GEOPHYSICAL FLUID DYNAMICS: Understanding (almost) everything with rotating shallow water models.

Hints and solutions to the problems

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Solutions to Problems of Chapter 2

1.1 Problem 1

To prove that

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}, \tag{1.1}$$

as applied to any dynamical variable $F(\mathbf{x},\, t)$, which is defined in the domain of the flow, we use the Euler-Lagrange duality and consider the space coordinates $\mathbf{x}$ as position of a fluid parcel passing through this point, and the Eulerian velocity $\mathbf{v}$ as the velocity of the parcel. We recall that the Lagrangian derivative is to be calculated along the parcel's trajectory, that is the change of $\mathbf{x}$ is $\Delta\mathbf{x} = \mathbf{v}\Delta t$, and hence

$$\frac{\Delta F}{\Delta t} = \frac{F(\mathbf{x} + \mathbf{v}\Delta t,\, t + \Delta t) - F(\mathbf{x},\, t)}{\Delta t}, \tag{1.2}$$

which, in the limit $\Delta t \to 0$ gives (1.1).

1.2 Problem 2

The demonstration follows by direct calculation of the Lagrangian derivative, and the use of the Lagrangian expression of divergence $\nabla \cdot \mathbf{v} = \frac{\partial \dot{X}}{\partial X} + \frac{\partial \dot{Y}}{\partial Y} + \frac{\partial \dot{Z}}{\partial Z}$:

$$\frac{d}{dt}\left(\rho \mathcal{J}\right) = \frac{d\rho}{dt}\mathcal{J} + \rho\frac{d\mathcal{J}}{dt} = \frac{d\rho_i}{dt} = 0, \tag{1.3}$$

$$\begin{aligned} \frac{d\mathcal{J}}{dt} &= \frac{\partial(\dot{X}, Y, Z)}{\partial(x, y, z)} + \frac{\partial(X, \dot{Y}, Z)}{\partial(x, y, z)} + \frac{\partial(X, Y, \dot{Z})}{\partial(x, y, z)} \\ &= \left(\frac{\partial(\dot{X}, Y, Z)}{\partial(X, Y, Z)} + ...\right)\mathcal{J} = \left(\frac{\partial \dot{X}}{\partial X} + \frac{\partial \dot{Y}}{\partial Y} + \frac{\partial \dot{Z}}{\partial Z}\right)\mathcal{J} \Rightarrow \end{aligned}$$

$$\frac{d\rho}{dt} + \rho\nabla \cdot \mathbf{v} = 0 \leftrightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{1.4}$$

1.3 Problem 3

In the direct calculation of the time-derivative, the fact that the contour is Lagrangian, that is its elements are moving with the fluid, should not be forgotten, which means that for an element of the contour $\mathbf{l} = d\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1$, where $\mathbf{x}_{1,2}$ are positions of neighbouring fluid parcels on the contour, $\frac{d}{dt}\mathbf{l} = \mathbf{v}_2 - \mathbf{v}_1 = d\mathbf{v}$

$$\frac{d\gamma}{dt} = \int_\Gamma \left(d\mathbf{l} \cdot \frac{d\mathbf{v}}{dt} + \mathbf{v} \cdot \frac{d}{dt}\mathbf{l} \right) = -\int_\Gamma \frac{\nabla P}{\rho} \cdot d\mathbf{l} + \int_\Gamma \mathbf{v} \cdot d\mathbf{v}. \tag{1.5}$$

The last integral in (1.5) is an integral of the full differential $d\frac{\mathbf{v}^2}{2}$ over the closed contour, and is therefore zero. In the case of barotropic fluid the integrand in the first integral becomes the differential of enthalpy, and the integral vanishes by the same reason.

1.4 Problem 4

The variation of the action with Lagrangian (2.26) gives:

$$\delta\mathcal{S} = \int dt \int d^3\mathbf{a} \left[\frac{d\mathbf{X}}{dt} \cdot \delta\mathbf{X} \; - P \left(\frac{\partial(\delta X, Y, Z)}{\partial(a,b,c)} + \frac{\partial(X, \delta Y, Z)}{\partial(a,b,c)} + \frac{\partial(X, Y, \delta Z)}{\partial(a,b,c)} \right) \right.$$
$$\left. - \; \delta P \left(\frac{\partial(X,Y,Z)}{\partial(a,b,c)} - 1 \right) \right]. \tag{1.6}$$

Vanishing $\delta\mathcal{S}$ for independent variations of δX, δY, δZ, and δP gives the equations of motion. The latter variation straightforwardly gives incompressibility condition in Lagrangian form:

$$\frac{\partial(X,Y,Z)}{\partial(a,b,c)} = 1. \tag{1.7}$$

In order to make appear variations of δX, δY, δZ instead of their derivatives, integrations by parts in t, a, b, and c in (1.6) under condition of vanishing variations at the boundaries are necessary.

In this way we get

$$\frac{d^2 X}{dt^2} = -\frac{\partial(P,Y,Z)}{\partial(a,b,c)} = -\frac{\partial(P,Y,Z)}{\partial(X,Y,Z)} = -\frac{\partial P}{\partial X}, \tag{1.8}$$

where (1.7) was used, and similarly for Y and Z. Replacing $\frac{d\mathbf{X}}{dt}$ by $\mathbf{v}$, and using Euler-Lagrange duality, we see that these equations are Euler equations for incompressible fluid with constant density.

1.5 Problem 5

In many textbooks, e.g. Landau & Lifshitz (1975), the Navier-Stokes equations are written in the coordinates using co-latitude θ instead of latitude ϕ. The coordinates are thus $(0 \le r \le \infty, 0 \le \theta \le \pi, 0 \le \lambda < 2\pi)$, and we will use them below. The formulas in coordinates $(0 \le r \le \infty, 0 \le \lambda < 2\pi, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2})$ are obtained by substitution $\theta = \frac{\pi}{2} - \phi$, and the change of sign of the corresponding component of velocity due to different orientation of the coordinate basis, see below. The right orthonormal coordinate basis is then $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\lambda}})$, where the unit vectors are pointed in the sense of increase of corresponding coordinates. The main technical difficulty of using curvilinear coordinates, with respect to Cartesian ones, is that the orientation of the basis depends on position. The position of a point is given by the radius-vector $\boldsymbol{r}$. An infinitesimal change of position is given by

$$d\boldsymbol{r} = dr\hat{\boldsymbol{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\lambda\hat{\boldsymbol{\lambda}}. \tag{1.9}$$

By definition, the velocity of the point is

$$\boldsymbol{v} = \frac{dr}{dt}\hat{\boldsymbol{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r\sin\theta\frac{d\lambda}{dt}\hat{\boldsymbol{\lambda}}. \tag{1.10}$$

The expressions in front of the unit vectors are respective components of the velocity

$$v_r = \frac{dr}{dt},\ v_\theta = r\frac{d\theta}{dt},\ v_\lambda = r\sin\lambda\frac{d\lambda}{dt}. \tag{1.11}$$

