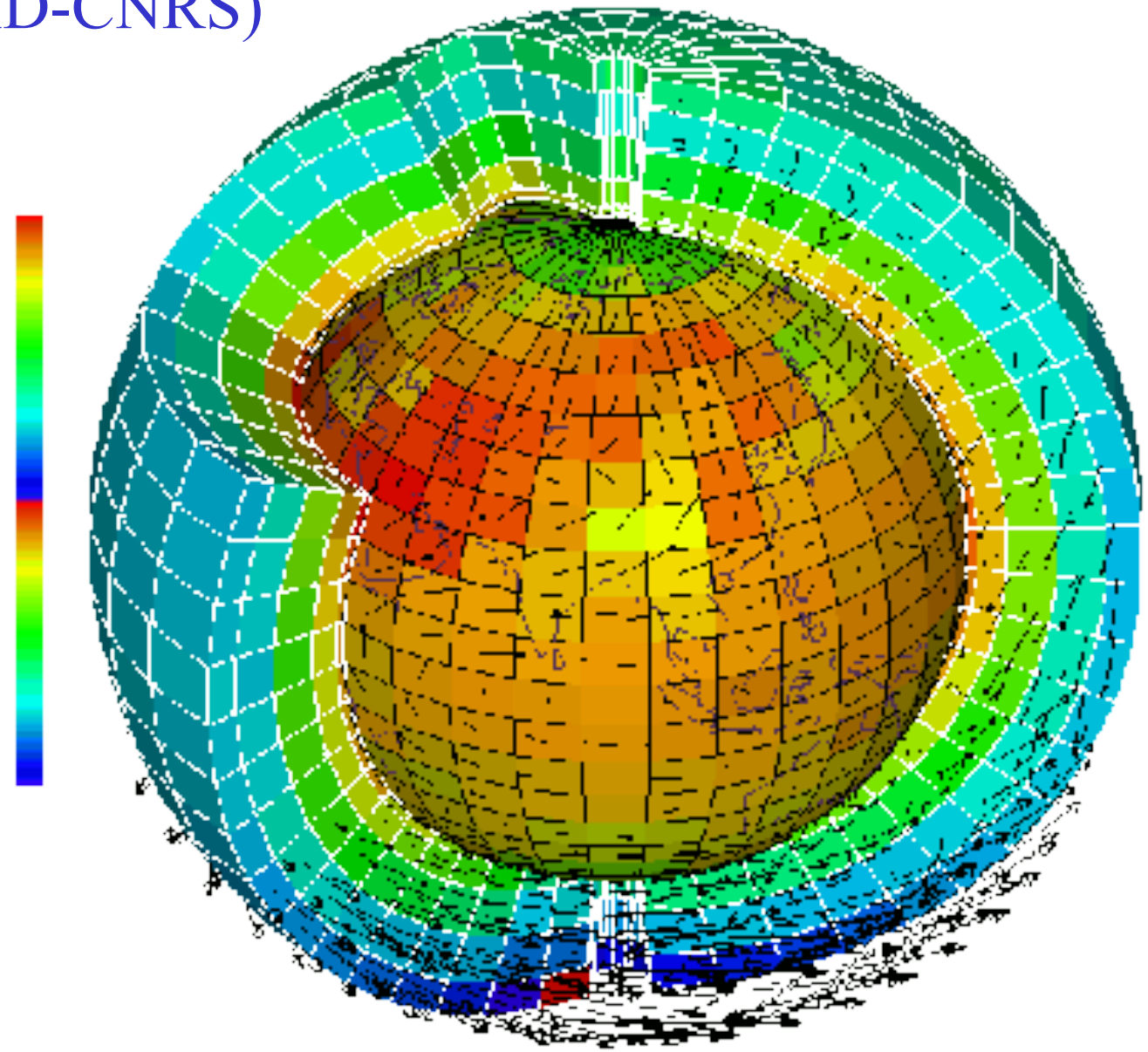
- Atmosphère et océan sont contenus dans une couche sphérique d'épaissseur négligeable devant le rayon de la Terre

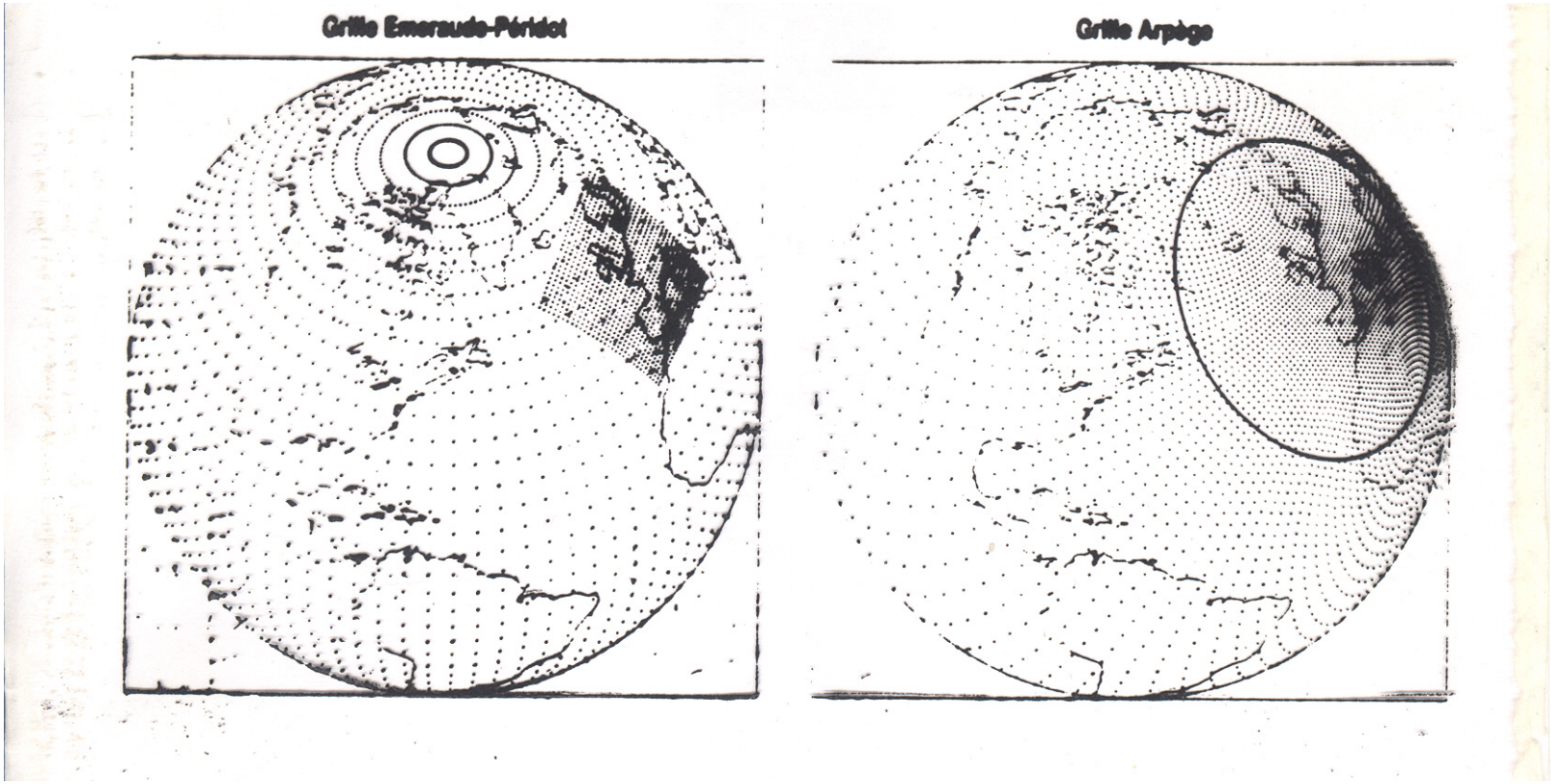
- …

$\Rightarrow$ Equations dites *primitives*

Modèles non-hydrostatiques, plus coûteux, sont utilisés pour la météorologie de petite échelle.

Schéma de principe d'un modèle atmosphérique (L. Fairhead /LMD-CNRS)





Grilles de modèles de Météo-France *(La Météorologie)*

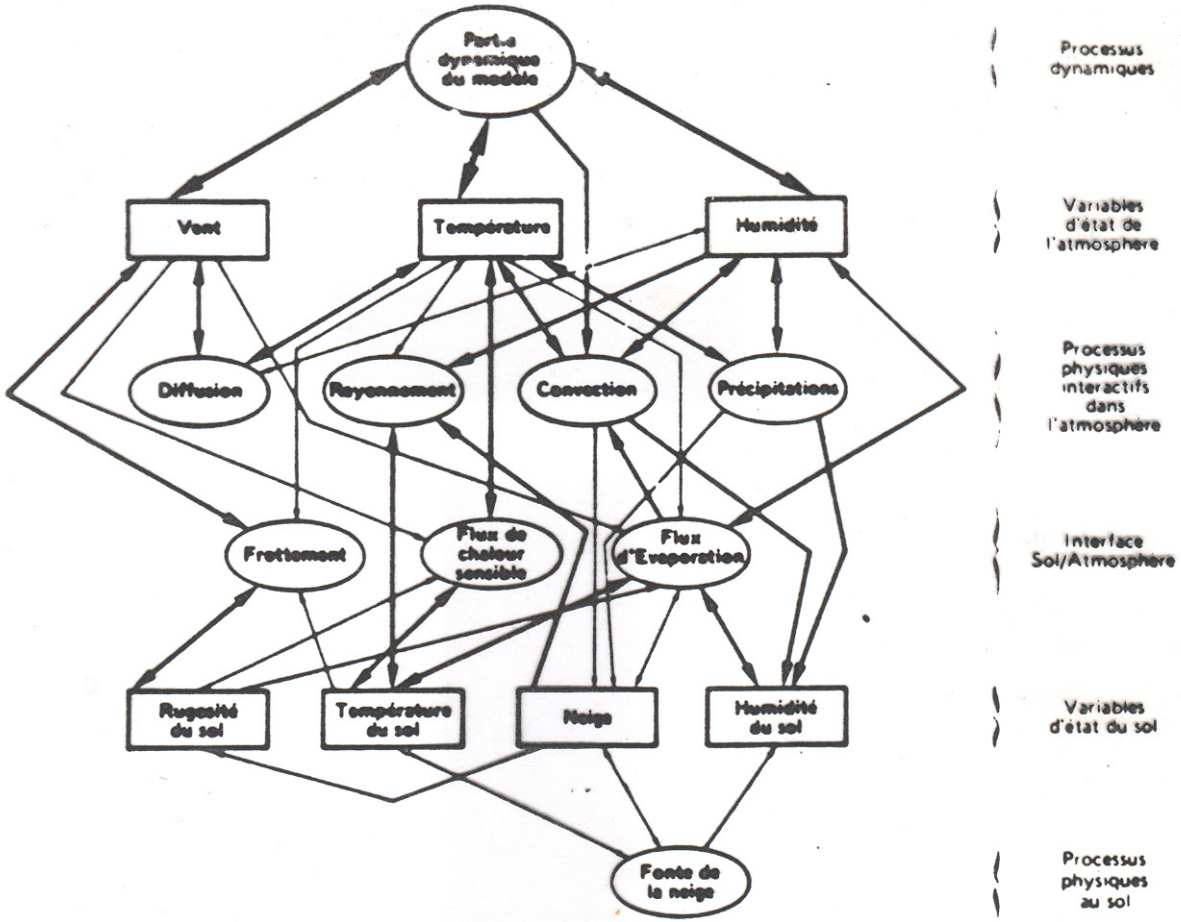
Discrétisation spatiale

Deux grandes classes de discrétisation

- discrétisation en points de grille (peu de volumes finis, en particulier pour l'atmosphère)

- discrétisation (semi-) spectrale, suivant les harmoniques sphériques. Seules les opérations linéaires relatives à la 'dynamique' sont effectuées dans l'espace spectral, les opérations non-linéaires et les opérations relatives à la 'physique' sont effectuées dans l'espace physique. Nécessité de passer en permanence d'un espace à l'autre. Possible grâce à l'utilisation des Transformées de Fourier Rapides (FFT)

5 - SCHEMA DES INTERACTIONS PHYSIQUES DANS LE MODELE



Centre Européen pour les Prévisions Météorologiques à Moyen Terme (CEPMMT, Reading, GB)

(European Centre for Medium-range Weather Forecasts, ECMWF)

Depuis le 26 Janvier 2010

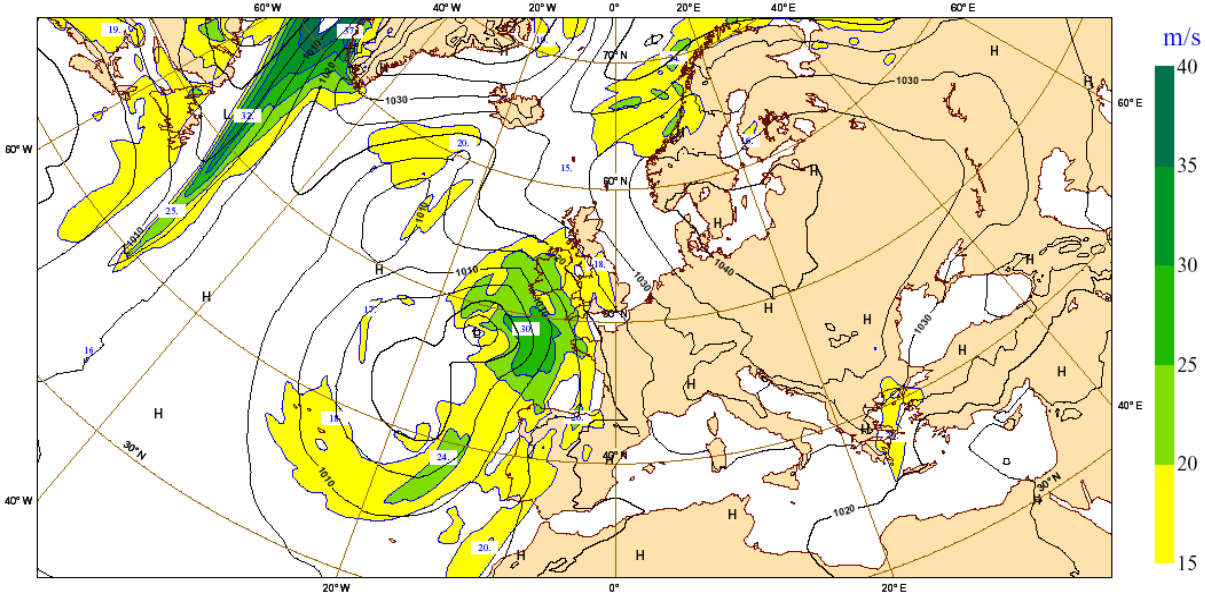
Troncature ‘triangulaire’ T1279 (résolution horizontale ≈ 16 kilomètres)

91 niveaux dans la direction verticale (0 - 80 km)

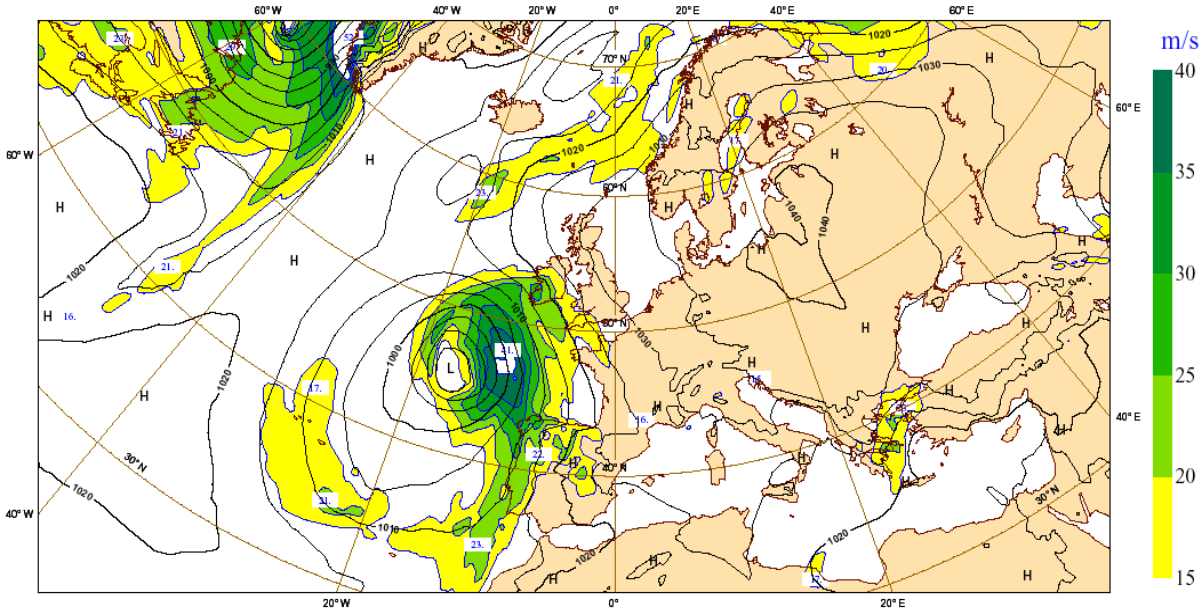
Dimension du vecteur d’état correspondant ≈ $1{,}5\ 10^9$