When calculating the acceleration $\frac{d\boldsymbol{v}}{dt}$ the changes of basis vectors with the change of position, i.e. in time, should be taken into account. As the length of the unit vectors does not change, the change of basis consists in a solid rotation. It is easy to check that it is given then by the following formulas:

$$\begin{cases} \frac{d\hat{\boldsymbol{r}}}{dt} &= \qquad \frac{v_\theta}{r}\hat{\boldsymbol{\theta}} + \frac{v_\lambda}{r}\hat{\boldsymbol{\lambda}}, \\ \frac{d\hat{\boldsymbol{\theta}}}{dt} &= -\frac{v_\theta}{r}\hat{\boldsymbol{\theta}} + \cot\theta\frac{v_\lambda}{r}\hat{\boldsymbol{\lambda}}, \\ \frac{d\hat{\boldsymbol{\lambda}}}{dt} &= -\frac{v_\theta}{r}\hat{\boldsymbol{\theta}} + \cot\theta\frac{v_\lambda}{r}\hat{\boldsymbol{\lambda}}, \end{cases} \tag{1.12}$$

and hence, the vector of acceleration takes the form

$$\begin{aligned} \frac{d\boldsymbol{v}}{dt} &= \frac{dv_r}{dt} - \left(\frac{v_\lambda^2 + v_\theta^2}{r}\right)\hat{\boldsymbol{r}} \\ &= \left(\frac{dv_\theta}{dt} + \frac{v_r v_\theta - v_\lambda^2\cot\lambda}{r}\right)\hat{\boldsymbol{\theta}} \\ &= \left(\frac{dv_\lambda}{dt} + \frac{v_r v_\lambda + v_\theta v_\lambda\cot\theta}{r}\right)\hat{\boldsymbol{\lambda}}. \end{aligned} \tag{1.13}$$

This determines the acceleration terms in Euler equations. The pressure gradient terms are obtained with the help of expression of the operator nabla in spherical coordinates:

$$\boldsymbol{\nabla} = \hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{\partial}{r\,\partial\theta} + \hat{\boldsymbol{\lambda}}\frac{\partial}{r\sin\theta\,\partial\lambda}, \tag{1.14}$$

which is also to be used in full (Lagrangian) time-derivatives in (1.13): $\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}$. The Coriolis acceleration is calculated from the decomposition of the angular velocity vector in the spherical coordinates basis: $\boldsymbol{\Omega} = \Omega\cos\theta\hat{\boldsymbol{r}} - \Omega\sin\theta\hat{\boldsymbol{\theta}}$, and the definition of the vector product in a right orthonormal basis:

$$\boldsymbol{\Omega}\wedge\boldsymbol{v} = \begin{vmatrix} \hat{\boldsymbol{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\lambda}} \\ \Omega\cos\theta & -\Omega\sin\theta & 0 \\ v_r & v_\theta & v_\lambda \end{vmatrix}. \tag{1.15}$$

All ingredients of the Euler equations are thus expressed in spherical coordinates. To obtain the continuity equation, the divergence of velocity should be expressed in spherical coordinates, taking into account that differentiations with respect to the coordinates are now applied not only to the components of velocity, but also to the basis vectors themselves. The formulas for the differentials of the basis vectors can be reconstructed from (1.12), by re-multiplying them by dt and recalling the definitions (2.15). This gives

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta v_\theta)}{\partial\theta} + \frac{\partial v_\lambda}{\partial\lambda}\right). \tag{1.16}$$

1.6 Problem 6

The demonstration is a straightforward calculation, taking the curl of the momentum equations, using the hydrodynamic identity, and retaining the dominant, in the domain of parameters of the primitive equations, terms in the curl.

1.7 Problem 7

By definition $q = \vec{\zeta}_a \cdot \nabla\rho$, and it is sufficient to prove that $\partial_t\vec{\zeta}_a \cdot \nabla\rho = -\vec{v}\cdot\left(\vec{\zeta}_a \cdot \nabla\rho\right)$. The left-hand side of the last equality is calculated using the equations of motion for $\vec{\zeta}_a$ and ρ, the fact that $\nabla\cdot\vec{\zeta}_a = \nabla\cdot\vec{v} = 0$, and the following identities of the vector analysis:

$$\nabla A \cdot \left(\nabla \wedge \vec{B}\right) = -\nabla\cdot\left(\nabla A \wedge \vec{B}\right), \tag{1.17}$$

$$\vec{A}\wedge\left(\vec{B}\wedge\vec{C}\right) = \vec{B}\left(\vec{A}\cdot\vec{C}\right) - \vec{C}\left(\vec{A}\cdot\vec{B}\right), \tag{1.18}$$

Proof

$$\begin{aligned}
\partial_t\left(\vec{\zeta}_a\cdot\nabla\rho\right) &= (\partial_t\vec{\zeta}_a)\cdot\nabla\rho + \vec{\zeta}_a\cdot\nabla(\partial_t\rho)\\
&= \nabla\rho\cdot\left(\nabla\wedge\left(\vec{v}\wedge\vec{\zeta}_a\right)\right) - \vec{\zeta}_a\cdot\nabla\left(\vec{v}\cdot\nabla\rho\right)\\
&= -\nabla\cdot\left(\nabla\rho\wedge\left(\vec{v}\wedge\vec{\zeta}_a\right)\right) - \vec{\zeta}_a\cdot\nabla\left(\vec{v}\cdot\nabla\rho\right)\\
&= -\nabla\cdot\left(\vec{v}\left(\vec{\zeta}_a\cdot\nabla\rho\right)\right) + \nabla\cdot\left(\vec{\zeta}_a\left(\vec{v}\cdot\nabla\rho\right)\right)\\
&- \vec{\zeta}_a\cdot\nabla\left(\vec{v}\cdot\nabla\rho\right) = -\vec{v}\cdot\nabla\left(\vec{\zeta}_a\cdot\nabla\rho\right).
\end{aligned} \tag{1.19}$$

□

1.8 Problem 8

For the internal waves of the form

$$(u, v, \phi) = \left(\hat{u}, \hat{v}, \hat{\phi}\right) e^{i(kx+ly+mw-\omega t)} + c.c., \tag{1.20}$$

with the dispersion relation:

$$\omega^2 = \left(N^2 \frac{k^2 + l^2}{m^2} + f^2\right), \tag{1.21}$$

polarisation relations are obtained by solving the inhomogeneous algebraic system of two Fourier-transformed momentum equations at a given amplitude of geopotential perturbation, which gives:

$$\hat{u} = \frac{\omega k + ilf}{\omega^2 - f^2} \hat{\phi} \tag{1.22}$$

$$\hat{v} = \frac{\omega l - ikf}{\omega^2 - f^2} \hat{\phi} \tag{1.23}$$

Perturbations of density and vertical velocity in terms of the perturbation of geopotential follow straightforwardly from the hydrostatic and linearised continuity equations.

Polarisation relation for the root $\omega = 0$ follows from the Fourier-transformed equation (2.80).

1.9 Problem 9

The fast way to show that inertia-gravity waves have no PV anomaly is to use the Fourier-transformed equation (2.80). The time-derivative in this equation becomes multiplication by ω, and the expression in brackets is the linearised PV anomaly. So if $\omega \neq 0$, which is the case for inertia-gravity waves, it is PV anomaly which should vanish. This can be alternatively shown by a direct calculations using polarisation relations of Problem 8.

1.10 Problem 10

As the horizontal momentum equations in (2.82) are the same as in (2.74), with the replacement $P \to \phi$ the polarisation relations for u and v are the same. On the contrary, the third polarisation relation, between w and P is different, and easy to find from the third equation in (2.80). Hence relations between the components of the horizontal velocity and the vertical velocity are different.

2

Solutions to Problems of Chapter 3

2.1 Problem 1

We give the derivation of the two-layer model with a free surface. The one-layer model is directly recovered from the equations for the upper layer below.