Pas de discrétisation temporelle : 10 minutes

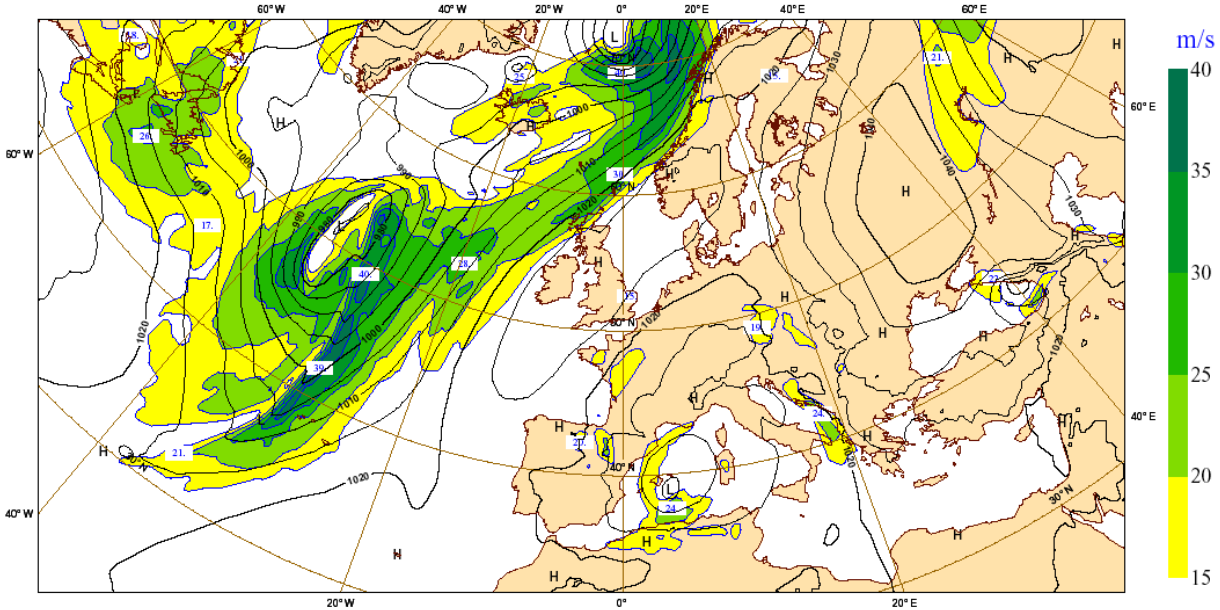
Monday 7 November 2011 00UTC ©ECMWF Forecast t+144 VT: Sunday 13 November 2011 00UTC
Surface: Mean sea level pressure / 850-hPa wind speed



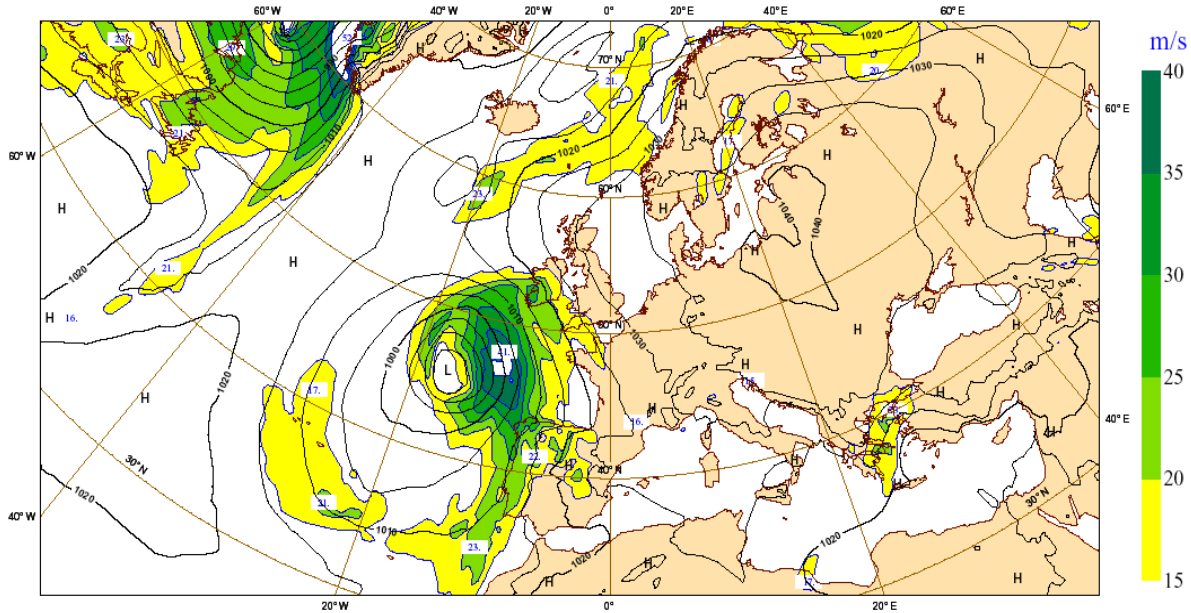
Sunday 13 November 2011 00UTC ©ECMWF Analysis t+000 VT: Sunday 13 November 2011 00UTC
Surface: Mean sea level pressure / 850-hPa wind speed



Monday 7 November 2011 00UTC ©ECMWF Analysis t+000 VT: Monday 7 November 2011 00UTC
Surface: Mean sea level pressure / 850-hPa wind speed



Sunday 13 November 2011 00UTC ©ECMWF Analysis t+000 VT: Sunday 13 November 2011 00UTC
Surface: Mean sea level pressure / 850-hPa wind speed



Résultats extraits de

Richardson *et al.*, 2010, *Verification statistics and evaluations of ECMWF forecasts in 2009-2010*, Memorandum Technique 635 CEPMMT, Reading, GB.

Disponible à l'adresse

http://www.ecmwf.int/publications/library/ecpublications/_pdf/tm/601-700/tm635.pdf

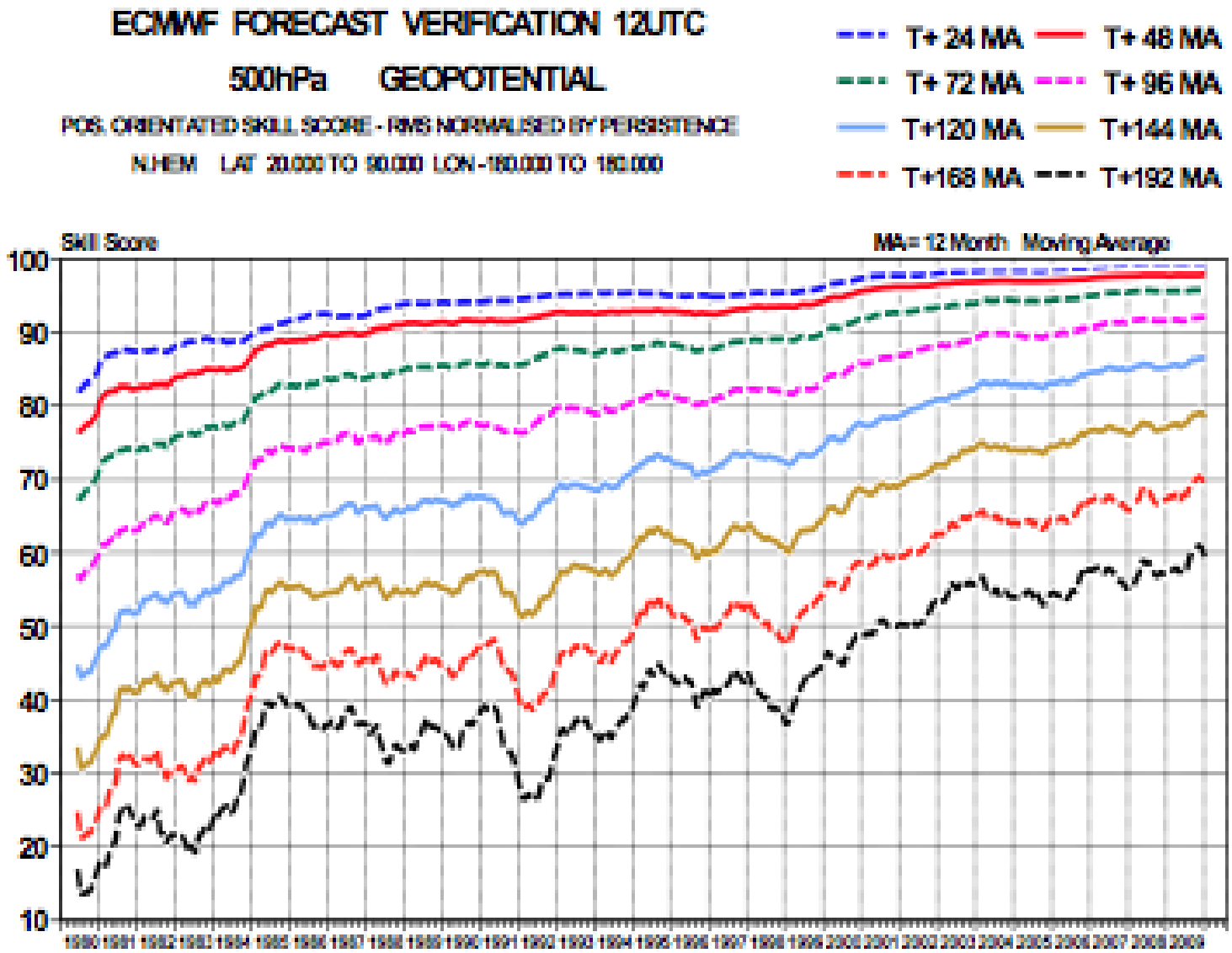


Figure 1: 500 hPa geopotential height skill score for Europe (top) and the northern hemisphere extra-tropics (bottom), showing 12-month moving averages for forecast ranges from 24 to 192 hours. The last point on each curve is for the 12-month period August 2009 - July 2010.

Persistence = 0 ; climatology = 50 at long range

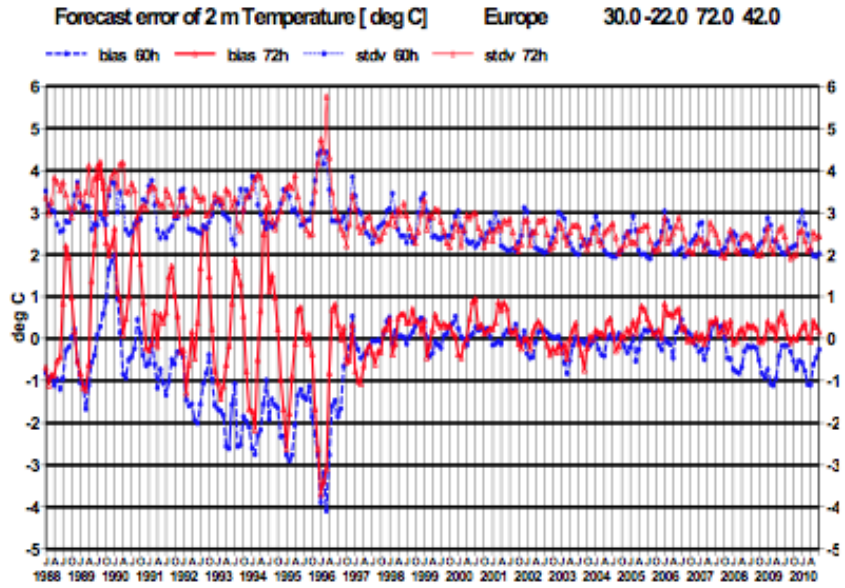


Figure 16: Verification of 2 metre temperature forecasts against European SYNOP data on the GTS for 60-hour (night-time) and 72-hour (daytime) forecasts. Lower pair of curves show bias, upper curves are standard deviation of error.

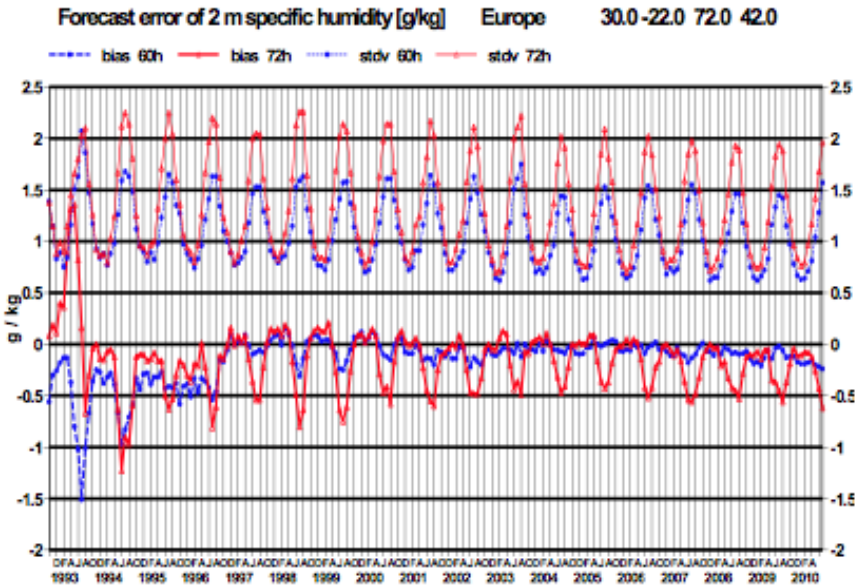
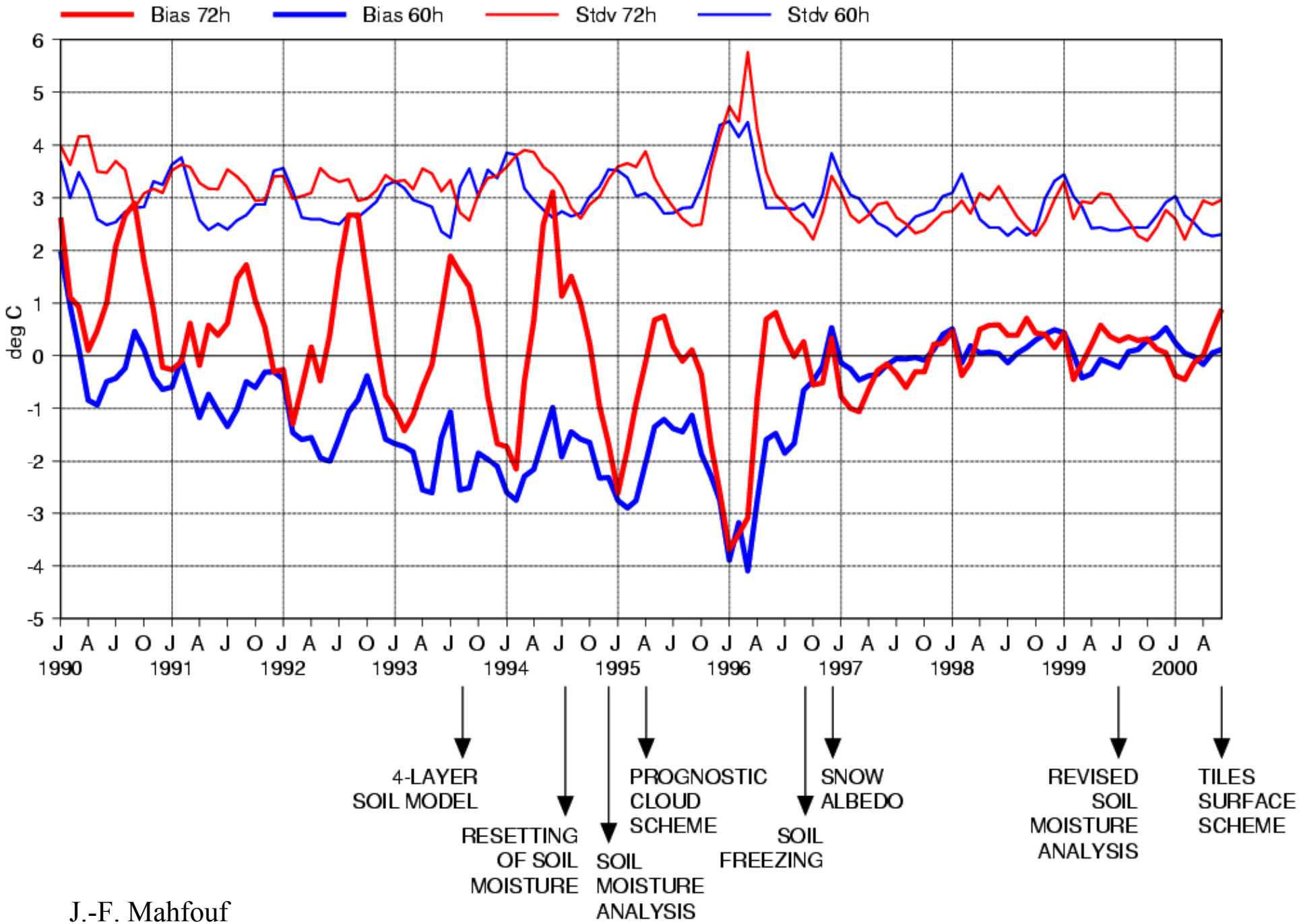


Figure 17: Verification of 2 metre specific humidity forecasts against European SYNOP data on the GTS for 60-hour (night-time) and 72-hour (daytime) forecasts. Lower pair of curves show bias, upper curves are standard deviation of error.


Bias 72h
Bias 60h
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4-LAYER SOIL MODEL
RESETTING OF SOIL MOISTURE
SOIL MOISTURE ANALYSIS
PROGNOSTIC CLOUD SCHEME
SOIL FREEZING
SNOW ALBEDO
REVISED SOIL MOISTURE ANALYSIS
TILES SURFACE SCHEME

J.-F. Mahfouf

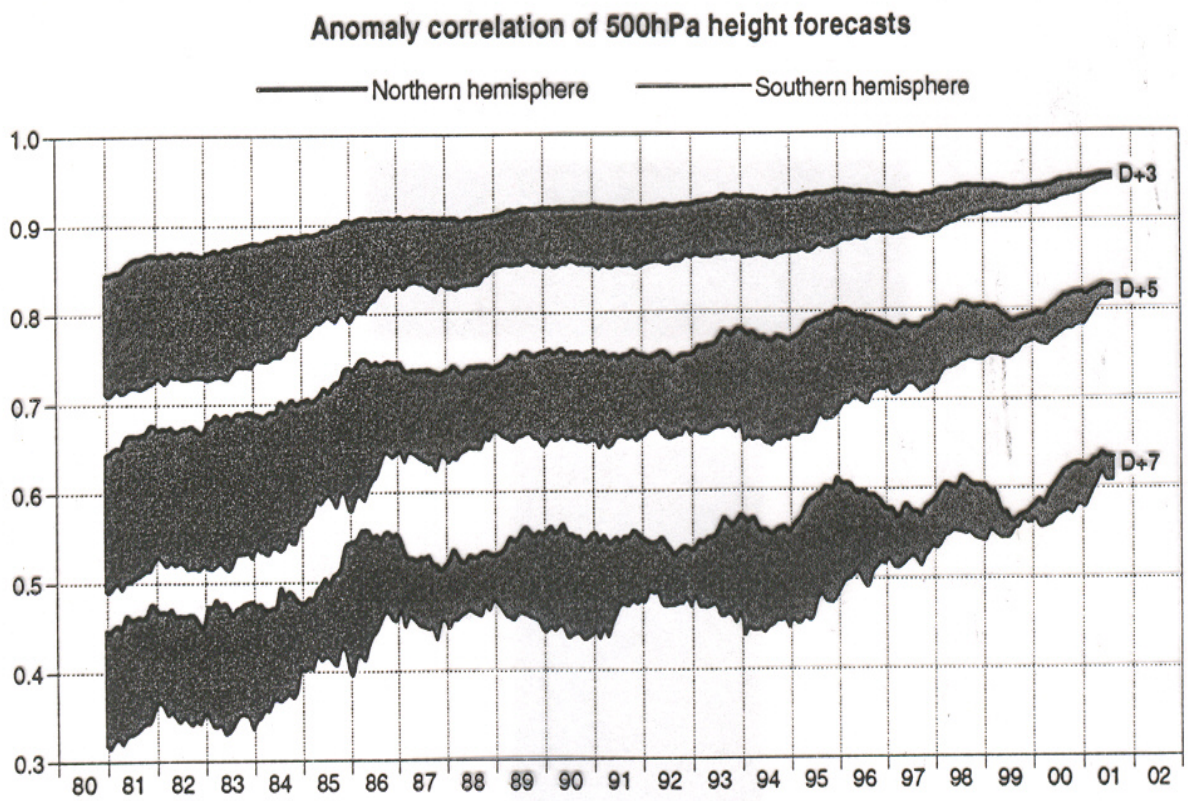


Fig 4. Anomaly correlation coefficients of 3-, 5- and 7-day ECMWF 500hPa height forecasts for the extratropical northern and southern hemispheres, plotted in the form of annual running means of archived monthly-mean scores for the period from January 1980 to August 2001. Values plotted for a particular month are averages over that month and the 11 preceding months. The shading shows the differences in scores between the two hemispheres at the forecast ranges indicated.

Simmons et Hollingsworth, 2002, *Q. J. R. Meteorol. Soc.*, **128**, 647-677

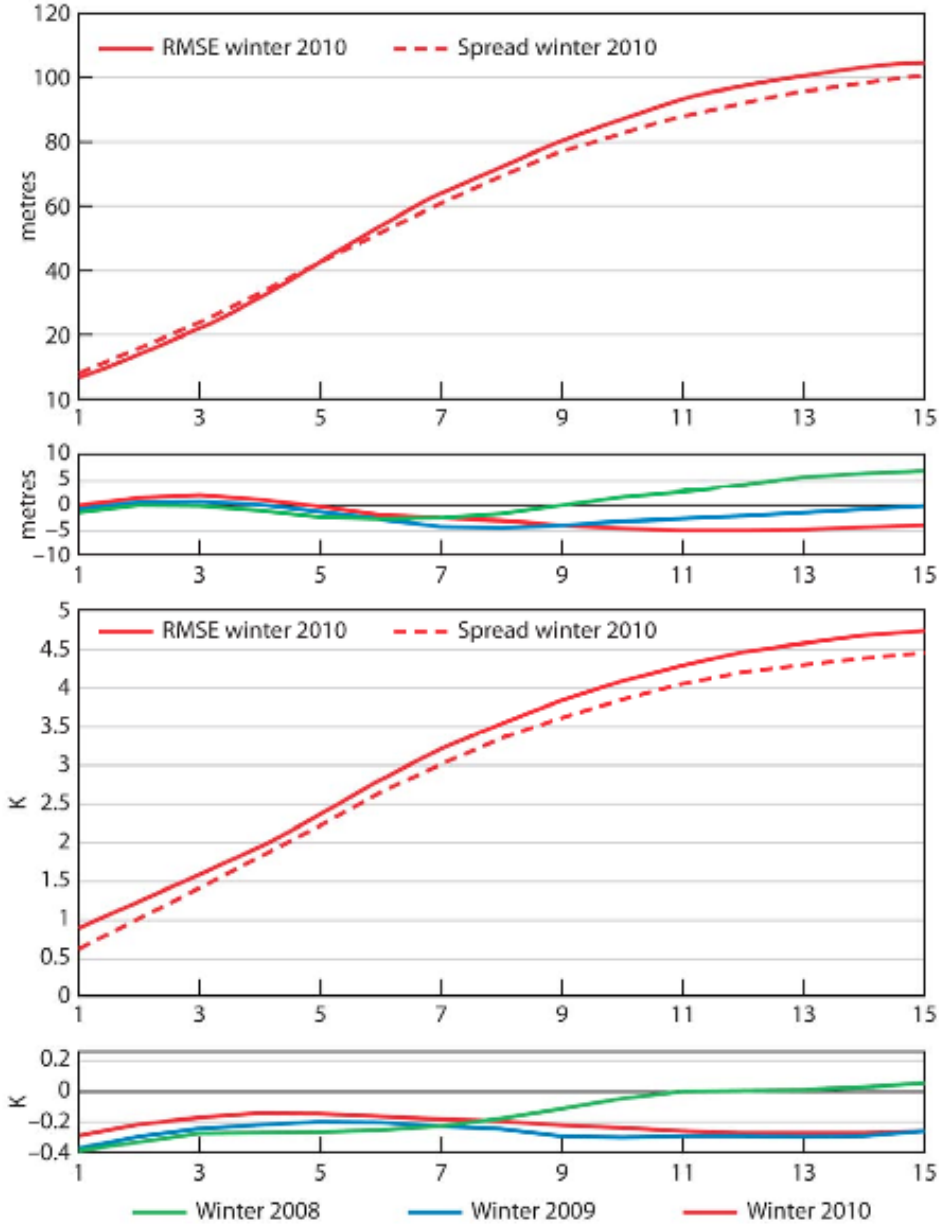


Figure 8: Ensemble spread (standard deviation, dashed lines) and root mean square error of ensemble-mean (solid lines) for winter 2009-2010(upper figure in each panel), complemented with differences of ensemble spread and root mean square error of ensemble-mean for last 3 winter seasons (lower figure in each panel, negative values indicate spread is too small); plots are for 500 hPa geopotential (top) and 850 hPa temperature (bottom) over the extra-tropical northern hemisphere for forecast days 1 to 15.

Problèmes restants

- Cycle de l'eau (évaporation, condensation, influence sur le rayonnement absorbé ou émis par l'atmosphère)

- Échanges avec l'océan ou la surface continentale (chaleur, eau, quantité de mouvement, …)

- …

Purpose of assimilation : reconstruct as accurately as possible the state of the atmospheric or oceanic flow, using all available appropriate information. The latter essentially consists of

- The observations proper, which vary in nature, resolution and accuracy, and are distributed more or less regularly in space and time.

- The physical laws governing the evolution of the flow, available in practice in the form of a discretized, and necessarily approximate, numerical model.

- 'Asymptotic' properties of the flow, such as, *e. g.*, geostrophic balance of middle latitudes. Although they basically are necessary consequences of the physical laws which govern the flow, these properties can usefully be explicitly introduced in the assimilation process.

Both observations and ‘model’ are affected with some uncertainty $\Rightarrow$ uncertainty on the estimate.

For some reason, uncertainty is conveniently described by probability distributions (don’t know too well why, but it works).

Assimilation is a problem in bayesian estimation.

Determine the conditional probability distribution for the state of the system, knowing everything we know

Assimilation is one of many '*inverse problems*' encountered in many fields of science and technology

- solid Earth geophysics
- plasma physics
- 'nondestructive' probing
- navigation (spacecraft, aircraft, ….)
- …

Solution most often (if not always) based on Bayesian, or probabilistic, estimation. 'Equations' are fundamentally the same.

Difficulties specific to assimilation of meteorological observations :

- Very large numerical dimensions ($n \approx 10^6$-10^9 parameters to be estimated, $p \approx 1$-3.10^7 observations per 24-hour period). Difficulty aggravated in Numerical Weather Prediction by the need for the forecast to be ready in time.

- Non-trivial, actually chaotic, underlying dynamics

At ECMWF, assimilation uses 32% of the computers resources allocated to operational prediction, including two daily 10-day 50-member ensemble predictions (J.-N. Thépaut, *pers. com.*)

$z_1 = x + \zeta_1$ density function $p_1(\zeta) \propto \exp[- (\zeta^2)/2s_1]$

$z_2 = x + \zeta_2$ density function $p_2(\zeta) \propto \exp[- (\zeta^2)/2s_2]$

$x = \xi \quad \Leftrightarrow \quad \zeta_1 = z_1 - \xi$ and $\zeta_2 = z_2 - \xi$

$$P(x = \xi \mid z_1, z_2) \propto p_1(z_1 - \xi)\, p_2(z_2 - \xi)$$
$$\propto \exp[- (\xi - x^a)^2/2p^a]$$

where $1/p^a = 1/s_1 + 1/s_2$, $x^a = p^a\,(z_1/s_1 + z_2/s_2)$

Conditional probability distribution of x, given z_1 and z_2 : $\mathcal{N}[x^a, p^a]$

$p^a < (s_1, s_2)$ independent of z_1 and z_2

$z_1 = x + \zeta_1$
$z_2 = x + \zeta_2$

Same as before, but ζ_1 and ζ_2 are now distributed according to exponential law with parameter a, *i. e.*

$$p(\zeta) \propto \exp[-|\zeta|/a] \quad ; \quad \mathrm{Var}(\zeta) = 2a^2$$

Conditional probability density function is now uniform over interval $[z_1, z_2]$, exponential with parameter $a/2$ outside that interval

$$E(x \mid z_1, z_2) = (z_1+z_2)/2$$

$\mathrm{Var}(x \mid z_1, z_2) = a^2\,(2\delta^3/3 + \delta^2 + \delta + 1/2) \,/\, (1 + 2\delta)$, with $\delta = |z_1 - z_2|/(2a)$
Increases from $a^2/2$ to ∞ as δ increases from 0 to ∞. Can be larger than variance $2a^2$ of original errors (probability 0.08)

(Entropy $-\int p \ln p$ always decreases in bayesian estimation)

Bayesian estimation

State vector x, belonging to *state space* $\mathcal{S}$ ($\dim \mathcal{S} = n$), to be estimated.

Data vector z, belonging to *data space* $\mathcal{D}$ ($\dim \mathcal{D} = m$), available.

$$z = F(x, \zeta) \tag{1}$$

where ζ is a random element representing the uncertainty on the data (or, more precisely, on the link between the data and the unknown state vector).

For example

$$z = \Gamma x + \zeta$$

Bayesian estimation (continued)

Probability that $x = \xi$ for given ξ ?

$$x = \xi \quad \Rightarrow \quad z = F(\xi, \zeta)$$

$$P(x = \xi \mid z) = P[z = F(\xi, \zeta)] \,/\, \int_{\xi'} P[z = F(\xi', \zeta)]$$

Unambiguously defined iff, for any ζ, there is at most one x such that (1) is verified.

$\Leftrightarrow$ data contain information, either directly or indirectly, on any component of x. *Determinacy* condition.

Bayesian estimation is however impossible in its general theoretical form in meteorological or oceanographical practice because

- It is impossible to explicitly describe a probability distribution in a space with dimension even as low as $n \approx 10^3$, not to speak of the dimension $n \approx 10^{6\text{-}9}$ of present Numerical Weather Prediction models.

- Probability distribution of errors on data very poorly known (model errors in particular).

One has to restrict oneself to a much more modest goal. Two approaches exist at present

- Obtain some ‘central’ estimate of the conditional probability distribution (expectation, mode, …), plus some estimate of the corresponding spread (standard deviations and a number of correlations).

- Produce an ensemble of estimates which are meant to sample the conditional probability distribution (dimension $N \approx O(10\text{-}100)$).

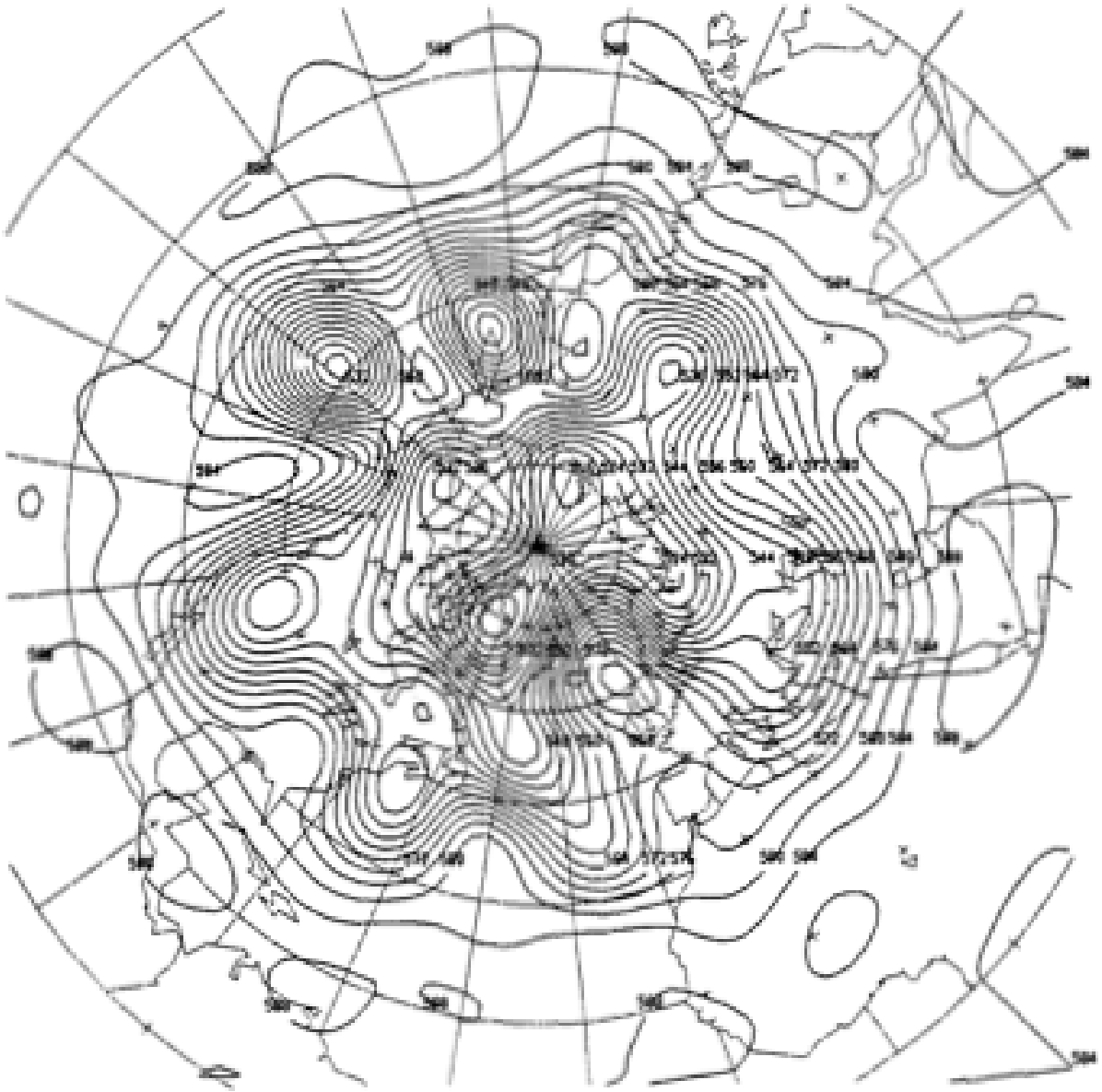


Figure 2. 500 mb height field produced by the operational analysis procedure of Direction de la Météorologie for 00 GMT, 26 April 1984. Units: dam, contour interval: 4 dam. The field has been truncated to the truncation of the model used for the experiments described in the article.