Conventions:

Upper layer - layer 1, position of the upper surface - z_2, position of the interface - z_1, position of the bottom, which is taken to be flat - $z_0 = \text{const}$, pressure over the upper surface $P_2 = \text{const}$, pressure at the interface - P_1, pressure at the bottom - P_0. Corresponding densities: $\rho_{1,2} = \text{const}$

2.1.1 Upper layer

Master equation:

$$\rho_1(z_2 - z_1)\left(\frac{d\mathbf{v}_1}{dt} + f\hat{\boldsymbol{z}} \wedge \mathbf{v}_1\right) =$$
$$-\nabla\left[-\rho_1 g\frac{(z_2 - z_1)^2}{2} + (z_2 - z_1)P_1\right] - \nabla z_1 P_1 + \nabla z_2 P_2 \tag{2.1}$$

After substitution of the hydrostatic relation

$$P_1 = P_2 + \rho_1 g(z_2 - z_1), \tag{2.2}$$

the right-hand side of (2.1) becomes, recalling that P_2 is constant:

$$\rho_1 g(z_2-z_1)\nabla(z_2-z_1)-P_2\nabla(z_2-z_1)-2\rho_1 g(z_2-z_1)\nabla(z_2-z_1)+P_2\nabla(z_2-z_1)-\rho_1 g(z_2-z_1)\nabla z_1 = \rho_1 g(z_2-z_1)\nabla z_2$$

Recalling that $z_2 - z_1 = h_1$, and $z_1 - z_0 = h_2$, where $h_{1,2}$ are respective thicknesses of the layers, we thus get

$$\frac{d\mathbf{v}_1}{dt} + f\hat{\boldsymbol{z}} \wedge \mathbf{v}_1 = -g\nabla(h_1 + h_2) \tag{2.3}$$

The equations for $h_{1,2}$ follow straightforwardly from the mass conservation in the layers, which becomes volume conservation at constant densities.

2.1.2 Lower layer

Master equation:

$$\rho_2(z_1 - z_0)\left(\frac{d\mathbf{v}_2}{dt} + f\hat{\boldsymbol{z}}\wedge\mathbf{v}_2\right) =$$
$$-\nabla\left[-\rho_2 g\frac{(z_1-z_0)^2}{2} + (z_1 - z_0)P_0\right] - \nabla z_0 P_0 + \nabla z_1 P_1 \qquad (2.4)$$

After substitution of the hydrostatic relation $P_0 = P_2 + \rho_2 g(z_1 - z_0)$ the right-hand side of (2.1) becomes, recalling that z_0 is constant:

$$\rho_2 g(z_1-z_0)\nabla(z_1-z_0) - P_1\nabla z_1 + P_1\nabla z_1 - (z_1-z_0)\nabla P_1 = -\rho_2 g(z_1-z_0)\nabla(z_1-z_0) - \rho_1 g(z_1-z_0)\nabla(z_2-z_1),$$

where we used (2.2) and condition $P_2 = \text{const}$. Recalling that $z_2 - z_1 = h_1$, and $z_1 - z_0 = h_2$, where $h_{1,2}$ are respective thicknesses of the layers, we thus get

$$\frac{d\mathbf{v}_2}{dt} + f\hat{\boldsymbol{z}}\wedge\mathbf{v}_2 = -g\nabla\left(h_2 + \frac{\rho_1}{\rho_2}h_1\right) \qquad (2.5)$$

2.2 Problem 2

2.2.1 Master equation and boundary conditions

Before solving this problem, the analogue of master equations should be derived for the primitive equations in pseudo-height coordinates for the atmosphere. The derivation goes along the same lines as in the oceanic case, but the incompressibility condition $\nabla\cdot\mathbf{v} + \partial_z w = 0$, where $\mathbf{v}$ is the horizontal velocity, should be averaged over the layer (z_2, z_1), giving, for the averaged horizontal velocity, without change of notation:

$$\partial_t(z_2 - z_1) + \nabla\cdot((z_2 - z_1)\mathbf{v}) = 0 \qquad (2.6)$$

Averaging of the momentum equation over the layer gives:

$$(z_2 - z_1)\left(\frac{d\mathbf{v}_1}{dt} + f\hat{\boldsymbol{z}}\wedge\mathbf{v}_1\right) =$$
$$-\nabla\left[-g\frac{\theta}{\theta_0}\frac{(z_2-z_1)^2}{2} + (z_2 - z_1)\,\phi|_{z_1}\right] - \nabla z_1 P_1 + \nabla z_2\,\phi|_{z_2} - \phi|_{z_1}\nabla z_1, \qquad (2.7)$$

where ϕ is geopotential, $\theta = \text{const}$ is the mean potential temperature in the layer, and θ_0 is a reference value.

As in the Problem 1 above, the master equation (2.7) is applied to the lower (1) and upper (2) layers. The difference, however, is that boundary conditions are not the same. The upper boundary z_2 is considered to be flat, while the lower boundary z_0 is free, but the geopotential at $z = z_0$ is constant.

2.2.2 Lower layer

The hydrostatic equation for the lower layer (layer 1) is

$$\phi_1 = \phi_0 + g\frac{\theta_1}{\theta_0}(z_1 - z_0), \tag{2.8}$$

where the subscript in geopotential indicates the corresponding vertical position, as for the pressure in Problem 1. Using this formula in the master equation applied to the lower layer we get

$$\frac{d\mathbf{v}_1}{dt} + f\hat{\boldsymbol{z}} \wedge \mathbf{v}_1 = +g\frac{\theta_1}{\theta_0}\nabla z_0 = -g\frac{\theta_1}{\theta_0}\nabla(h_1 + h_2), \tag{2.9}$$

where we used that $z_2 = h_1 + h_2 + z_0 = \text{const}$.

2.2.3 Upper layer

The hydrostatic equation for the upper layer (layer 2) is

$$\phi_2 = \phi_1 + g\frac{\theta_2}{\theta_0}(z_2 - z_1). \tag{2.10}$$

Applying (2.7) we get:

$$\rho_1(z_2 - z_1)\left(\frac{d\mathbf{v}_2}{dt} + f\hat{\boldsymbol{z}} \wedge \mathbf{v}_2\right) =$$
$$-\nabla\left[-g\frac{\theta_2}{\theta_0}\frac{(z_2 - z_1)^2}{2} + (z_2 - z_1)\phi_1\right] - \nabla z_1\phi_1 + \nabla z_2\phi_2, \tag{2.11}$$

and using the boundary conditions and (2.8) we obtain

$$\frac{d\mathbf{v}_2}{dt} + f\hat{\boldsymbol{z}} \wedge \mathbf{v}_1 = -g\frac{\theta_2}{\theta_0}\nabla h_2 - g\frac{\theta_1}{\theta_0}\nabla h_1. \tag{2.12}$$

We can either consider now that the mean potential temperature in the lower layer has the reference value: $\theta_1 = \theta_0$, or introduce the reduced gravity $g' = g\frac{\theta_1}{\theta_0}$. In both cases the stratification parameter $\alpha = \frac{\theta_2}{\theta_1} > 1$ appears in the equation for the upper layer.

The continuity equations for thicknesses $h_{1,2}$ of both layers follow directly from mass conservation.

2.3 Problem 3

As there is no dependence on the vertical coordinate in the RSW equations, we only need the polar coordinates $0 \leq r < \infty$, $0 \leq \theta < 2\pi$ in the plane.The corresponding orthogonal unit vectors are $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$. Similarly to Problem 5 in Chapter 1 of this manual, the infinitesimal change of the radius vector $\mathbf{r} = r\hat{\boldsymbol{r}}$ is

$$d\boldsymbol{r} = dr\hat{\boldsymbol{r}} + rd\theta\hat{\boldsymbol{\theta}}, \tag{2.13}$$

and velocity is

$$\boldsymbol{v} = \frac{dr}{dt}\hat{\boldsymbol{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}, \tag{2.14}$$

The expressions in front of the unit vectors are respective components of the velocity

$$v_r = \frac{dr}{dt},\ v_\theta = r\frac{d\theta}{dt}. \tag{2.15}$$

While calculating the derivative of the velocity, that is acceleration, a change in orientation of the basis with the change of position should be taken into account. It is easy to see that it is only the change of polar angle θ that changes orientation, and

$$d\hat{\boldsymbol{r}} = d\theta\hat{\boldsymbol{\theta}}, \quad d\hat{\boldsymbol{\theta}} = -d\theta\hat{\boldsymbol{r}} \tag{2.16}$$

Hence

$$\begin{aligned}\frac{d\boldsymbol{v}}{dt} &= \frac{dv_r}{dt} - \left(\frac{v_\theta^2}{r}\right)\hat{\boldsymbol{r}} \\ &= \left(\frac{dv_\theta}{dt} + \frac{v_r v_\theta}{r}\right)\hat{\boldsymbol{\theta}}.\end{aligned} \tag{2.17}$$

The operator nabla in polar coordinates takes the form:

$$\boldsymbol{\nabla} = \hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{\partial}{r\,\partial\theta}, \tag{2.18}$$

which is to be used in the gradient and divergence operators. The divergence of velocity thus becomes

$$\nabla \cdot \mathbf{v} = \frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta}. \tag{2.19}$$

Finally the components of the Coriolis acceleration in the cylindrical basis $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{z}}$ are

$$2\boldsymbol{\Omega} \wedge \boldsymbol{v} = \begin{vmatrix} \hat{\boldsymbol{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{z}} \\ 0 & 0 & 2\Omega \\ v_r & v_\theta & 0 \end{vmatrix}. \tag{2.20}$$

Injecting all these ingredients in the RSW equations gives these equations in polar coordinates, which were used in applications to vortex stability in the book.

2.4 Problem 4

The demonstration is provided by a straightforward calculation of the time-derivative of the energy density using the expressions for time-derivative of height and velocity following from the equations of motion:

$$\partial_t e = \partial_t h\left(\frac{v_k v_k}{2} + gh\right) + hv_k\partial_t v_k = -\partial_i(hv_i)\left(\frac{v_k v_k}{2} + gh\right) + (hv_i)\partial_i\left(\frac{v_k v_k}{2} + gh\right), \tag{2.21}$$

where we used tensor notation $\partial_k = \frac{\partial}{\partial x_k}$, $x_1 = x$, $x_2 = y$, the corresponding expression for advective derivative $\partial_t + v_k\partial_k$, and Einstein's rule of summation over repeting indices. It is easy to see that the left-hand side of (2.21) is a divergence of $h\mathbf{v}\left(\frac{\mathbf{v}^2}{2} + gh\right)$.

2.5 Problem 5

The vorticity equation is derived by cross-differentiation of the the equations for u and v, respectively by y and x, which eliminates h, and rearranging the remaining terms.

2.6 Problem 6

As the integration is over the fixed domain of the flow, the time-derivative of $\mathcal{C}_\mathcal{F}$ is equal to the integral of the time-derivative of the integrand, which is

$$\partial_t \left(h\mathcal{F}(q)\right) = \partial_t h\mathcal{F}(q) + h\mathcal{F}'(q)\partial_t q = -\nabla\cdot(h\mathbf{v})\mathcal{F}(q) - h\mathbf{v}\cdot\nabla q\mathcal{F}'(q) = -\nabla\cdot\left(h\mathbf{v}\mathcal{F}(q)\right). \tag{2.22}$$

Using the Gauss theorem the integral over the domain of the divergence of the vector field $h\mathbf{v}\mathcal{F}$ is transformed into the total flux through the boundary, which is zero for isolated system.

2.7 Problems 7 and 8

These problems parallel the corresponding problems for the primitive equations, and the methods of solution are the same, as well as the polarisation relations, with the replacement $\phi \to gh$.

2.8 Problem 9

The equations are straightforwardly obtained by taking weighted sums and differences of the corresponding equations in (3.43).

2.9 Problem 10

The invariance of the action with respect to time-shifts implies that the expression (3.57) vanishes. This expression is the well-known form of energy conservation in field theory. The first term represents the time invariance of energy, as the time-derivative, with a fixed boundary, can be taken out of the integral, and if no-flux boundary conditions are imposed, the other term vanish, afer applying Gauss theorem

$$\frac{d}{dt}\int dxdy\left(\frac{\delta\mathcal{L}}{\delta\dot{X}}\dot{X} + \frac{\delta\mathcal{L}}{\delta\dot{Y}}\dot{Y} - \mathcal{L}\right) = 0. \tag{2.23}$$

The Lagrangian energy thus is

$$E = \int dxdy\left(\frac{\delta\mathcal{L}}{\delta\dot{X}}\dot{X} + \frac{\delta\mathcal{L}}{\delta\dot{Y}}\dot{Y} - \mathcal{L}\right) = \int dxdy\left(\frac{\dot{X}^2 + \dot{Y}^2}{2} + g\frac{h(X,Y)}{2}\right). \tag{2.24}$$

In Eulerian description, the coordinates are X, Y themselves, and the corresponding change of integration variables should be made in (2.25): $(x, y) \to (X, Y)$ with the

Jacobian which is $\frac{h}{H_0}$, as the Lagrangian Labels can be supposed to correspond to the flat surface $h = H_0$, without loss of generality. Thus

$$E = \frac{1}{H_0} \int dX dY \, h(X,Y) \left(\frac{\dot{X}^2 + \dot{Y}^2}{2} + g \frac{h(X,Y)}{2} \right), \tag{2.25}$$

and we recognise in the integrand in new variables the Eulerian energy density of (3.27), (3.28). The other terms in (3.57) can be transformed similarly and, together with replacing the Lagrangian time derivative by its Eulerian expression, give (3.28).

2.10 Problem 11

The Lagrangian (3.52) is obviously invariant with respect to translations $X \to X + \Delta_X$, $Y \to Y + \Delta_Y$ with constant $\Delta_{X,Y}$, as it depends only on derivatives of X and Y. Let us consider, for simplicity, the one-dimensional RSW and translations in x-direction: $X \to X + \Delta$. $X' = X_x$ will denote the x-derivative of X. The demonstration in the two-dimensional case is the same, with more cumbersome formulae. The formal calculation of the variation of the action gives (cf. the derivation of the equations of motion in the main text):

$$\delta \mathcal{S} = \rho \int dt \int dx \, \delta \mathcal{L} =$$

$$\rho \int dt \int dx \left[\left(\frac{\delta \mathcal{L}}{\delta X} - \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{X}} - \frac{\partial}{\partial x} \frac{\delta \mathcal{L}}{\delta X'} \right) \Delta + \frac{d}{dt} \left(\frac{\delta \mathcal{L}}{\delta \dot{X}} \Delta \right) + \frac{\partial}{\partial x} \left(\frac{\delta \mathcal{L}}{\delta X'} \Delta \right) \right] \tag{2.26}$$

and the expression in the right-hand side should vanish. The first bracket has equations of motion inside, and vanishes. hence the invariance of action at any Δ implies

$$\frac{d}{dt} \left(\frac{\delta \mathcal{L}}{\delta \dot{X}} \right) + \frac{\partial}{\partial x} \left(\frac{\delta \mathcal{L}}{\delta X'} \right) = 0, \tag{2.27}$$

which gives

$$\ddot{X} + g \frac{\partial h}{\partial X} = 0, \tag{2.28}$$

following the same manipulation with the space derivative of h as those used in (3.54), (3.55).

3
Solutions to Problems of Chapter 4

3.1 Problem 1

The demonstration follows the same lines as in Problem 8 to Chapter 3: the linearised potential vorticity has the same form in the presence of the boundary, and non-zero wave-frequency implies vanishing of the anomaly of PV. A straightforward calculation with the polarisation relation for Kelvin waves gives the same.

3.2 Problem 2

See Chapter 12.2. These are baroclinic and barotropic Kelvin waves, and to obtain corresponding solutions in the two-layer model, it is sufficient to apply the procedure used in the one-layer model to (3.45) and (3.46).