Courtier and Talagrand, *QJRMS*, 1987

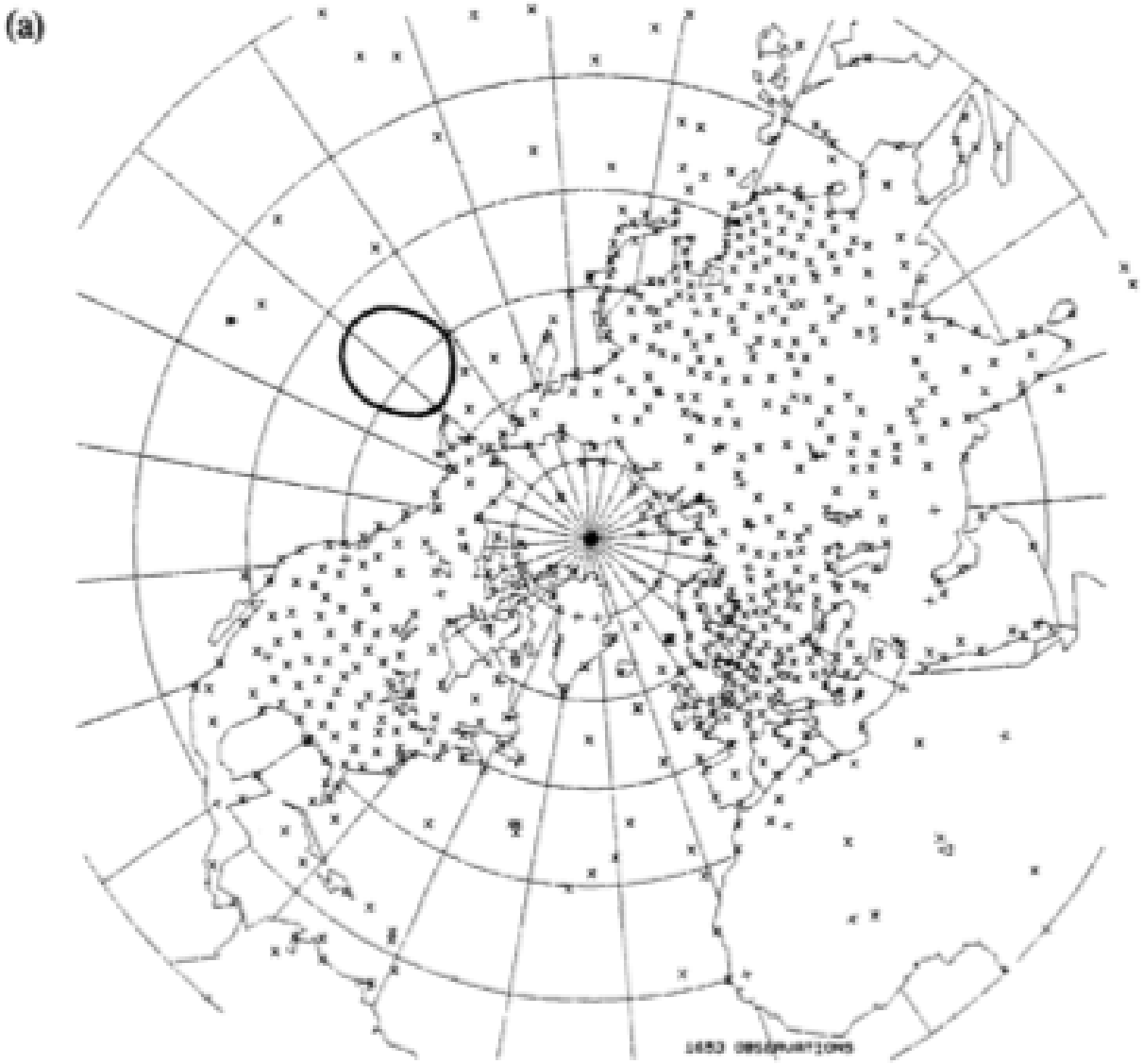
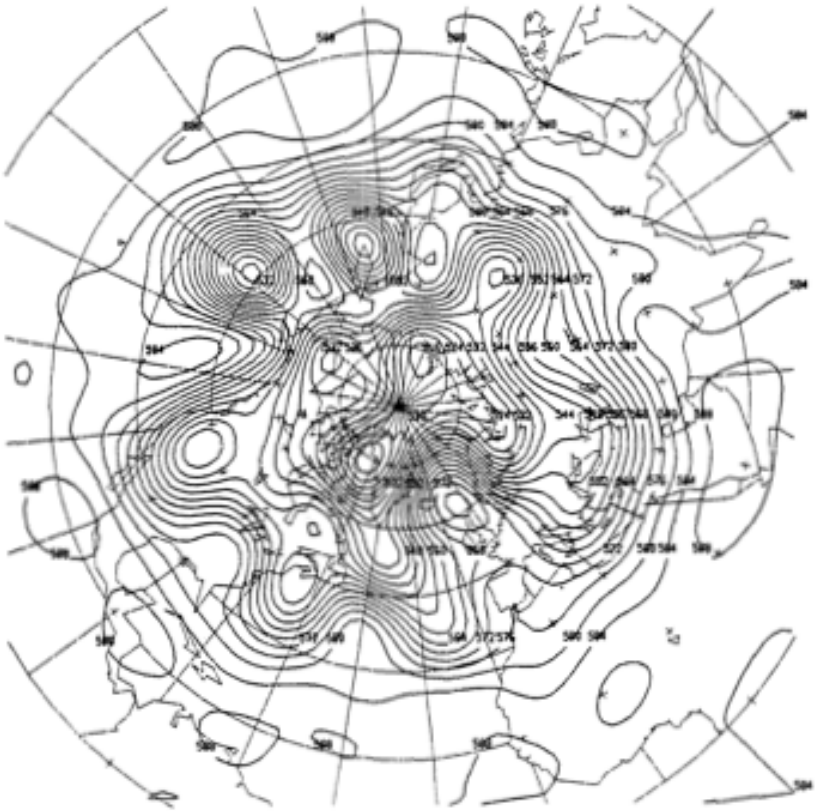


Figure 1. Geographical distribution of the observations used for the assimilation experiments. (a): geopotential observations; (b): wind observations. At most of the points plotted, several observations were made at successive synoptic hours. On each of the two charts, the heavy line delineates the Aleutian depression (see Figure 2).

Courtier and Talagrand, *QJRMS*, 1987



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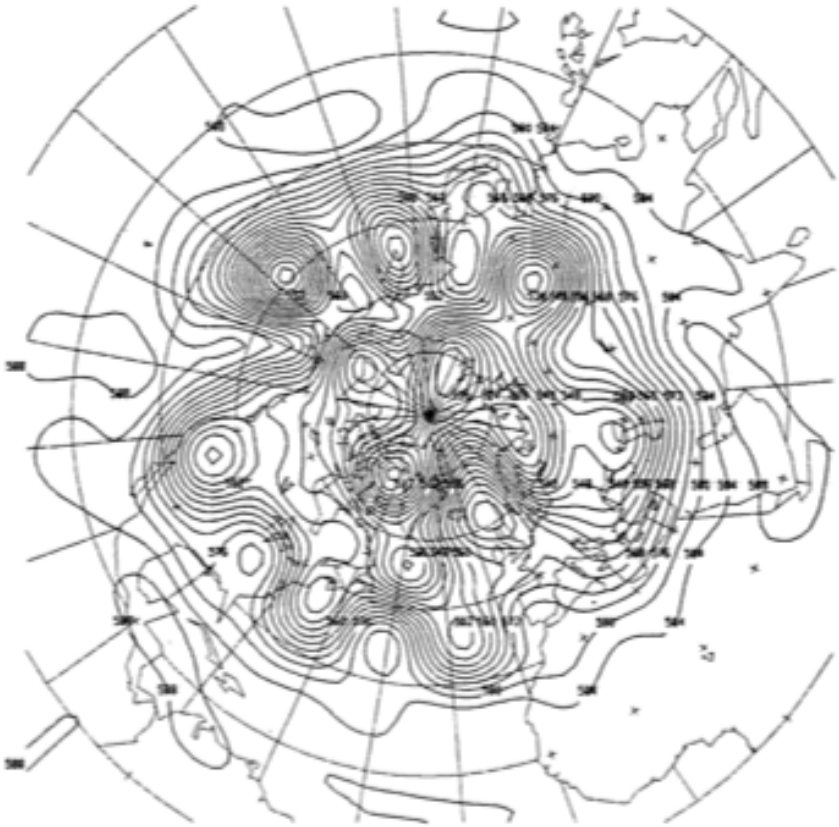


Figure 3. 500 mb height field produced for 00 GMT, 26 April 1984, by the variational analysis minimizing the distance function defined by Eqs. (1)–(2) over a 24-hour period. Units: dam; contour interval: 4 dam.

500-hPa geopotential field as determined by : (left) operational assimilation system of French Weather Service (3D, primitive equation) and (right) experimental variational system (2D, vorticity equation)

Courtier and Talagrand, *QJRMS*, 1987

Random vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} = (x_i)$ (*e. g.* pressure, temperature, abundance of given chemical compound at n grid-points of a numerical model)

- Expectation $E(\boldsymbol{x}) \equiv [E(x_i)]$; centred vector $\boldsymbol{x}' \equiv \boldsymbol{x} - E(\boldsymbol{x})$

- Covariance matrix

$$E(\boldsymbol{x}'\boldsymbol{x}'^{\mathrm{T}}) = [E(x_i' x_j')]$$

  dimension nxn, symmetric non-negative (strictly definite positive except if linear relationship holds between the x_i's with probability 1).

- Two random vectors
  $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$
  $\boldsymbol{y} = (y_1, y_2, \ldots, y_p)^{\mathrm{T}}$

$$E(\boldsymbol{x}'\boldsymbol{y}'^{\mathrm{T}}) = E(x_i' y_j')$$

  dimension nxp

Random function $\Phi(\xi)$ (field of pressure, temperature, abundance of given chemical compound, … ; ξ is now spatial and/or temporal coordinate)

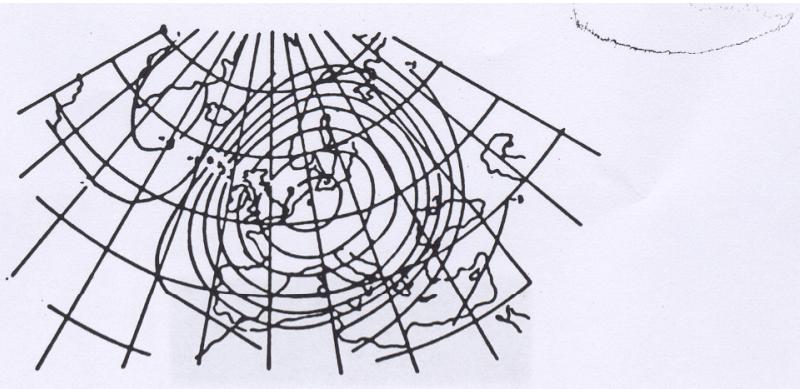
- Expectation $E[\Phi(\xi)]$; $\Phi'(\xi) \equiv \Phi(\xi) - E[\Phi(\xi)]$
- Variance $Var[\varphi(\xi)] = E\{[\varphi'(\xi)]^2\}$

- Covariance function

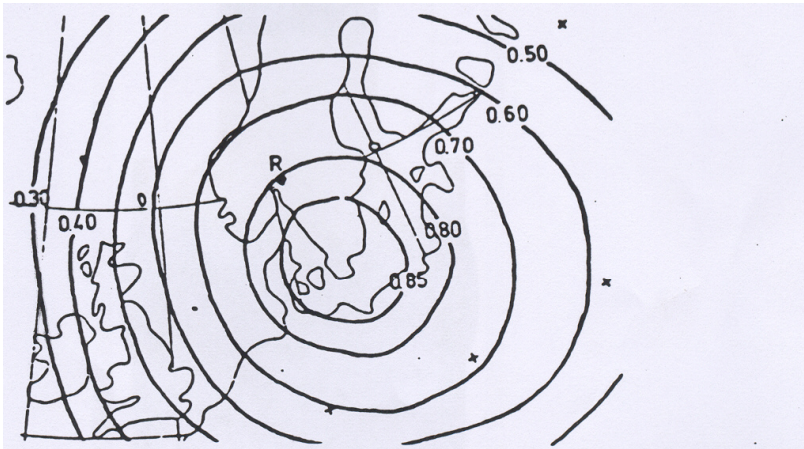
$$(\xi_1, \xi_2) \rightarrow C_\Phi(\xi_1, \xi_2) \equiv E[\Phi'(\xi_1)\ \Phi'(\xi_2)]$$

- Correlation function

$$Cor_\varphi(\xi_1, \xi_2) \equiv E[\Phi'(\xi_1)\ \Phi'(\xi_2)] \,/\, \{Var[\Phi(\xi_1)]\ Var[\Phi(\xi_2)]\}^{1/2}$$



.: Isolines for the auto-correlations of the 500 mb geopotential between the station in Hannover and surrounding stations.
From Bertoni and Lund (1963)



: Isolines of the cross-correlation between the 500 mb geopotential in station 01 384 (R) and the surface pressure in surrounding stations.

After N. Gustafsson

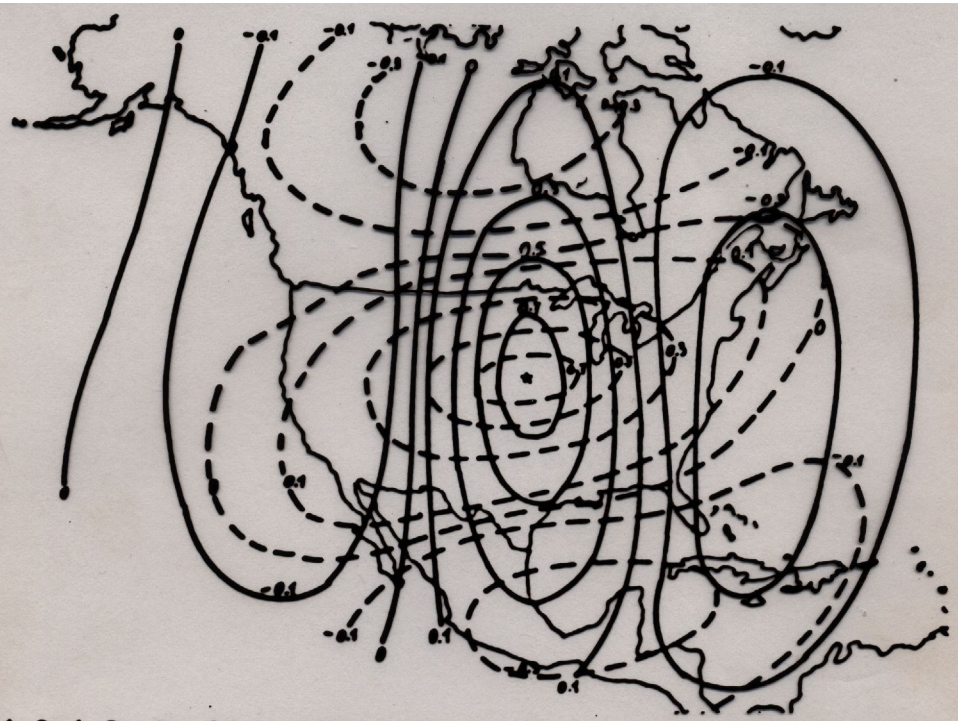


Figure 4.2.4.3: Isolines for the auto-correlation of the 500 mb u-wind component (dashed line) and the auto-correlation of the 500 mb v-wind component (full line). The "star" indicates the position of the reference station. (From Buel (1972).

After N. Gustafsson

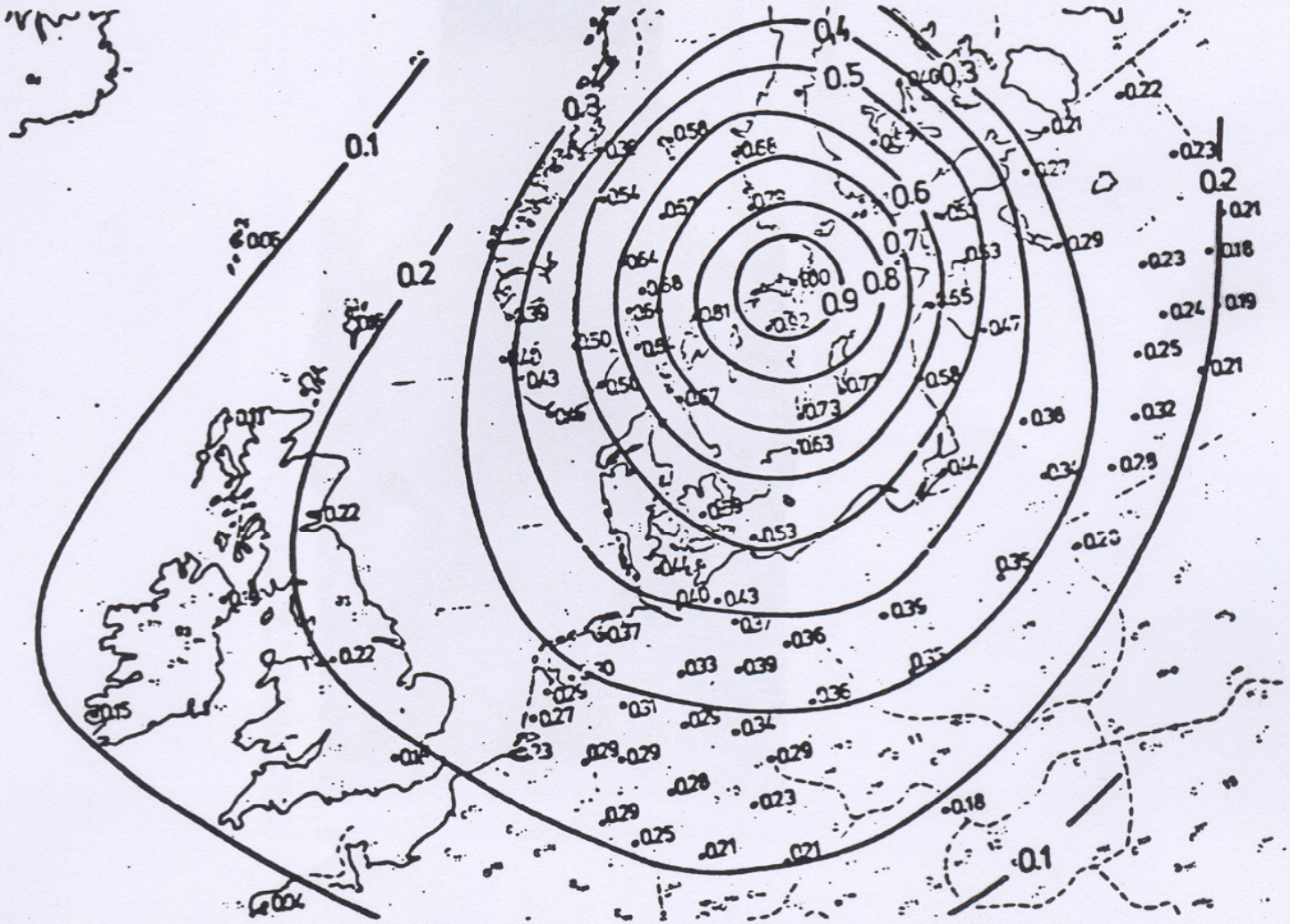


Figure 5.1.1.4.1 Auto-correlation of errors in 12h numerical forecasts of surface pressure in a reference station (Stockholm) and other stations.