3.3 Problem 3

We take $f = 1$ to simplify the formulae. Solution for the Kelvin wave is obtained at $n = 0$ and $l < 0$. From (4.30) we get

$$p + \frac{1}{2} - \sqrt{l^2 - \frac{l}{\omega} + \frac{1}{4}}, \tag{3.1}$$

where p is given by (4.28). Exponentiating (3.1) twice, we get

$$(\omega^2 - 1)(\omega^4 - l^2 - 2l) = 0. \tag{3.2}$$

The roots $\omega^2 = 1$ should be discarded, as in order to eliminate $\bar{u}_0$ and $\bar{v}_0$ in favour of $\bar{\eta}_0$ in (4.15) we have to suppose $\omega^2 \neq 1$. The positive root at negative l gives the Kelvin-wave branch of Fig. 4.3. The solution for $\bar{\eta}_0(x)$ readily follows from (4.32) and consists of a single exponent. The corresponding polarisation relations are straightforwardly obtained from (4.15). An important difference with the abrupts shelf is that in spite that the other characteristics of the wave are similar, the across-coast component of velocity is not zero. This can be checked directly in (4.15): if a solution is sought with $\bar{u}_0 \equiv 0$ a contradiction arises, as the first two equations lead to a solution for $\bar{\eta}_0(x)$ in a form of an exponential, the last equation does not.

3.4 Problem 4

To find the wave solutions over escarpment in form of the step function, which we will orient along the x-axis in the f-plane and place at $y = 0$, without loss of generality, we linearise the RSW equations at each side of the escarpment, and eliminate all variables in favour of the deviation of the free surface η. The only difference between the equations in the lower and upper half-planes is in the mean depth: $h = H_{\pm}$ respectively. By looking for solutions in the form $(u_{\pm}, v_{\pm}, \eta_{\pm}) = (\bar{u}_{\pm}(y), \bar{v}_{\pm}(y), \bar{\eta}_{\pm}(y))e^{i(kx-\omega t)}$ we thus get

$$gH_{\pm}\bar{\eta}''_{\pm} + (\omega^2 - f^2 - gH_{\pm}k^2)\bar{\eta}''_{\pm} = 0, \tag{3.3}$$

with

$$(\omega^2 - f^2)\bar{v}_{\pm} + i\omega g\bar{\eta}'_{\pm} + ikfg\bar{\eta}_{\pm} = 0, \tag{3.4}$$

where prime denotes y- differentiation. For decaying off the escarpment solutions $\bar{\eta}_{\pm} = A_{\pm}e^{\mp a_{\pm}}$, $a_{\pm} > 0$ we get

$$\omega^2 - f^2 - gH_{\pm}k^2 + gH_{\pm}a_{\pm}^2 = 0. \tag{3.5}$$

It remains to match the solutions in the lower and upper half-planes using boundary conditions at the escarpment, that is at $y = 0$. They are 1) continuity of pressure, $\eta_+ = \eta_-$, and 2) continuity of the transverse mass flux $H_+v_+ = H_-v_-$. The first immediately gives $A_+ = A_-$, and the second, with the help of (3.4), gives

$$\omega(a_+H_+ + a_-H_-) = kf(H_+ - H_-). \tag{3.6}$$

with $a_{\pm} > 0$ satisfying (3.5). This is the dispersion relation in question. The phase velocity $c = \frac{\omega}{k}$ is dermined by the orientation of the step, such as the waves always run leaving shallow water on their right, as coastal Kelvin waves. The dispersion relation simplifies in the case of long waves $k \to 0$, and becomes

$$\omega = f\frac{k}{|k|}\frac{H_+ - H_-}{H_+ + H_-} \tag{3.7}$$

3.5 Problem 5

Erratum: The second inequality in the upper line of (4.48) should be opposite. The one-

dimensional Green's functions (4.46), (4.47) are calculated by the method of residues in both cases. We recall that the method is based on the Cauchy formula for any function $f(z)$ of the complex variable, which is analytic inside the contour C on the complex plane, and a point a inside the contour:

$$\oint_C dz\,\frac{f(z)}{z-a} = 2\pi i f(a) \tag{3.8}$$

If a lies outside the contour, the integral is zero.

The integrand in (4.45) is represented as $e^{ik(x-x')}\frac{1}{2}\sqrt{\frac{1}{\mathcal{U}}}\left(\frac{1}{k-\sqrt{\frac{1}{\mathcal{U}}}} - \frac{1}{k+\sqrt{\frac{1}{\mathcal{U}}}}\right)$, and the contour is chosen to consist of the real axis and demi-circle of radius tending to infinity either in upper, or in lower half-plane, depending on the sign of $x - x'$, to render the integrand in the Cauchy formula non-singular inside the contour. The application of (3.8) gives the result (4.46) directly for negative $\mathcal{U}$, as the poles of the integrand lie on the imaginary axis of the complex plane. For positive $\mathcal{U}$ the poles of the integrand lie on the real axis, and additional hypotheses are needed to prescribe how to circumvent them. It is clear that the result of the integration, whatever the rule gives either zero or oscillating in $x-x'$, that is non-decaying, function. As usual in such cases the boundary conditions of causality should be imposed. Causality means that there is no reaction before action, and that the response of the flow to the mountain is downstream, not upstream. This is achieved by chifting the integration contour along the real axis downwards, i.e. adding $-i\epsilon$, $\epsilon \to 0$ in the denominator of the integrand, which leads to (4.47).

The two-dimensional Green's function (4.48) is obtained by Fourier-transforming the equation, renormalising one of spatial variables with $\sqrt{F^2-1}$ which leads to the integral of the form

$$\int dkdl\, \frac{e^{i(kx+ly)}}{k^2 - l^2 - \text{const}}$$

The integration over k goes as in the one-dimensional example above, and condition $x > 0$ arises as in (4.47). The remaining integration in l is not elementary and the integral can be found in tables of integrals.

3.6 Problem 6

Ordinary differential equation in (4.56) is inhomogeneous, with constant coefficients, if PV is constant, $Q(y) = Q_0$. According to standard rules, its general solution is a general solution of the corresponding homogeneous equation $H = Ae^{\sqrt{Q_0}y} + Be^{-\sqrt{Q_0}y}$ plus a particular solution of the inhomogeneous equation $H = \frac{1}{Q_0}$. The constants A, B are determined from the boundary conditions.

3.7 Problem 7

Although the problem (4.64) is linear, the presence of y in the coefficients leads to technical difficulties while consecutively eliminating variables to obtain a single equation. The safe way of doing it is 1) to express u_x in terms of other variables, and differentiate the two remaining equations in x, in order to use the expression for u_x. The x-momentum equation gives $(\partial^2_{xx} - \partial^2_{tt})\eta$ as a function of various derivatives of v and y. Acting by the operator $\partial^2_{xx} - \partial^2_{tt}$ upon the remaining equations gives the result, up to an extra differentiation by x in the left-hand side, which can be removed without loss of generality for wave solutions.

3.8 Problem 8

Solution is obtained by straightforward calculations. For Kelvin waves it is given in the text. For Yanai waves $u = h$ by construction, and the dispersion relation follows from (4.45). For other waves, once a solution for v is found in terms of Gauss-Hermite functions, the zonal momentum and continuity equations, after Fourier-transformation in x and t give a pair of algebraic equations for u and η. For example, for Rossby waves we get:

$$(u, v, \eta) = (U(y), V(y), H(y))\, e^{i(kx-\omega t)}, \tag{3.9}$$

with

$$
\begin{aligned}
V(y) &= \phi_n(y),\ U(y) = -i\frac{k\phi_n'(y) - \omega_n y\phi_n(y)}{\omega_n^2 - k^2}, \\
H(y) &= i\frac{\omega_n\phi_n'(y) - ky\phi_n(y)}{\omega_n^2 - k^2}.
\end{aligned}
\tag{3.10}
$$

3.9 Problem 9

Erratum: nabla is missing in the last term in the first line of eq. (4.90), this term should read $-q\mathbf{v}\cdot\nabla\mathbf{v}$

The derivation is a direct computations, taking first the weighted sum and the difference of the momentum equations in the layers, in order to obtain the momentum equations for barotropic and baroclinic components, respectively, cross-differentiating the equations for barotropic u and barotropic v to get an equation for ψ, and expressing velocity components in nonlinear terms through the inverse transformation (4.86), and (4.88) - see also solution of Problem 7 to Chapter 5.

3.10 Problem 10

We first rewrite the Eulerian RSW equations in cylindrical coordinates (r, θ) and assume exact axial symmetry:

$$
\begin{aligned}
(\partial_t + u_r\,\partial_r)u_r - u_\theta\left(f + \frac{u_\theta}{r}\right) + g\partial_r h &= 0\,, \\
(\partial_t + u_r\,\partial_r)u_\theta + u_r\left(f + \frac{u_\theta}{r}\right) &= 0\,, \\
\partial_t h + \frac{1}{r}\partial_r(r\,u_r\,h) &= 0\,.
\end{aligned}
\tag{3.11}
$$

Here u_r, u_θ are the radial and azimuthal components of velocity. Note that the adjusted stationary state changes character as compared to the rectilinear case: it verifies conditions of the ***cyclo-geostrophic balance***, and not of the purely geostrophic one:

$$
u_\theta\left(f + \frac{u_\theta}{r}\right) = \partial_r h,\ u_r = 0. \tag{3.12}
$$

Multiplying the second equation in (3.11) by r, we recover the conservation of angular momentum:

$$
(\partial_t + u_r\,\partial_r)\left(ru_\theta + f\frac{r^2}{2}\right) = 0\,, \tag{3.13}
$$

which replaces the conservation of geostrophic momentum in the plane-parallel case. Equations (3.11) can be rewritten using the Lagrangian coordinate $R(r, t)$. Integrating (3.13) gives:

$$
R(r,t)\,u_\theta(r,t) + f\frac{R^2(r,t)}{2} = r\,u_{\theta I}(r) + f\frac{r^2}{2} \equiv G(r)\,, \tag{3.14}
$$

where $u_{\theta I}$ is the initial azimuthal velocity profile. Using the above expression we get:

$$u_\theta \left(f + \frac{u_\theta}{R}\right) = \frac{1}{R}\left(G - f\frac{R^2}{2}\right)\left(f + \frac{G}{R^2} - \frac{f}{2}\right)$$
$$= \frac{1}{R^3}\left(G^2 - \frac{f^2 R^4}{4}\right) . \quad (3.15)$$

The mass conservation is expressed by the following relation:

$$h(r,t)\, R(r,t)\, dR \; = \; h_I(r)\, r\, dr \, . \quad (3.16)$$

With the help of (3.15), (3.16), and the definition $\dot{R}(r,t) = u_r(r,t)$ the radial momentum equation becomes:

$$\ddot{R} + \frac{f^2}{4} R - \frac{1}{R^3} G^2 + \frac{1}{\partial_r R}\partial_r \left(\frac{r\, h_I}{R\, \partial_r R}\right) = 0 \, . \quad (3.17)$$

to be solved with initial conditions $R(r,0) = r, \; \dot{R}(r,0) = u_{r_I}$.

A direct check shows that

$$h_I(r) = \frac{h_0}{2}\left(1 - \frac{r^2}{L^2}\right), \quad R(r,t) = r, \quad u_{\theta I}(r) = r\Omega, \quad (3.18)$$

where Ω is determined by cyclo-geostrophic balance

$$\Omega(\Omega + f) r - g\frac{h_0}{L^2} r = 0, \quad (3.19)$$

is an exact solution of (3.17). We could linearise the (3.17) about this solution to find small oscillations, but it is possible to do better and find nonlinear oscillation which give an exact solution (1). Indeed, if we make the same ansatz with *arbitrary* constant Ω, and time-dependent radial position

$$h_I(r) = \frac{h_0}{2}\left(1 - \frac{r^2}{L^2}\right), \quad R(r,t) = r\phi(t), \quad u_{\theta I}(r) = r\Omega, \; \Omega = \text{const}, \quad (3.20)$$

non-dimensionalise the system in the obvious way, with L as the length scale, f^{-1} as the time scale, introduce the Burger number γ, and denote $M = \frac{1}{2} + \frac{\Omega}{f}$ we get:

$$\ddot{\phi} + \frac{\phi}{4} - \frac{M^2}{\phi^3} - \frac{\gamma}{\phi^3} = 0, \quad (3.21)$$

to be solved with initial conditions $\phi(0) = 1$, $\dot{\phi}(0) = u_{r_I}$. A drastic simplification of this equation is provided by the substitution $\phi^2 = \chi$ which immediately gives the equation of the harmonic oscillator with shifted equilibrium position:

$$\ddot{\chi} + \chi - 4E = 0, \quad E = \frac{u_{r_I}^2}{2} + \frac{1}{8} + \frac{M^2 + \gamma}{2} > 0. \quad (3.22)$$

A "pulson" solution satisfying the initial conditions $\chi(0) = 1$, $\dot{\chi}(0) = 2u_{r_I}$ is given by

$$\chi(t) = 4E + (1 - 4E)\cos t + (2u_{r_I} + 1 - 4E)\sin t. \quad (3.23)$$

and represents nonlinear inertial oscillations, as its frequency is always one (in non-dimensional terms). A small-amplitude version of this solution is obtained by straightforward linearisation.

The details of solution can be found in (Zeitlin 2010).

4

Solutions to Problems of Chapter 5

4.1 Problem 1

Introducing parameters, as in the main text, we write down the definition of PV:

$$q = \frac{v_x - u_y + f_0 + \beta y}{H_0(1 + \lambda\eta)}. \tag{4.1}$$

The QG scaling is: U for both components of velocity, L for both coordinates, and all nondimensional parameters small and of the same order: $\lambda \sim \beta L \sim \epsilon = \frac{U}{f_0 L}$. The PV thus becomes

$$q = \frac{f_0}{H_0}\frac{1 + \epsilon(v_x - u_y + y}{1 + \epsilon\eta}. \tag{4.2}$$

Expanding the denominator in Taylor series, and retaining the leading terms gives

$$q = \frac{f_0}{H_0}\left(1 + \epsilon(v_x - u_y + y - \eta)\right), \tag{4.3}$$

where the leading-order in ϵ expressions $u = -\eta_y$, $v = \eta_x$ should be used.

With the QG scaling for time: $T = L/U$, the leading-order in ϵ term in the advection operator is $\frac{d}{dt} = \partial_t + \eta_x\partial_y - \eta_y\partial_x$, and applying it to (4.3) we recover the QG equation.

4.2 Problem 2

We take as dispersion relation $\omega = -\beta\frac{k}{k^2+l^2}$, where the wave vector is $\mathbf{k} = (k,\, l)$. The presence of an additional constant in the denominator of the dispersion relation does not change the analysis of phase and group velocities. The expression for the phase velocity is

$$\mathbf{c} = \frac{\omega\mathbf{k}}{\mathbf{k}^2} = \left(-\beta\frac{k^2}{(k^2+l^2)^2},\; 2\beta\frac{kl}{(k^2+l^2)^2}\right),\quad \mathbf{k} = (k,\, l). \tag{4.4}$$

Its remarkable property is that the zonal phase velocity is negative-definite, i.e. the waves propagate always to the West.

The expression for the group velocity is

$$\mathbf{c}_g = (\partial_k\omega, \partial_l\omega) = \left(-\beta\frac{k^2 - l^2}{(k^2+l^2)^2}, -\beta\frac{kl}{(k^2+l^2)^2}\right). \tag{4.5}$$

The zonal group velocity is not sign-definite and, at a given l, is positive for short waves (large enough k), and negative for long waves (small enough k). So although waves are always propagating westward, the modulations (as well as energy, momentum, etc) can propagate both ways, depending on the wavelength.