After N. Gustafsson

Optimal Interpolation

Random field $\Phi(\xi)$

Observation network $\xi_1, \xi_2, \ldots, \xi_p$

For one particular realization of the field, observations

$$y_j = \Phi(\xi_j) + \varepsilon_j \quad , \; j = 1, \ldots, p \qquad ,$$

making up vector $\boldsymbol{y} = (y_j)$

Estimate $x = \Phi(\xi)$ at given point ξ, in the form

$$x^a = \alpha + \Sigma_j \, \beta_j y_j \; = \alpha + \boldsymbol{\beta}^{\mathrm{T}} \boldsymbol{y} \qquad ,$$

where $\boldsymbol{\beta} = (\beta_j)$

α and the β_j's being determined so as to minimize the expected quadratic estimation error

$E[(x\text{-}x^a)^2]$

Optimal Interpolation (continued 1)

Solution

$$x^a = E(x) + E(x'\boldsymbol{y}'^{\mathrm{T}})\,[E(\boldsymbol{y}'\boldsymbol{y}'^{\mathrm{T}})]^{-1}\,[\boldsymbol{y} - E(\boldsymbol{y})]$$

i.e.,

$$\boldsymbol{\beta} = [E(\boldsymbol{y}'\boldsymbol{y}'^{\mathrm{T}})]^{-1}\,E(x'\boldsymbol{y}')$$

$$\alpha = E(x) - \boldsymbol{\beta}^{\mathrm{T}}E(\boldsymbol{y})$$

Estimate is unbiased $\quad E(x-x^a) = 0$

Minimized quadratic estimation error

$$E[(x-x^a)^2] = E(x'^2) - E(x'\boldsymbol{y}'^{\mathrm{T}})\,[E(\boldsymbol{y}'\boldsymbol{y}'^{\mathrm{T}})]^{-1}\,E(\boldsymbol{y}'x')$$

Estimation made in terms of deviations from expectations x' and $\boldsymbol{y}'$.

Optimal Interpolation (continued 2)

$$x^a = E(x) + E(x'\boldsymbol{y}'^{\mathrm{T}})\,[E(\boldsymbol{y}'\boldsymbol{y}'^{\mathrm{T}})]^{-1}\,[\boldsymbol{y} - E(\boldsymbol{y})]$$

$$y_j = \boldsymbol{\Phi}(\xi_j) + \varepsilon_j$$

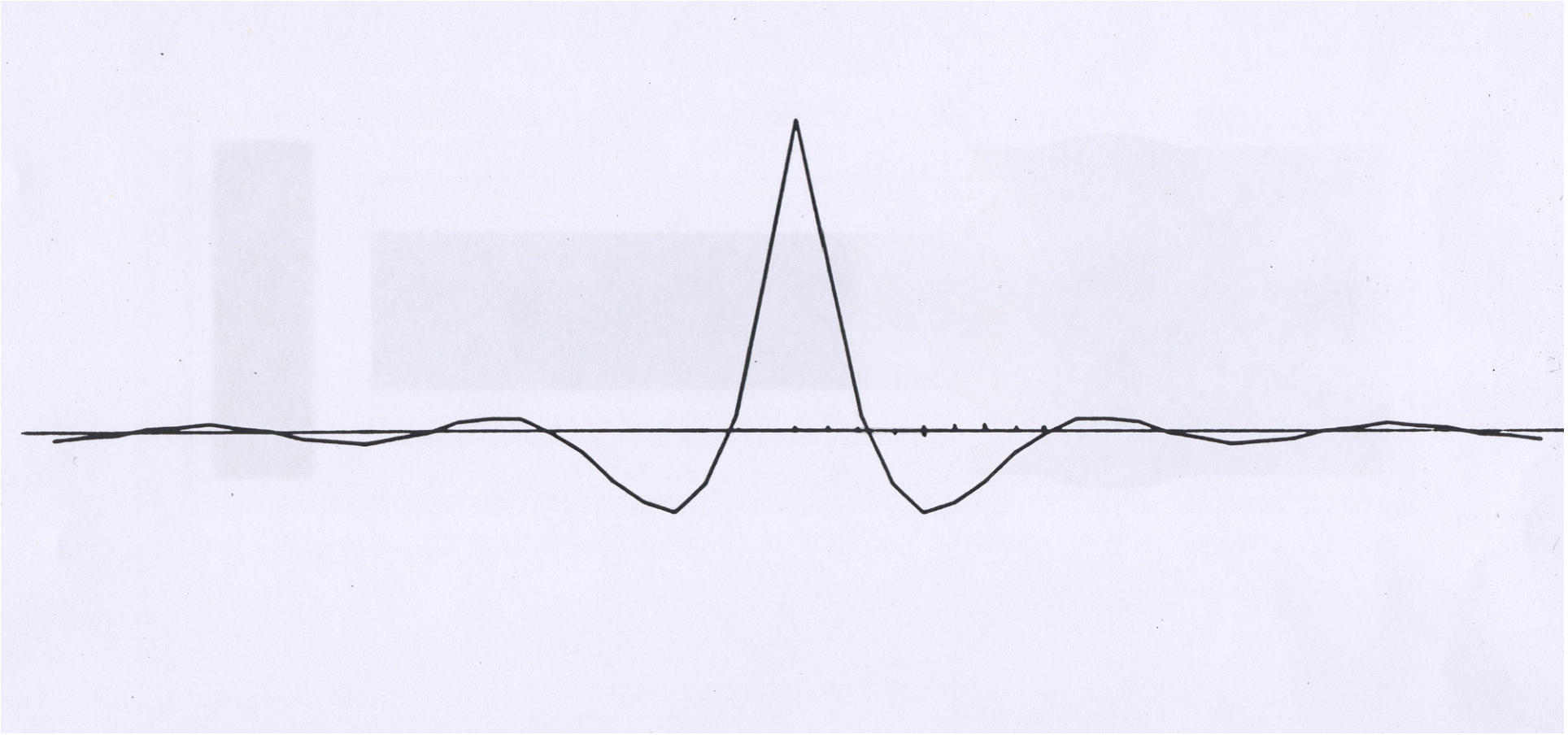
$$E(y_j'y_k') = E[\boldsymbol{\Phi}'(\xi_j) + \varepsilon_j'][\boldsymbol{\Phi}'(\xi_k) + \varepsilon_k']$$

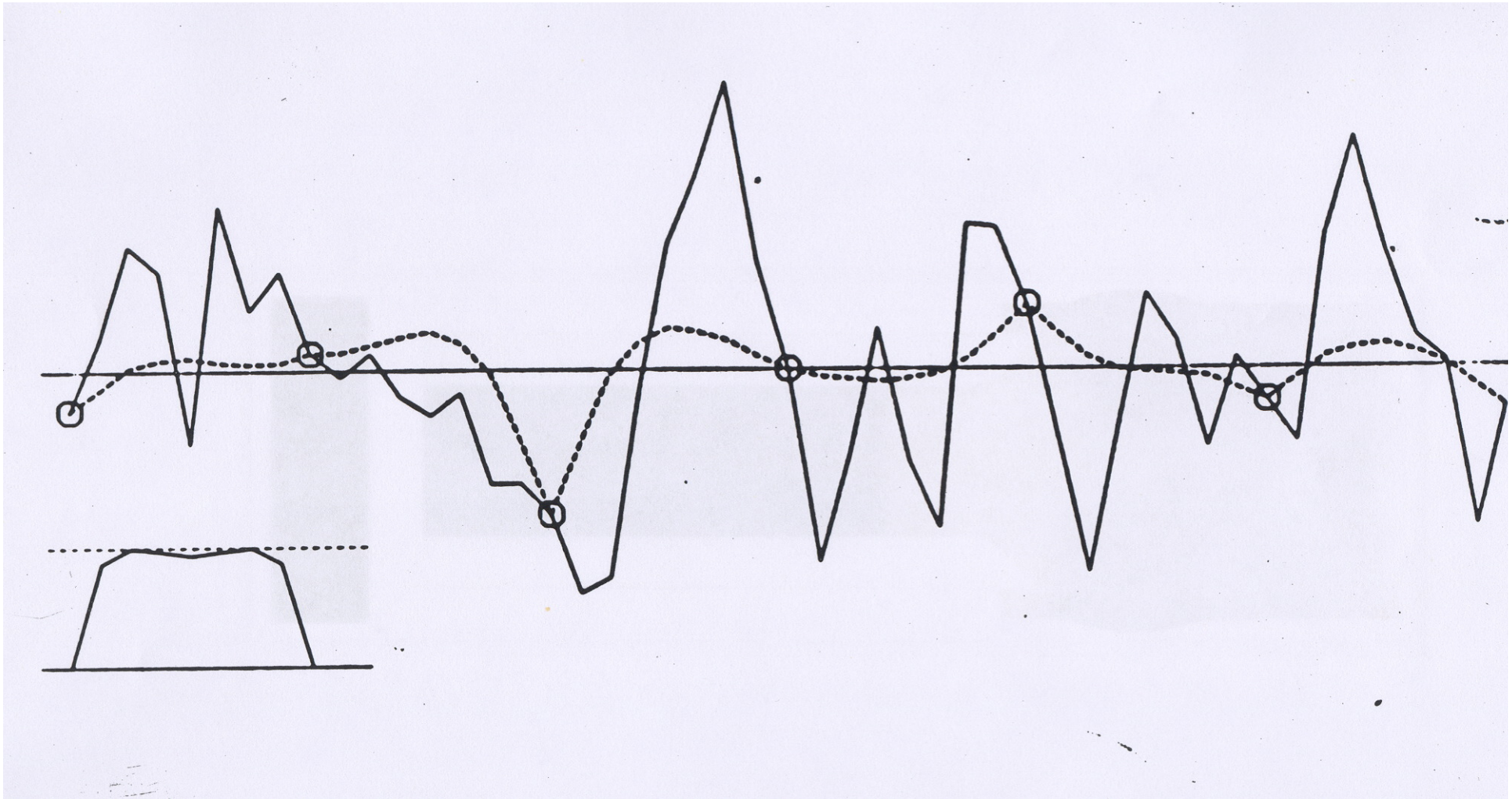
If observation errors ε_j are mutually uncorrelated, have common variance s, and are uncorrelated with field $\boldsymbol{\Phi}$, then

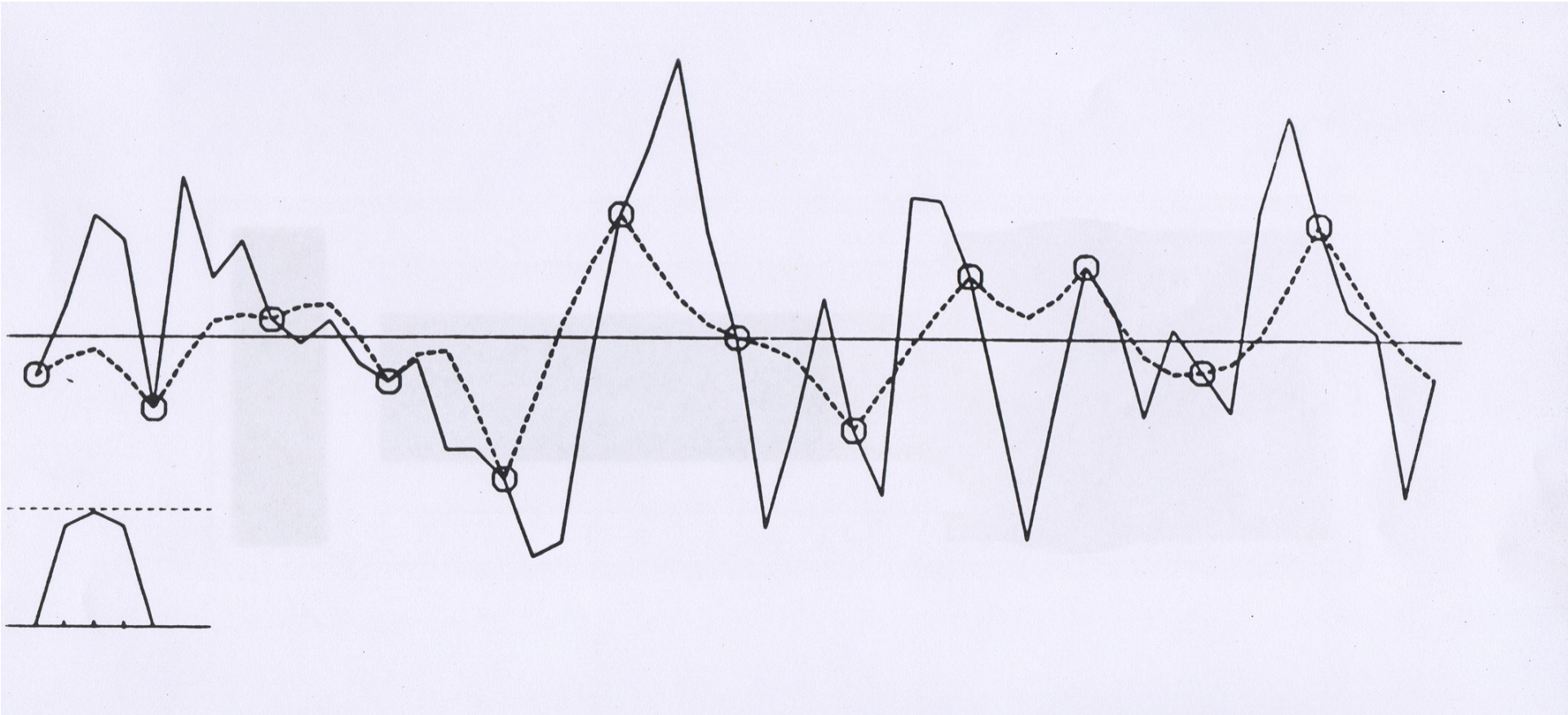
$$E(y_j'y_k') = C_{\boldsymbol{\Phi}}(\xi_j, \xi_k) + s\delta_{jk}$$

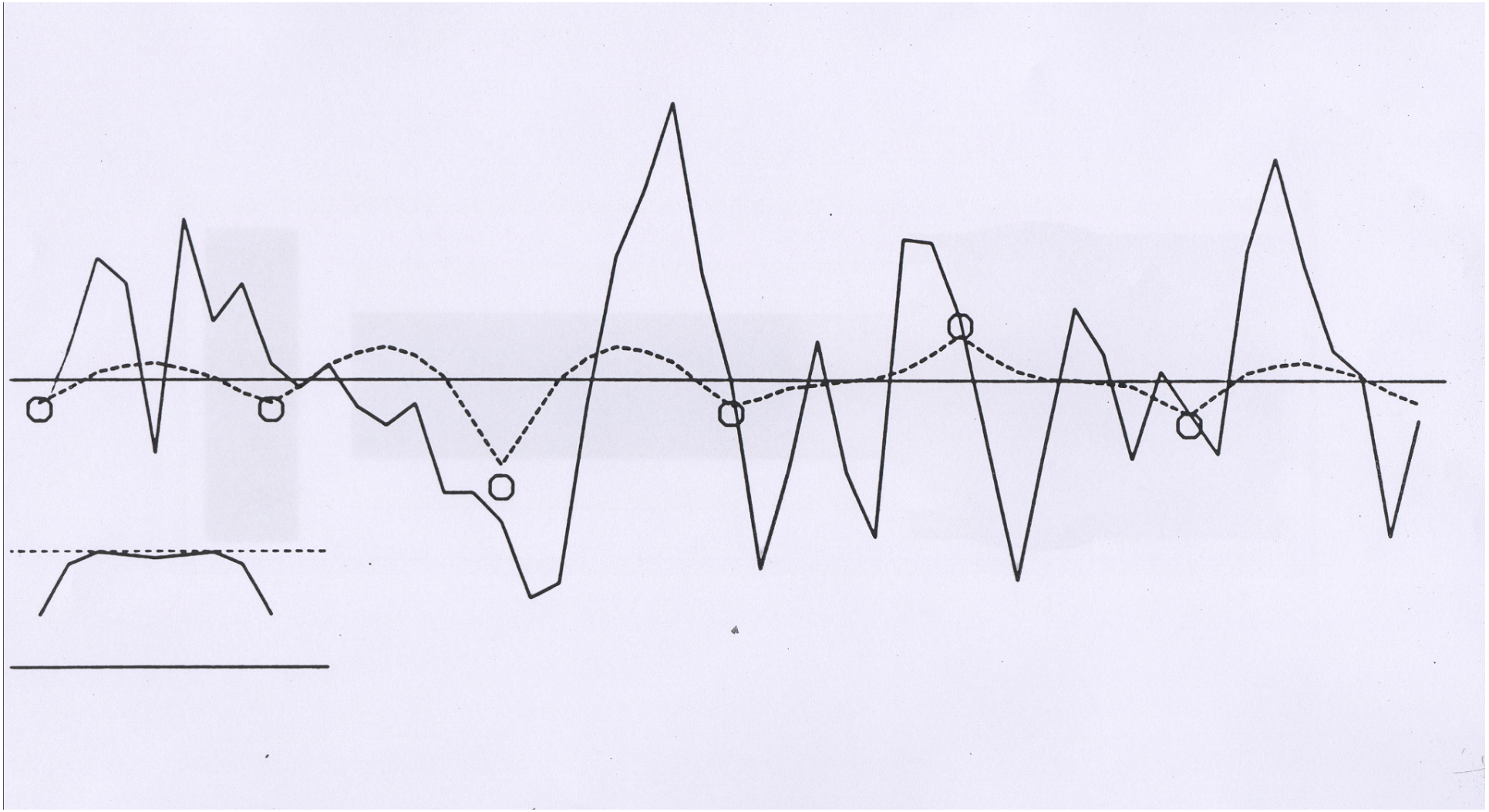
and

$$E(x'y_j') = C_{\boldsymbol{\Phi}}(\xi, \xi_j)$$









Optimal Interpolation (continued 3)

$$x^a = E(x) + E(x'\boldsymbol{y}'^{\mathrm{T}})\,[E(\boldsymbol{y}'\boldsymbol{y}'^{\mathrm{T}})]^{-1}\,[\boldsymbol{y} - E(\boldsymbol{y})]$$

Vector

$$\boldsymbol{\mu} = (\mu_j) \equiv [E(\boldsymbol{y}'\boldsymbol{y}'^{\mathrm{T}})]^{-1}\,[\boldsymbol{y} - E(\boldsymbol{y})]$$

is independent of variable to be estimated

$$x^a = E(x) + \Sigma_j\, \mu_j\, E(x' y_j')$$

$$\begin{aligned}\Phi^a(\xi) &= E[\Phi(\xi)] + \Sigma_j\, \mu_j\, E[\Phi'(\xi)\, y_j'] \\ &= E[\Phi(\xi)] + \Sigma_j\, \mu_j\, C_\Phi(\xi, \xi_j)\end{aligned}$$

Correction made on background expectation is a linear combination of the p functions

$$E[\Phi'(\xi)\, y_j'].\, E[\Phi'(\xi)\, y_j'] \;[\, = C_\Phi(\xi, \xi_j)\,]$$

considered as a function of estimation position ξ, is the *representer* associated with observation y_j.