4.3 Problem 3

Following the same lines as in the problem 1 above, we get for the PV

$$q = \frac{f_0}{H_0}\frac{1+\epsilon(v_x - u_y + y}{h}, \tag{4.6}$$

as in the frontal regime it does not make sense to split h into the mean and a perturbation. Hence, unlike the QG regime the zero-order PV is not constant. However, it is not advected by the geostrophic velocity, as $\mathcal{J}(h, \frac{1}{h}) \equiv 0$. So the first-slow-time derivative of h is zero, and we need to introduce a slower time $\tau = \epsilon t$. By the same reason we have to keep the first ageostrophic correction to advection, which is given by the equation (5.14) in the main text, where the time-derivative ∂_t should be omitted by the just explained reason. Retaining the leading -order terms in the resulting expression for the Lagrangian conservation of PV:

$$\left(u^{(0)}\partial_x + v^{(0)}\partial_y + \epsilon(\partial_\tau + u^{(1)}\partial_x + v^{(1)}\partial_y)\right)\frac{1+\epsilon(\partial_x v^{(0)} - \partial_y u^{(0)} + y)}{h} = 0, \tag{4.7}$$

with

$$v^{(0)} = h_x,\, u^{(0)} = -h_y, \quad v^{(1)} = -\mathcal{J}(h, h_y) - yh_x,\, u^{(1)} = -\mathcal{J}(h, h_x) + yh_y, \tag{4.8}$$

and calculating derivatives leads to the FG equation, up to multiplication by $\frac{1}{h^2}$.

4.4 Problem 4

It is easy to see that the linearisation over the state of rest with constant thickness $h = H + \eta$ leads to a simple-wave equation resulting from the first and the last terms in (5.38):

$$\eta_t - H\eta_x = 0. \tag{4.9}$$

It describes non-dispersive waves propagating westward. This is a long-wave limit of the Rossby-wave equation, which is consistent with the nature of the FG regime applicable to the scales which are much larger than the deformation radius.

4.5 Problem 5

Erratum: in equation (5.49) a plus/minus sign between the two brackets inside the square brackets is missing in the first print.

The polarisation relation is given by any of the two equations in (5.47) after Fourier-transformation in space and time. After substitution of the expressions for ω , it gives that $\pi_1 = \pi_2$ for $\omega = \omega_{bt}$. This means that geostrophic velocities are the same in the layers, which is the essence of the barotropic motion. For $\omega = \omega_{bc}$ the pressures and the geostrophic velocities are opposite.

4.6 Problem 6

The most unstable mode corresponds to the maximum of the growth rate, i.e. of the imaginary part of the eigenfrequency corresponding to (5.54). It is clear from (5.54) that the maximum of the growth rate is achieved at the minimal possible meridional wavenumber k_y. Although the boundary conditions were not discussed in Chapter 5.3.3 of the main text, it is clear that we need some, at least in y. As the interface between the layers has a constant inclination, the available potential energy of the system is infinite, if there are no lateral boundaries. A natural configuration is a zonal channel, cf. Chapter 10 of the main text, where the baroclinkc instability is treated in more detail. So the meridional component of velocity v should vanish at the boundary, which means, in turn, that the Fourier-transforms of $\pi_{1,2}$ in x and t vanish. This implies that zero-mode with constant in y-direction $\pi_{1,2}$ is to be discarded, and the meridional wavenumber k_y is discrete and non-zero. Let the minimal value of k_y be l_{min}. We give a solution of the problem in the case $U_1 = -U_2 = U/2$, and $F_1 = F_2 = F$ (layers of the same depth in the reference frame moving with the mean zonal velocity of the system). The expression for the growth rate then is

$$\sigma(k,l) = k\frac{U}{2}\frac{\sqrt{(k^2+l^2)^2-4F^2}}{k^2+l^2+F^2}, \tag{4.10}$$

where we denote k_x by k.

Differentiating this expression with respect to k gives

$$\frac{4F^4-(k^2+l^2)^3-F^2(3k^4+l^2(-4+l^2)+4k^2(1+l^2))}{(F^2+k^2+l^2)^2\sqrt{4F^2-(k^2+l^2)^2}}, \tag{4.11}$$

and zeroes of the numerator give extrema of the growth rate. As is easy to see, e.g. by plotting σ, the growth rate has a single maximum. The corresponding value of $k = k_0$ is given by the real root of the cubic equation, which can be obtained by using Cardano formulas, or numerically. For example, for $l_{min} = 1,\ F = 1$, the zone of unstable k is the interval[0, 1], with k monotonously increasing from zero to $k_0 \approx 0.65$, and then decreasing to zero, with $\sigma_{max} = 0.19U$ in non-dimensional terms.

4.7 Problem 7

By introducing baroclinic and barotropic components of velocity

$$\mathbf{v}_{bt} = \frac{1}{H}(h_1\mathbf{v}_1 + h_2\mathbf{v}_2), \quad \mathbf{v}_{bc} = \mathbf{v}_1 - \mathbf{v}_2, \tag{4.12}$$

where the subscripts 1(2) refer to upper (lower) layer and $H = \text{const}$ is the total thickness of the system, it is easy to see that $\mathbf{v}_{bt}$ is divergenceless, and its components can be expressed in terms of stream-function ψ, as usual. Taking the curl of the barotropic momentum equation on the f- plane gives

$$\partial_t\nabla^2\psi + \mathcal{J}(\psi,\nabla^2\psi) + (\partial^2_{xx}-\partial^2_{yy})\left(\frac{h(H-h)}{H^2}uv\right) - \partial^2_{xy}\left(\frac{h(H-h)}{H^2}(u^2-v^2)\right) = 0, \tag{4.13}$$

where $h \equiv h_1$, and we omitted the subscript bc at the components of the baroclinic velocity u, v, for which we have:

$$\partial_t u + \mathcal{J}(\psi, u) - \left(u\partial^2_{xy} + v\partial^2_{y^2}\right)\psi + \frac{H-2h}{H}\left(u\partial_x + v\partial_y\right)u - \frac{u}{H}\left(u\partial_x + v\partial_y\right)h - fv + g\partial_x h = 0, \tag{4.14}$$

$$\partial_t v + \mathcal{J}(\psi, v) - \left(u\partial^2_{x^2} + v\partial^2_{xy}\right)\psi + \frac{H-2h}{H}\left(u\partial_x + v\partial_y\right)v - \frac{v}{H}\left(u\partial_x + v\partial_y\right)h + fu + g\partial_y h = 0. \tag{4.15}$$

The mass conservation in the upper layer takes the form:

$$\partial_t h + \mathcal{J}(\psi, h) + \partial_x\left(\frac{h(H-h)}{H}u\right) + \partial_y\left(\frac{h(H-h)}{H}v\right) = 0. \tag{4.16}$$

Defining the baroclinic Rossby number ϵ, as usual, with the help of the typical baroclinic velocity and horizontal scale, choosing the aspect ratio and the barotropic Rossby number both to be of the order ϵ^2, and the time-scale to be of the order of ϵ^2, as well, we get from (4.14) and (4.15) the expressions for the u and v in terms of h and ψ, as an expansion in ϵ about the geostrophic wind. By injecting them into (4.13) and (4.16) we get a system of coupled equations for h and ψ, where h can be expressed in terms of deviation of the interface η as $h = 1 - \eta$.

A detailed derivation can be found in (Benilov & Reznik 1996).

4.8 Problem 8

Erratum: $h^{(0)}$ should be understood as its value at the coast, that is a vertical bar with subscript $x = 0$ should be inserted before the equality sign. Equation (5.68)

is obtained by straightforward substitution of expressions for zero-order fields, i.e. geostrophic balance equations, in the right-hand side of (5.67).

Equation (5.72), where $h^{(0)}$ should be understood as its value at the coast, is obtained similarly, with substitution of the zero-order expression for $v^{(0)}$, and $u^{(0)} = 0$ from (5.65) and elementary integrations, with integrations by parts when necessary.