Optimal Interpolation (continued 4)

Univariate interpolation. Each physical field (*e. g.* temperature) determined from observations of that field only.

Multivariate interpolation. Observations of different physical fields are used simultaneously. Requires specification of cross-covariances between various fields.

Cross-covariances between mass and velocity fields can simply be modelled on the basis of geostrophic balance.

Cross-covariances between humidity and temperature (and other) fields still a problem.

Schlatter's (1975) multivariate covariances

Specified as multivariate 2-point functions.

Not easy to ensure that specified functions are actually valid covariances.

Used in OI and related observation-space methods.

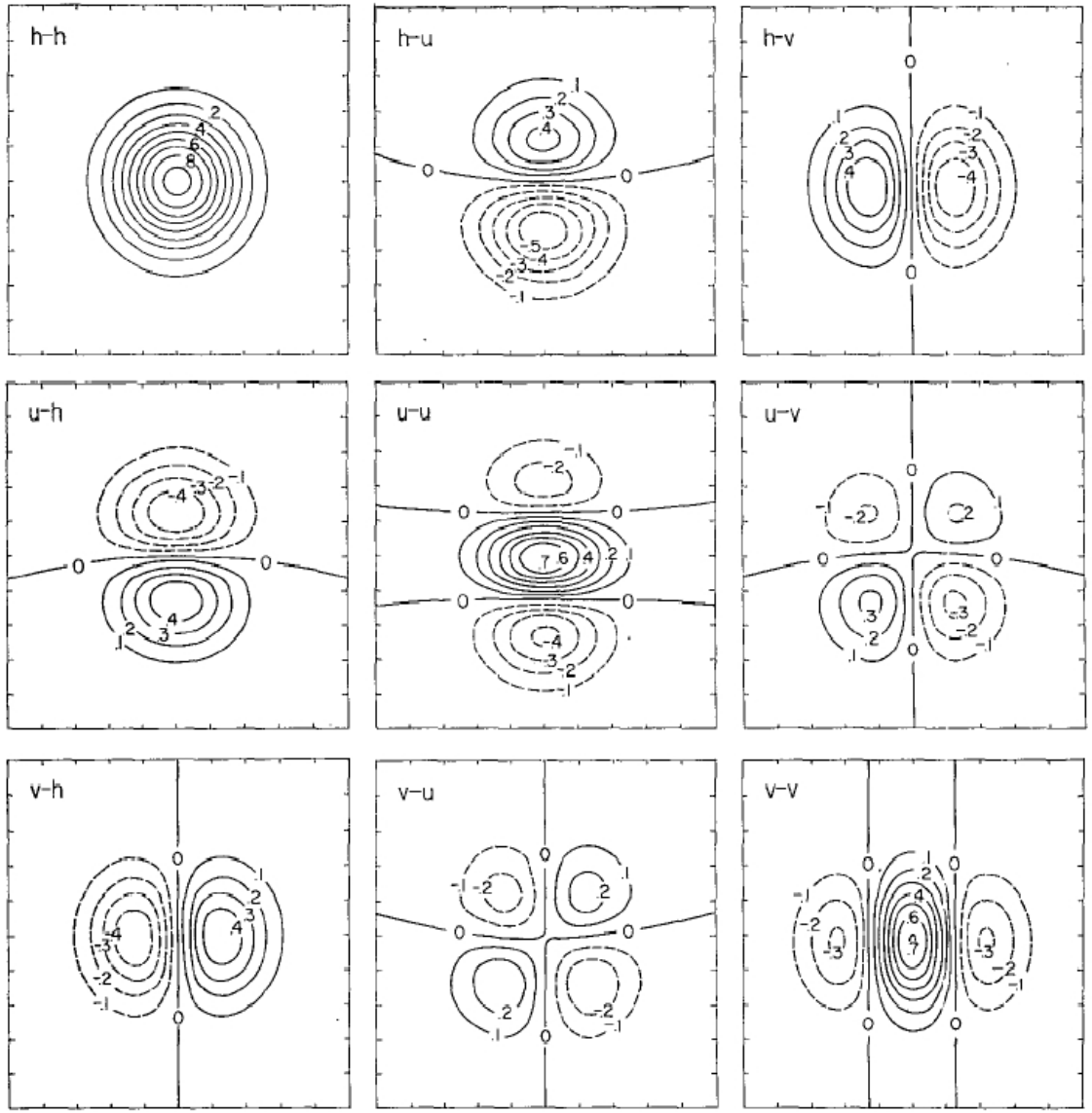


FIG. 3. Correlations among the variables h, u, and v based upon the expression $\mu = 0.95 \exp(-1.24s^2)$ for height-height correlation and the geostrophic relations. Diagrams centered at 110°W, 35°N. Tick marks 500 km apart.

Courtesy A. Lorenc

Best Linear Unbiased Estimate

State vector x, belonging to *state space* $\mathcal{S}$ ($\dim \mathcal{S} = n$), to be estimated.

Available data in the form of

- A '*background*' estimate (*e. g.* forecast from the past), belonging to *state space*, with dimension n

  $$x^b = x + \zeta^b$$

- An additional set of data (*e. g.* observations), belonging to *observation space*, with dimension p

  $$y = Hx + \varepsilon$$

  H is known linear *observation operator*.

Assume probability distribution is known for the couple (ζ^b, ε).

Assume $E(\zeta^b) = 0$, $E(\varepsilon) = 0$, $E(\zeta^b \varepsilon^{\mathrm{T}}) = 0$ (not restrictive)

Set $E(\zeta^b \zeta^{b\mathrm{T}}) = P^b$ (also often denoted B), $E(\varepsilon\varepsilon^{\mathrm{T}}) = R$

Best Linear Unbiased Estimate (continuation 1)

$$\boldsymbol{x}^b = \boldsymbol{x} + \boldsymbol{\xi}^b \qquad (1)$$

$$\boldsymbol{y} = H\boldsymbol{x} + \boldsymbol{\varepsilon} \qquad (2)$$

A probability distribution being known for the couple $(\boldsymbol{\xi}^b, \boldsymbol{\varepsilon})$, eqs (1-2) define probability distribution for the couple $(\boldsymbol{x}, \boldsymbol{y})$, with

$E(\boldsymbol{x}) = \boldsymbol{x}^b$, $\boldsymbol{x}' = \boldsymbol{x} - E(\boldsymbol{x}) = -\boldsymbol{\xi}^b$

$E(\boldsymbol{y}) = H\boldsymbol{x}^b$, $\boldsymbol{y}' = \boldsymbol{y} - E(\boldsymbol{y}) = \boldsymbol{y} - H\boldsymbol{x}^b = \boldsymbol{\varepsilon} - H\boldsymbol{\xi}^b$

$\boldsymbol{d} \equiv \boldsymbol{y} - H\boldsymbol{x}^b$ is called the *innovation vector*.

Best Linear Unbiased Estimate (continuation 2)

Apply formulæ for Optimal Interpolation

$$\boldsymbol{x}^a = \boldsymbol{x}^b + P^b H^{\mathrm{T}} [HP^b H^{\mathrm{T}} + R]^{-1} (\boldsymbol{y} - H\boldsymbol{x}^b)$$
$$P^a = P^b - P^b H^{\mathrm{T}} [HP^b H^{\mathrm{T}} + R]^{-1} HP^b$$

x^a is the *Best Linear Unbiased Estimate* (*BLUE*) of x from x^b and y.

Equivalent set of formulæ

$$\boldsymbol{x}^a = \boldsymbol{x}^b + P^a H^{\mathrm{T}} R^{-1} (\boldsymbol{y} - H\boldsymbol{x}^b)$$
$$[P^a]^{-1} = [P^b]^{-1} + H^{\mathrm{T}} R^{-1} H$$

Matrix $K = P^b H^{\mathrm{T}} [HP^b H^{\mathrm{T}} + R]^{-1} = P^a H^{\mathrm{T}} R^{-1}$ is *gain matrix*.

If probability distributions are *globally* gaussian, *BLUE* achieves bayesian estimation, in the sense that $P(\boldsymbol{x} \mid \boldsymbol{x}^b, \boldsymbol{y}) = \mathcal{N}[\boldsymbol{x}^a, \boldsymbol{P}^a]$.

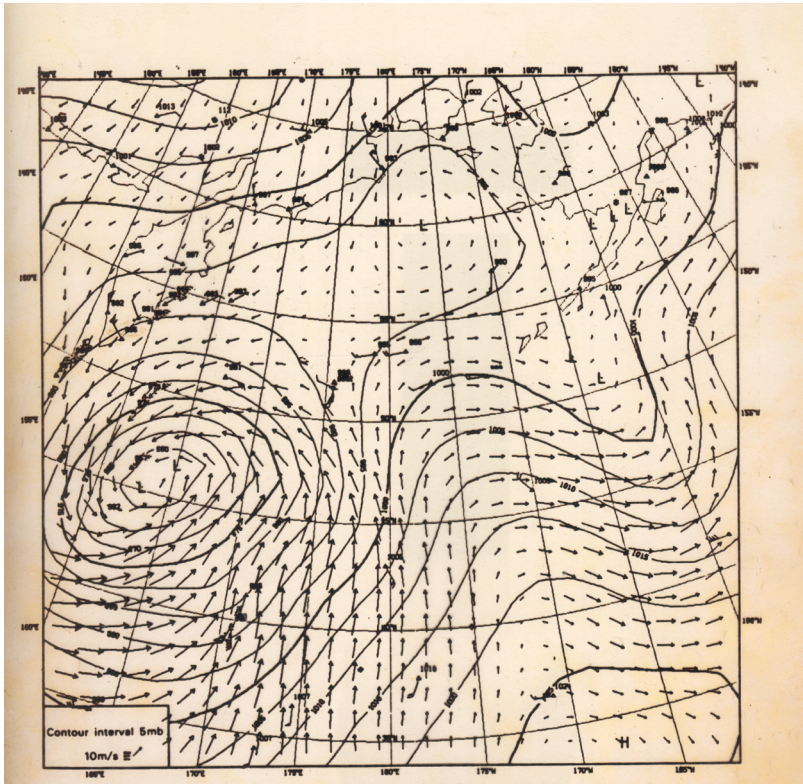


Fig. 4 Sea level pressure and wind forecast corresponding to the central area of Fig. 1, with plotted surface observations of pressure and wind (each fleche = 5 m/s).

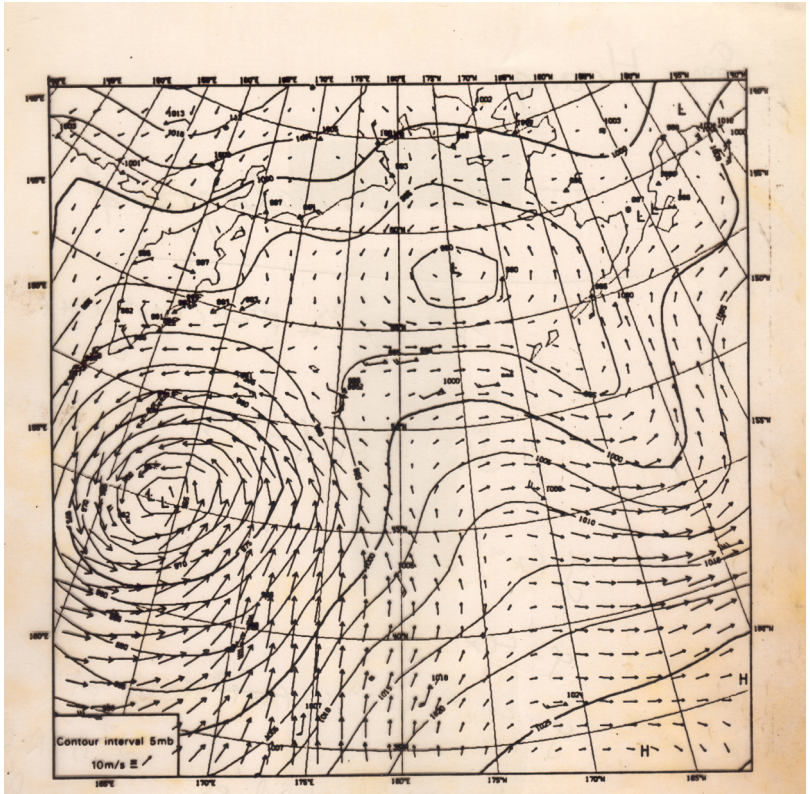


Fig. 5 As Fig. 4 for the analysis in the data assimilation cycle

After A. Lorenc

Best Linear Unbiased Estimate (continuation 4)

Variational form of the *BLUE*

BLUE x^a minimizes following scalar *objective function*, defined on state space

$\xi \in \mathcal{S} \rightarrow$

$$\mathcal{J}(\xi) = (1/2)\,(x^b - \xi)^{\mathrm{T}}\,[P^b]^{-1}\,(x^b - \xi) + (1/2)\,(y - H\xi)^{\mathrm{T}}\,R^{-1}\,(y - H\xi)$$

$$= \qquad \mathcal{J}_b \qquad + \qquad \mathcal{J}_o$$

'3D-Var'

Can easily, and heuristically, be extended to the case of a nonlinear observation operator H.

Used operationally in USA, Australia, China, …

Question. How to introduce temporal dimension in estimation process ?

- Logic of Optimal Interpolation can be extended to time dimension.

- But we know much more than just temporal correlations. We know explicit dynamics.

Real (unknown) state vector at time k (in format of assimilating model) x_k. Belongs to state space $\mathcal{S}$ (dim $\mathcal{S} = n$)

Evolution equation

$$x_{k+1} = M_k(x_k) + \eta_k$$

M_k is (known) model, η_k is (unknown) model error

Sequential Assimilation

- Assimilating model is integrated over period of time over which observations are available. Whenever model time reaches an instant at which observations are available, state predicted by the model is updated with new observations.

Variational Assimilation

- Assimilating model is globally adjusted to observations distributed over observation period. Achieved by minimization of an appropriate scalar *objective function* measuring misfit between data and sequence of model states to be estimated.

- Observation vector at time k

$$y_k = H_k x_k + \varepsilon_k \qquad k = 0, \ldots, K$$

$$E(\varepsilon_k) = 0 \quad ; \quad E(\varepsilon_k \varepsilon_j^{\mathrm{T}}) = R_k \, \delta_{kj}$$

H_k linear

- Evolution equation

$$x_{k+1} = M_k x_k + \eta_k \qquad k = 0, \ldots, K\text{-}1$$

$$E(\eta_k) = 0 \quad ; \quad E(\eta_k \eta_j^{\mathrm{T}}) = Q_k \, \delta_{kj}$$

M_k linear

- $E(\eta_k \varepsilon_j^{\mathrm{T}}) = 0$ (errors uncorrelated in time)

At time k, background $x^b{}_k$ and associated error covariance matrix $P^b{}_k$ known

- Analysis step

$$x^a{}_k = x^b{}_k + P^b{}_k H_k^{\mathrm{T}} [H_k P^b{}_k H_k^{\mathrm{T}} + R_k]^{-1} (y_k - H_k x^b{}_k)$$
$$P^a{}_k = P^b{}_k - P^b{}_k H_k^{\mathrm{T}} [H_k P^b{}_k H_k^{\mathrm{T}} + R_k]^{-1} H_k P^b{}_k$$

- Forecast step

$$x^b{}_{k+1} = M_k x^a{}_k$$
$$\begin{aligned} P^b{}_{k+1} &= E[(x^b{}_{k+1} - x_{k+1})(x^b{}_{k+1} - x_{k+1})^{\mathrm{T}}] = E[(M_k x^a{}_k - M_k x_k - \eta_k)(M_k x^a{}_k - M_k x_k - \eta_k)^{\mathrm{T}}] \\ &= M_k E[(x^a{}_k - x_k)(x^a{}_k - x_k)^{\mathrm{T}}] M_k^{\mathrm{T}} - E[\eta_k (x^a{}_k - x_k)^{\mathrm{T}}] - E[(x^a{}_k - x_k)\eta_k^{\mathrm{T}}] + E[\eta_k \eta_k^{\mathrm{T}}] \\ &= M_k P^a{}_k M_k^{\mathrm{T}} + Q_k \end{aligned}$$

At time k, background x^b_k and associated error covariance matrix P^b_k known

- Analysis step

$$x^a_k = x^b_k + P^b_k H_k^{\mathrm{T}} [H_k P^b_k H_k^{\mathrm{T}} + R_k]^{-1} (y_k - H_k x^b_k)$$
$$P^a_k = P^b_k - P^b_k H_k^{\mathrm{T}} [H_k P^b_k H_k^{\mathrm{T}} + R_k]^{-1} H_k P^b_k$$

- Forecast step

$$x^b_{k+1} = M_k x^a_k$$
$$P^b_{k+1} = M_k P^a_k M_k^{\mathrm{T}} + Q_k$$

Kalman filter (KF, Kalman, 1960)

Must be started from some initial estimate (x^b_0, P^b_0)

If all operators are linear, and if errors are uncorrelated in time, Kalman filter produces at time k the *BLUE* ${x^b}_k$ (resp. ${x^a}_k$) of the real state x_k from all data prior to (resp. up to) time k, plus the associated estimation error covariance matrix ${P^b}_k$ (resp. ${P^a}_k$).

If in addition errors are gaussian, the corresponding conditional probability distributions are the respective gaussian distributions

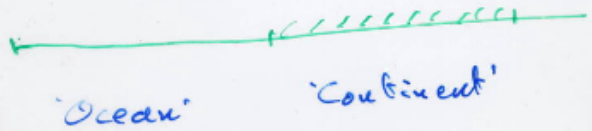
$\mathcal{N}([{x^b}_k, {P^b}_k]$ and $\mathcal{N}([{x^a}_k, {P^a}_k]$.

A didactic example (Ghil et al.)

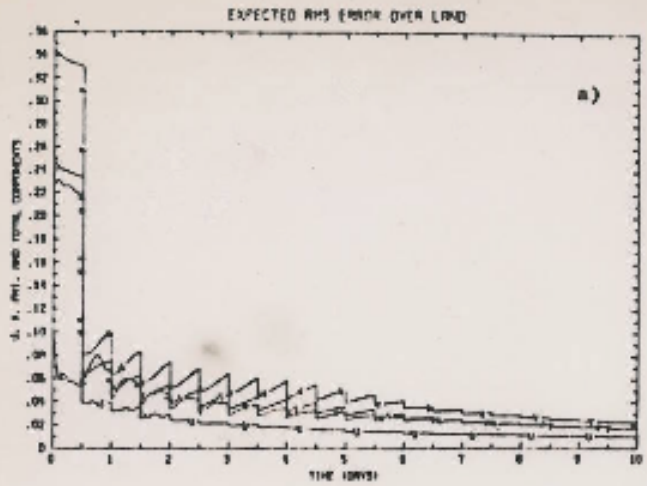
Barotropic model

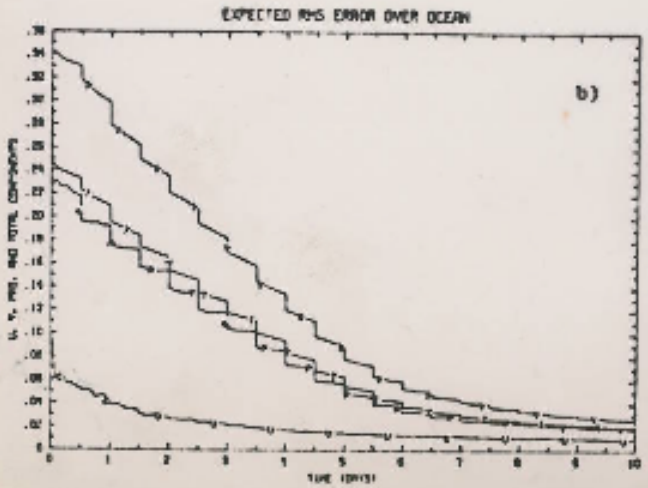
$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + \mathrm{div}(\varphi \underline{U}) = 0 \\[2ex] \dfrac{\partial \underline{U}}{\partial t} + \mathrm{grad}\left(\varphi + \frac{1}{2} U^2\right) + \underline{k} \times (f + \xi) \underline{U} = 0 \end{cases}$$

One dimension, periodic



Linearized (conserves energy)





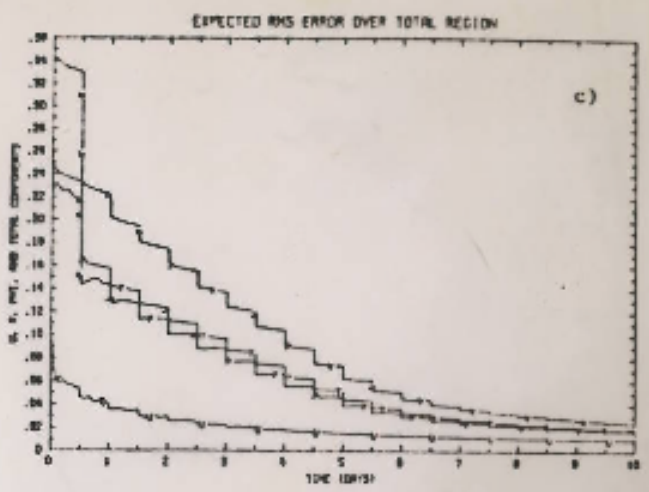


Fig. 2

The components of the total expected rms error (Erms), (trace: $P_k)^{1/2}$, in the estimation of solutions to the stochastic-dynamic system (Ψ,H), with Ψ given by (3.6) and $H = (I\ 0)$. System noise is absent, $Q = 0$. The filter used is the standard K-B filter (2.11) for the model.

a) Erms over land; b) Erms over the ocean; c) Erms over the entire L-domain

In each one of the figures, each curve represents one component of the total Erms error. The curves labelled U, V, and P represent the u component, v component and ϕ component, respectively. They are found by summing the diagonal elements of P_k which correspond to u, v, and ϕ, respectively, dividing by the number of terms in the sum, and then taking the square root. In a) the summation extends over land points only, in b) over ocean points only, and in c) over the entire L-domain. The vertical axis is scaled in such a way that 1.0 corresponds to an Erms error of v_{max} for the U and V curves, and of ϕ_0 for the P curve. The observational error level is 0.089 for the U and V curves, and 0.080 for the P curve. The curves labelled T represent the total Erms error over each region. Each T curve is a weighted average of the corresponding U, V, and P curves, with the weights chosen in such a way that the T curve measures the error in the total energy $u^2 + v^2 + \phi^2/\gamma$, conserved by the system (3.1). The observational noise level for the T curve is then 0.088. Notice the immediate error decrease over land and the gradual decrease over the ocean. The total estimation error tends to zero.

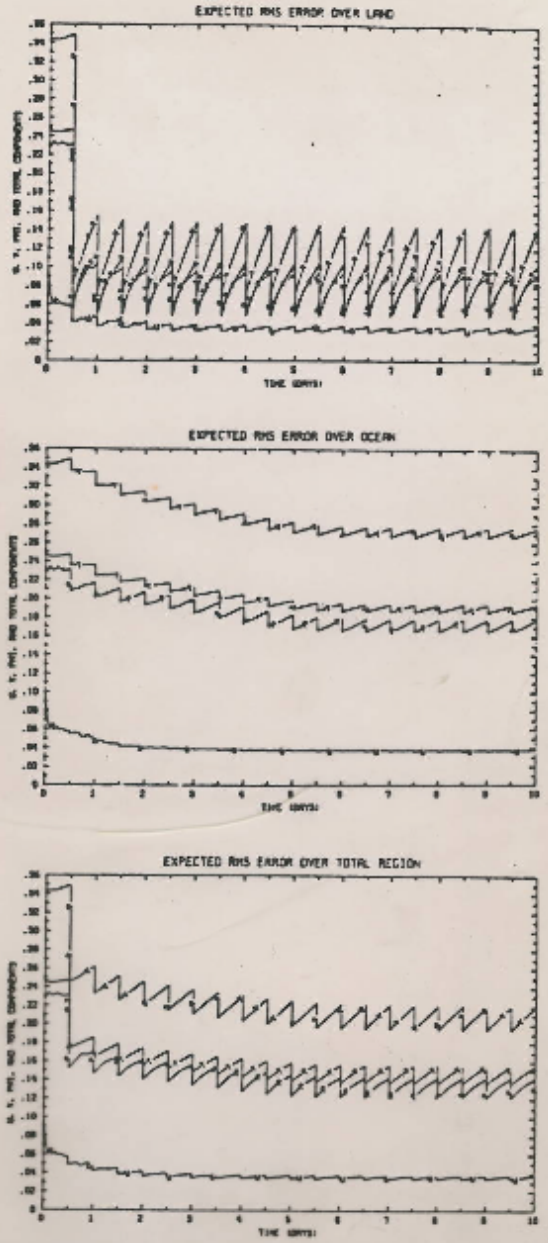


Fig. 6 This figure and the following ones show the properties of the estimated algorithm (2.11) in the presence of system noise, $Q \neq 0$. This figure gives the Erms estimation error, and is homologous to Fig. 2. Notice the sharper increase of error over land between synoptic times, and the convergence of each curve to a periodic, nonzero function.

Nonlinearities ?

Model is usually nonlinear, and observation operators (satellite observations) tend more and more to be nonlinear.

- Analysis step

$$x^a_k = x^b_k + P^b_k H_k'^{\mathrm{T}} [H_k' P^b_k H_k'^{\mathrm{T}} + R_k]^{-1} [y_k - H_k(x^b_k)]$$
$$P^a_k = P^b_k - P^b_k H_k'^{\mathrm{T}} [H_k' P^b_k H_k'^{\mathrm{T}} + R_k]^{-1} H_k' P^b_k$$

- Forecast step

$$x^b_{k+1} = M_k(x^a_k)$$
$$P^b_{k+1} = M_k' P^a_k M_k'^{\mathrm{T}} + Q_k$$

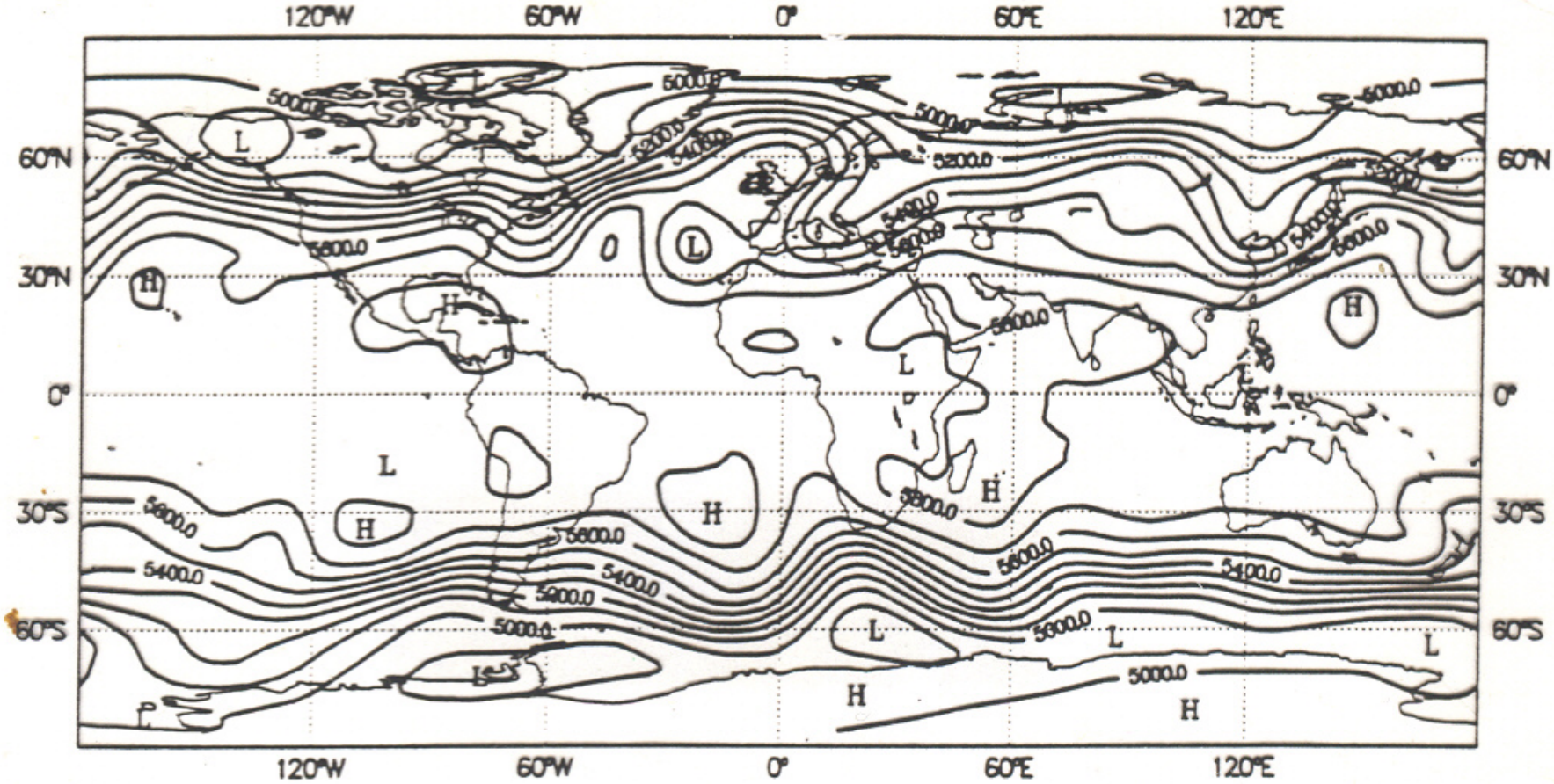
Extended Kalman Filter (EKF, heuristic !)

Costliest part of computation

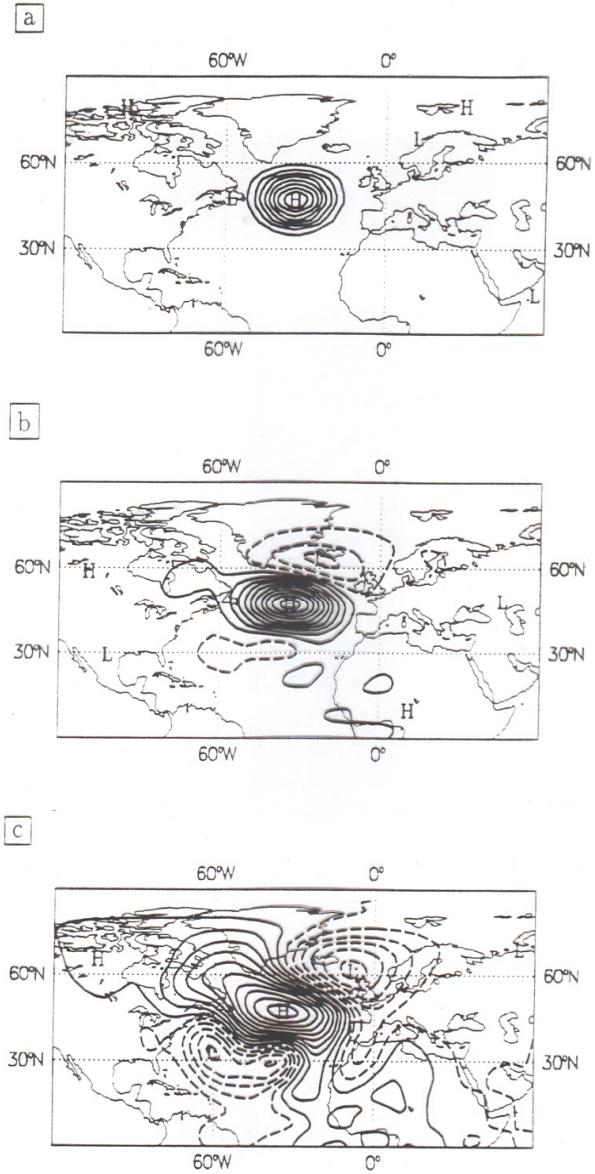
$$P^b{}_{k+1} = M_k P^a{}_k M_k{}^{\mathrm{T}} + Q_k$$

Multiplication by M_k = one integration of the model between times k and $k+1$.
Computation of $M_k P^a{}_k M_k{}^{\mathrm{T}} \approx 2n$ integrations of the model

Need for determining the temporal evolution of the uncertainty on the state of the system is the major difficulty in assimilation of meteorological and oceanographical observations



Analysis of 500-hPa geopotential for 1 December 1989, 00:00 UTC
(ECMWF, spectral truncation T21, unit *m*. After F. Bouttier)



Temporal evolution of the 500-hPa geopotential autocorrelation with respect to point located at 45N, 35W. From top to bottom: initial time, 6- and 24-hour range. Contour interval 0.1. After F. Bouttier.

Two solutions :

- Low-rank filters (Heemink, Pham, …)

  Reduced Rank Square Root Filters, Singular Evolutive Extended Kalman Filter, ….

- Ensemble filters (Evensen, Anderson, Kalnay, …)

  Uncertainty is represented, not by a covariance matrix, but by an ensemble of point estimates in state space which are meant to sample the conditional probability distribution for the state of the system (dimension $N \approx O(10\text{-}100)$).

  Ensemble is evolved in time through the full model, which eliminates any need for linear hypothesis as to the temporal evolution.

How to update predicted ensemble with new observations ?

Predicted ensemble at time t : $\{x^b_i\}$, $\quad i = 1, \ldots, N$
Observation vector at same time : $y = Hx + \varepsilon$

- Gaussian approach

  Produce sample of probability distribution for real observed quantity Hx
  $y_i = y - \varepsilon_i$
  where ε_i is distributed according to probability distribution for observation error ε
  .

  Then use Kalman formula to produce sample of 'analysed' states

  $$x^a_i = x^b_i + P^b H^T [HP^bH^T + R]^{-1} (y_i - Hx^b_i) , \qquad i = 1, \ldots, N \qquad (2)$$

  where P^b is covariance matrix of predicted ensemble $\{x^b_i\}$.

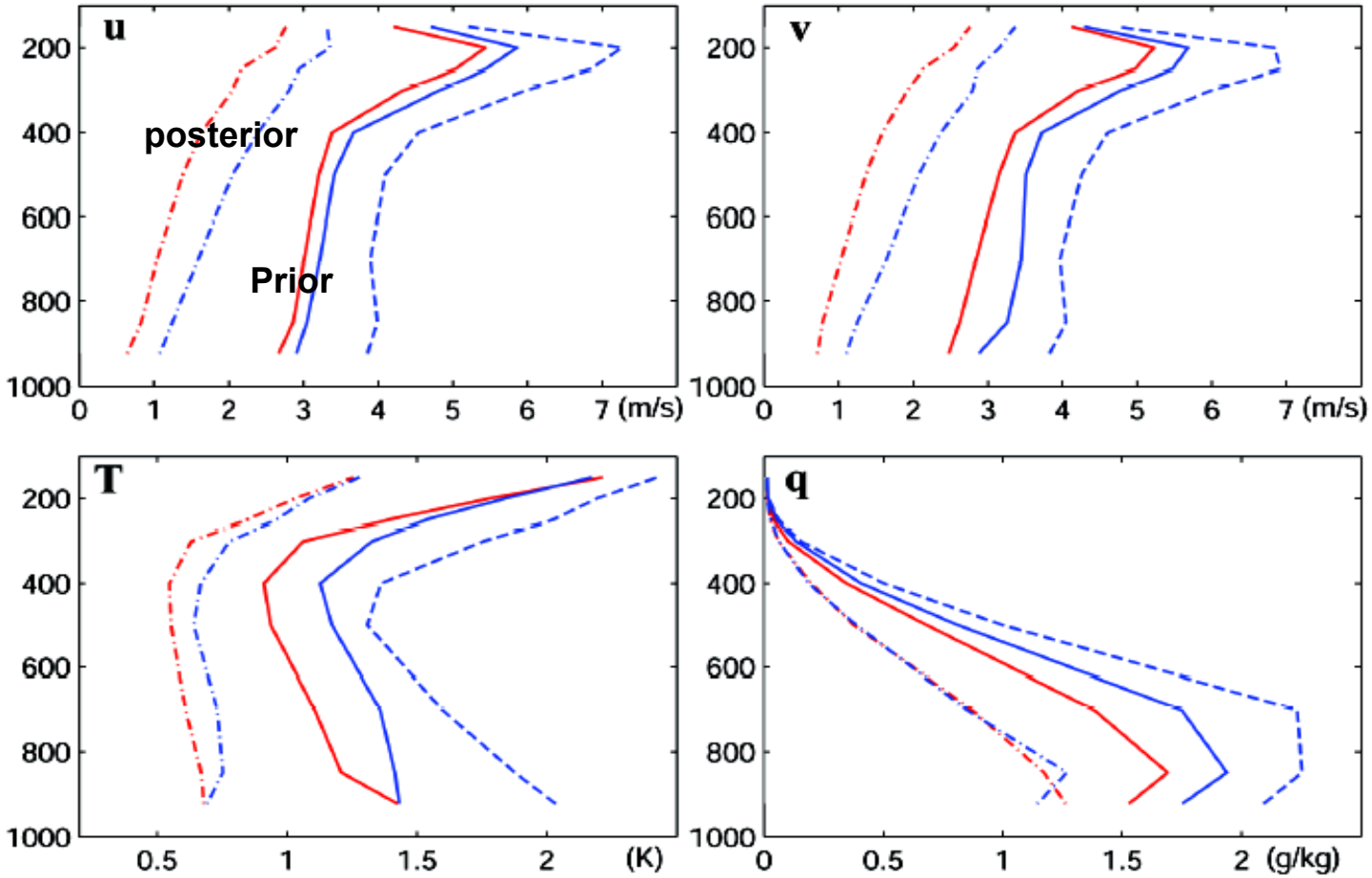
  Remark. If P^b was exact covariance matrix of background error, (2) would achieve Bayesian estimation, in the sense that $\{x^a_i\}$ would be a sample of conditional probability distribution for x, given all data up to time t.

Called *Ensemble Kalman Filter*. Has now become one of the two major powerful algorithms for assimilation of meteorological and oceanographical observations.

Local Ensemble Transform Kalman Filter (*LETKF*, Kalnay and colleagues)

Month-long Performance of EnKF vs. 3Dvar with WRF

— EnKF — 3DVar (prior, solid; posterior, dotted)



Better performance of EnKF than 3DVar also seen in both 12-h forecast and posterior analysis in terms of root-mean square difference averaged over the entire month

(Meng and Zhang 2007c, MWR, in review)

Variational Assimilation

- Observation vector at time k

$$y_k = H_k x_k + \varepsilon_k \qquad k = 0, \ldots, K$$
$$E(\varepsilon_k) = 0 \quad ; \quad E(\varepsilon_k \varepsilon_j^{\mathrm{T}}) \equiv R_k \, \delta_{kj}$$

- Evolution equation

$$x_{k+1} = M_k x_k + \eta_k \qquad k = 0, \ldots, K\text{-}1$$
$$E(\eta_k) = 0 \quad ; \quad E(\eta_k \eta_j^{\mathrm{T}}) \equiv Q_k \, \delta_{kj}$$
$$E(\eta_k \varepsilon_j^{\mathrm{T}}) = 0$$

- Background estimate at time 0

$$x^b{}_0 = x_0 + \zeta^b{}_0$$
$$E(\zeta^b{}_0) = 0 \quad ; \quad E(\zeta^b{}_0 \, \zeta^b{}_0{}^{\mathrm{T}}) \equiv P^b{}_0$$
$$E(\zeta^b{}_0 \varepsilon_k^{\mathrm{T}}) = 0 \quad ; \quad E(\zeta^b{}_0 \eta_k^{\mathrm{T}}) = 0$$

- Errors uncorrelated in time

Variational assimilation leads to the following *weak constraint* objective function

$(\xi_0, \xi_1, .., \xi_K) \rightarrow$

$$\mathcal{J}(\xi_0, \xi_1, ..., \xi_K)$$
$$= (1/2)\,(x^b{}_0 - \xi_0)^{\mathrm{T}}\,[P^b{}_0]^{-1}\,(x^b{}_0 - \xi_0)$$
$$+ (1/2)\,\Sigma_{k=0,\,\ldots,\,K}\,(y_k - H_k\xi_k)^{\mathrm{T}}\,R_k^{-1}\,(y_k - H_k\xi_k)$$
$$+ (1/2)\,\Sigma_{k=0,\,\ldots,\,K\text{-}1}\,(\xi_{k+1} - M_k\xi_k)^{\mathrm{T}}\,Q_k^{-1}\,(\xi_{k+1} - M_k\xi_k)$$

If model errort is ignored (Q_k=0), problem reduces to minimizing

$$\xi_0 \rightarrow \mathcal{J}(\xi_0) = (1/2)\,(x^b{}_0 - \xi_0)^{\mathrm{T}}\,[P^b{}_0]^{-1}\,(x^b{}_0 - \xi_0)$$
$$+\ (1/2)\,\Sigma_{k=0,\ldots,K}\,(y_k - H_k\xi_k)^{\mathrm{T}}\,R_k^{-1}\,(y_k - H_k\xi_k)$$

subject to

$$\xi_{k+1} = M_k\xi_k \qquad , \qquad k = 0, \ldots, K\text{-}1$$

Strong constraint four-dimensional variational assimilation, or strong constraint *4D-Var*

Used operationally in several meteorological centres (Météo-France, UK Meteorological Office, Canadian Meteorological Centre, Japan Meteorological Agency, …) and, until recently, at ECMWF. The latter now has a ‘weak constraint’ component in its operational system.

$$\mathcal{J}(\xi_0) = (1/2)\,(x_0^b - \xi_0)^{\mathrm{T}}\,[P_0^b]^{-1}\,(x_0^b - \xi_0) + (1/2)\,\Sigma_k[y_k - H_k\xi_k]^{\mathrm{T}}\,R_k^{-1}\,[y_k - H_k\xi_k]$$

Background is not necessary, if observations are in sufficient number to overdetermine the problem. Nor is strict linearity.

Minimization achieved by iterative algorithm, each step of which requires the explicit knowledge of the local gradient $\nabla_u \mathcal{J} \equiv (\partial\mathcal{J}/\partial u_i)$ of $\mathcal{J}$ with respect to u.

Gradient computed by *adjoint method*, which proceeds, in the space of partial derivatives, in reverse order of direct computations.

How to numerically compute the gradient $\nabla_u \mathcal{J}$?

Direct perturbation, in order to obtain partial derivatives $\partial \mathcal{J} / \partial u_i$ by finite differences ? That would require as many explicit computations of the objective function $\mathcal{J}$ as there are components in u. Practically impossible.

Adjoint Method

Input vector $\boldsymbol{u} = (u_i)$, $\dim \boldsymbol{u} = n$

Numerical process, implemented on computer (*e. g.* integration of numerical model)

$$\boldsymbol{u} \rightarrow \boldsymbol{v} = \boldsymbol{G}(\boldsymbol{u})$$

$\boldsymbol{v} = (v_j)$ is *output vector* , $\dim \boldsymbol{v} = m$

Perturbation $\delta \boldsymbol{u} = (\delta u_i)$ of input. Resulting first-order perturbation on $\boldsymbol{v}$

$$\delta v_j = \Sigma_i \, (\partial v_j / \partial u_i) \; \delta u_i$$

or, in matrix form

$$\delta \boldsymbol{v} \; = \; \boldsymbol{G}' \, \delta \boldsymbol{u}$$

where $\boldsymbol{G}' \equiv (\partial v_j / \partial u_i)$ is local matrix of partial derivatives, or jacobian matrix, of $\boldsymbol{G}$.

Adjoint Method (continued 1)

$$\delta \boldsymbol{v} \;=\; \boldsymbol{G}'\,\delta \boldsymbol{u} \qquad \text{(D)}$$

Scalar function of output

$$\mathcal{J}(\boldsymbol{v}) \;=\; \mathcal{J}[\boldsymbol{G}(\boldsymbol{u})]$$

Gradient $\nabla_{\boldsymbol{u}}\mathcal{J}$ of $\mathcal{J}$ with respect to input $\boldsymbol{u}$?

'Chain rule'

$$\partial\mathcal{J}/\partial u_i = \Sigma_j\, \partial\mathcal{J}/\partial v_j\,(\partial v_j/\partial u_i)$$

or

$$\nabla_{\boldsymbol{u}}\mathcal{J} \;=\; \boldsymbol{G}'^{\mathrm{T}}\,\nabla_{\boldsymbol{v}}\mathcal{J} \qquad \text{(A)}$$

Adjoint Method (continued 2)

$\boldsymbol{G}$ is the composition of a number of successive steps

$$\boldsymbol{G} = \boldsymbol{G}_N \circ \ldots \circ \boldsymbol{G}_2 \circ \boldsymbol{G}_1$$

'Chain rule'

$$\boldsymbol{G}' = \boldsymbol{G}_N' \ldots \boldsymbol{G}_2' \, \boldsymbol{G}_1'$$

Transpose

$$\boldsymbol{G}'^{\mathrm{T}} = \boldsymbol{G}_1'^{\mathrm{T}} \, \boldsymbol{G}_2'^{\mathrm{T}} \ldots \boldsymbol{G}_N'^{\mathrm{T}}$$

Transpose, or *adjoint*, computations are performed in reversed order of direct computations.

If $\boldsymbol{G}$ is nonlinear, local jacobian $\boldsymbol{G}'$ depends on local value of input $\boldsymbol{u}$. Any quantity which is an argument of a nonlinear operation in the direct computation will be used again in the adjoint computation. It must be kept in memory from the direct computation (or else be recomputed again in the course of the adjoint computation).

If everything is kept in memory, total operation count of adjoint computation is at most 4 times operation count of direct computation (in practice about 2).

Adjoint Approach

$$\mathcal{J}(\xi_0) = (1/2)\,(x_0^b - \xi_0)^{\mathrm{T}}\,[P_0^b]^{-1}\,(x_0^b - \xi_0) + (1/2)\,\Sigma_k[y_k - H_k\xi_k]^{\mathrm{T}}\,R_k^{-1}\,[y_k - H_k\xi_k]$$

subject to $\xi_{k+1} = M_k\xi_k\,, \qquad k = 0, \ldots, K\text{-}1$

Control variable $\qquad \xi_0 = \boldsymbol{u}$

Adjoint equation

$$\lambda_K = \qquad H_K^{\mathrm{T}}\,R_K^{-1}\,[H_K\,\xi_K - y_K]$$

$$\lambda_k = M_k^{\mathrm{T}}\lambda_{k+1} + H_k^{\mathrm{T}}\,R_k^{-1}\,[H_k\,\xi_k - y_k] \qquad k = K\text{-}1, \ldots, 1$$

$$\lambda_0 = M_0^{\mathrm{T}}\lambda_1 \quad + H_0^{\mathrm{T}}\,R_0^{-1}\,[H_0\,\xi_0 - y_0] \;\; + \;\; [P_0^b]^{-1}\,(\xi_0 - x_0^b)$$

$$\nabla_u\mathcal{J} = \lambda_0$$

Result of direct integration (ξ_k), which appears in quadratic terms in expression of objective function, must be kept in memory from direct integration.

Adjoint Approach (continued 2)

Nonlinearities ?

$$\mathcal{J}(\xi_0) = (1/2)\,(x_0^b - \xi_0)^{\mathrm{T}}\,[P_0^b]^{-1}\,(x_0^b - \xi_0) + (1/2)\,\Sigma_k[y_k - H_k(\xi_k)]^{\mathrm{T}}\,R_k^{-1}\,[y_k - H_k(\xi_k)]$$

subject to $\xi_{k+1} = M_k(\xi_k)\,, \qquad k = 0, \ldots, K\text{-}1$

Control variable $\qquad \xi_0 = \boldsymbol{u}$

Adjoint equation

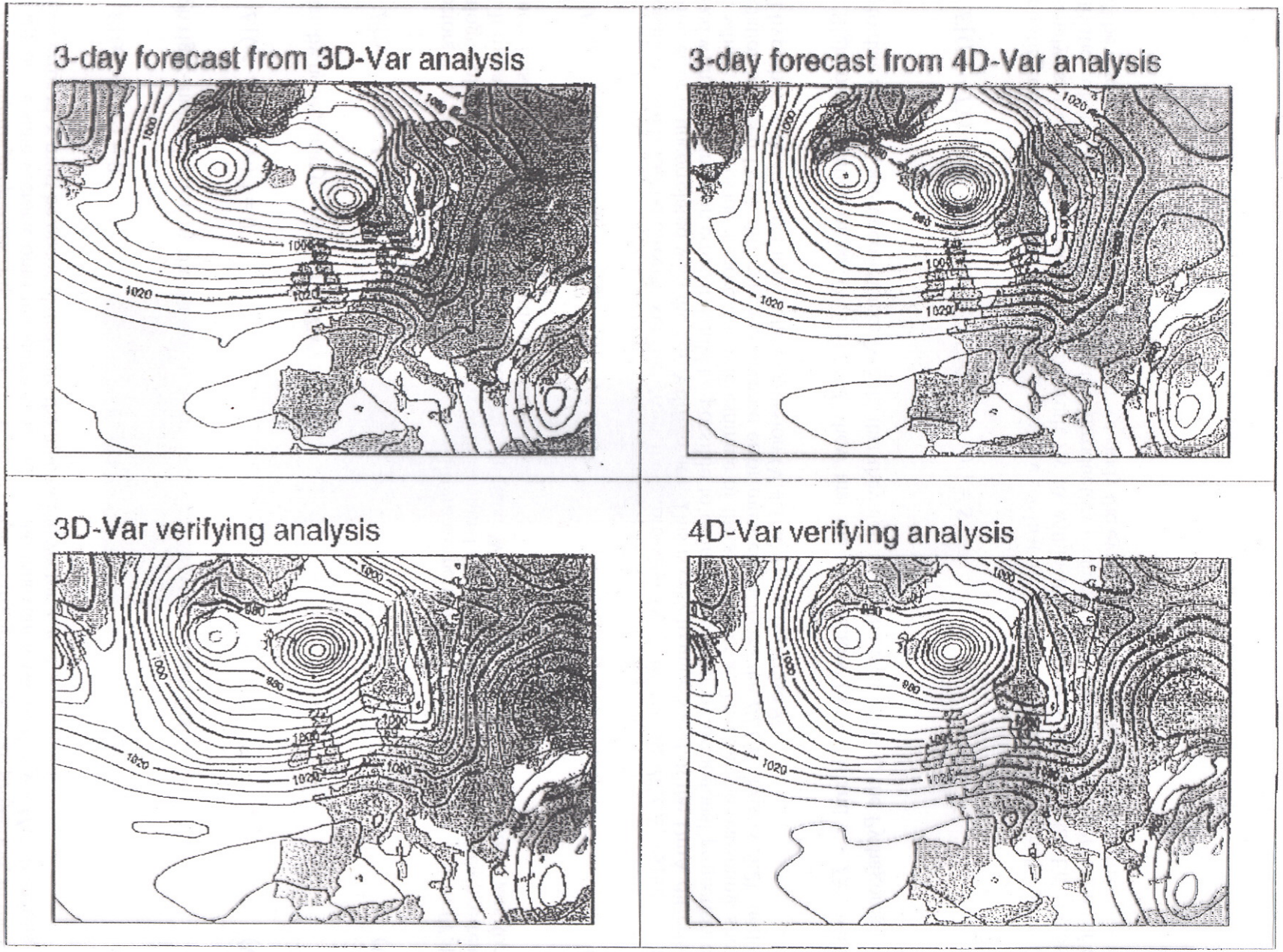
$$\lambda_K = H_K'^{\mathrm{T}}\,R_K^{-1}\,[H_K(\xi_K) - y_K]$$

$$\lambda_k = M_k'^{\mathrm{T}}\lambda_{k+1} + H_k'^{\mathrm{T}}\,R_k^{-1}\,[H_k(\xi_k) - y_k] \qquad k = K\text{-}1, \ldots, 1$$

$$\lambda_0 = M_0'^{\mathrm{T}}\lambda_1 + H_0'^{\mathrm{T}}\,R_0^{-1}\,[H_0(\xi_0) - y_0] + [P_0^b]^{-1}\,(\xi_0 - x_0^b)$$

$$\nabla_u \mathcal{J} = \lambda_0$$

Not heuristic (it gives the exact gradient $\nabla_u \mathcal{J}$), and really used as described here.



ECMWF, Results on one FASTEX case (1997)

Buehner (2008)

For the same numerical cost, and in meteorologically realistic situations, Ensemble Kalman Filter and Variational Assimilation produce results of similar quality.

Conclusions

Assimilation, which originated from the need of defining initial conditions for numerical weather forecasts, has progressively extended to many diverse applications

- Oceanography
- Atmospheric chemistry (both troposphere and stratosphere)
- Oceanic biogeochemistry
- Ground hydrology
- Terrestrial biosphere and vegetation cover
- Glaciology
- Magnetism (both planetary and stellar)
- Plate tectonics
- Planetary atmospheres (Mars, …)
- Reassimilation of past observations (mostly for climatological purposes, ECMWF, NCEP/NCAR)
- Identification of source of tracers
- Parameter identification
- *A priori* evaluation of anticipated new instruments
- Definition of observing systems *(Observing Systems Simulation Experiments)*
- Validation of models
- Sensitivity studies (adjoints)
- …

It has now become a major tool of numerical environmental science

Assimilation is related to

- Estimation theory
- Probability theory
- Atmospheric and oceanic dynamics
- Atmospheric and oceanic predictability
- Instrumental physics
- Optimisation theory
- Control theory
- Algorithmics and computer science
- …