4.9 Problem 9

Imposing a rigid lid in shallow water equations means neglecting free-surface elevation in the mass conservation equation, and replacing it by barotropic pressure in the momentum equations. The continuity equation becomes a time-independent constraint

$$(Hu)_x + (Hv)_y = 0, \tag{4.17}$$

and can be explicitly resolved with the help of stream-function ψ:

$$Hu = -\psi_y, \quad Hv = +\psi_x \tag{4.18}$$

As suggested in the Problem, we will work with one-dimensional topography $H = H(x)$ and get the following expression for the divergence:

$$H = H(x) \Rightarrow u_x + v_y = -\frac{H'(x)}{H(x)}u = D. \tag{4.19}$$

The relative orticity becomes

$$\zeta = v_x - u_y = \left(\frac{\psi_x}{H(x)}\right)_x + \frac{\psi_{yy}}{H(x)}, \tag{4.20}$$

and vorticity equation $\frac{d}{dt}(\zeta+f)+(\zeta+f)D = 0$ becomes an equation for ψ. Its inearised vertion on the f- plane is:

$$\zeta_t + fD = 0 \ \leftrightarrow \ \left[\left(\frac{\psi_x}{H(x)}\right)_x + \frac{\psi_{yy}}{H(x)}\right]_t + f\frac{H'(x)}{H^2(x)}\psi_y = 0. \tag{4.21}$$

Looking for wave solutions $\psi = \phi(x)e^{i(\omega t - ly)} + \text{c.c.}$, we get:

$$\left(\frac{\phi'}{H}\right)' - \frac{l^2}{H}\phi - \frac{f}{c}\frac{H'}{H^2}\phi = 0, \quad c = \frac{\omega}{l}. \tag{4.22}$$

which is a Sturm - Liouville problem for eigenfunctions ϕ_n and eigenvalues $c_n(l)$. It can be shown that for escarpment-like topography the spectrum is discrete and corresponds to trapped waves with a number of nodes $n = 0, 1, 2, ...$

4.10 Problem 10

Let us consider the Korteweg-deVries equation in a most general form:

$$\partial_t u + c_0 \partial_x u + \alpha u \partial_x u + \beta \partial^3_{xxx} u = 0. \tag{4.23}$$

We are looking for solutions propagating without change of form:

$$u = u(x - Vt) \ \Rightarrow$$

$$(c_0 - V)u' + \alpha u u' + \beta u''' = 0, \tag{4.24}$$

which are localised (solitary waves, solitons). With this hypothesis we integrate (4.24) once:

$$(c_0 - V)u + \alpha\frac{u^2}{2} + \beta u'' = 0, \tag{4.25}$$

then multiply by u' and integrate once more:

$$(c_0 - V)\frac{u^2}{2} + \alpha\frac{u^3}{6} + \beta\frac{u'^2}{2} = 0. \tag{4.26}$$

This equation can be interpreted as an equation of a material point moving in a potential, given by the first two terms in (4.26), with zero total energy, and integrated by separation of variables giving the solution $u(x - Vt) = \frac{3}{\alpha}\frac{V - c_0}{\cosh^2\sqrt{\frac{V-c_0}{4\beta}}(x - Vt)}$, where $V = c_0 + \frac{\alpha}{3}u_{max}$, and thus the speed of solution depends on amplitude, which is a purely nonlinear effect.

5

Solutions to Problems of Chapter 6

5.1 Problem 1

The demonstration is given by calculating the circulation $\int_C udx + vdy$ of the velocity field corresponding to the Green's function of the source situated at the origin: $\mathbf{x}' = 0$:

$$v = \frac{x}{2\pi(x^2 + y^2)}, \quad u = -\frac{y}{2\pi(x^2 + y^2)}$$

over a circle C, by passing to polar coordinates r, θ with the help of relations $dy = \cos\theta d\theta$, $dx = -\sin\theta d\theta$. The circulation thus is equal to one. By Stokes theorem, this is equal to the integral of corresponding vorticity over the area of the circle. As is easy to see, the vorticity is zero everywhere except the origin, and thus is given by the delta-function.

5.2 Problem 2

The demonstration is the same as in Problem 6 in Chapter 2

5.3 Problem 3

Erratum: An overall minus sign is missing in the expression of energy (6.20).

The conservation of energy follows directly from its conservation in the continuous system. The demonstration for two other supposed integrals of motion consists in taking the time derivative of each of them, in substituting the expression for time-derivatives (6.19), and using relabelling of indices and symmetry properties, e. g. $\sum_{ij} \kappa_i \kappa_j \mathbf{x}_{ij} = \sum_{ji} \kappa_j \kappa_i \mathbf{x}_{ji} = -\sum_{ij} \kappa_i \kappa_j \mathbf{x}_{ij}$, and hence $\sum_{ij} \kappa_i \kappa_j \mathbf{x}_{ij} = 0$.

5.4 Problem 4

The solution of this problem can be found in the main text: equations (6.84) and (6.23), and can be obtained by straightforward application of (6.19) to the two-vortex system.

5.5 Problem 5

The solution of this problem is geometric and does not necessitate any calculations: it is sufficient to plot the vectors of velocity field due to two other vortices at the location of a given vortex, that is at the vertex of the triangle, sum them up and show that the resulting vector is perpendicular to the radius-vector giving the position of the vortex.

5.6 Problem 6

The solution of the problem readily follows from the analysis of equations (6.23) and (6.24): if a wall is put at the axis, we get a solution of the Euler equations in a half-plane, as the boundary condition of zero normal velocity is identically satisfied. Thus, a vortex near the wall forms a dipole with its image, and moves along the wall.

5.7 Problem 7

Erratum: There is an overall minus sign missing in the first print in the expression of vorticity in (6.14).

First of all we notice that the Jacobian of the complex Lagrangian mapping:

$$Z = \alpha\xi e^{i\omega t} + \beta\xi^*, \quad Z^* = \alpha^*\xi^* e^{i\omega t} + \beta^*\xi, \tag{5.1}$$

is $|\alpha|^2 - |\beta|^2 \neq 1$, and this should be taken into account while calculating various physical quantities, like vorticity.

It is easy to see that (5.1) satisfies (6.13) identically, as in the complex form the equations of motion contain Jacobians of Z and $\ddot{Z}^*$, and its complex conjugate, and their respective contributions cancel each other, as $\ddot{Z}$ contains only ξ, and no ξ^*.

By changing variables $(X, Y) \to (Z, Z^*)$ in (6.14), and renormalising with the above Jacobian gives

$$\zeta = \frac{2\omega|\alpha|^2}{|\alpha|^2 - |\beta|^2} = \text{const} \tag{5.2}$$

The proposed solution of 2D Euler equatons, thus, has uniform vorticity inside the unit circle in the ξ - plane.

Taking the real and imaginary part of Z given by (5.1) at $\xi = e^{i\theta}$ gives a parametric equations of the ellipse, whose orientation is defined by t, at fixed ω;

Matching with the outer solution at the unit circle is straightforward as at the unit circle $\xi^* = \xi^{-1}$. Note that in order to ensure matching the complex velocity should be understood as $V = \dot{X} + i\dot{Y} = u + iv$ here. The proposed velocity field has zero vorticity out of the unit circle in the ξ- plane.

5.8 Problem 8

Erratum: The sign in the equation (5.36) in the main text was erroneous in the first print. The factor 2π in (5.35), (5.36) can be removed by renormalisation, to get the standard expressions for Green's functions.

We are looking for steady solutions $\psi = \psi(x - Ut)$, moving with constant zonal velocity U, of the barotropic beta-plane equation

$$\partial_t \nabla^2\psi + \mathcal{J}(\psi, \nabla^2\psi) + \beta\partial_x\psi = 0, \tag{5.3}$$

which can be, thus rewritten as

$$-U\partial_{x'}\left(\nabla^2\psi - \frac{\beta}{U}\psi\right) - \mathcal{J}\left(\psi, \nabla^2\psi - \frac{\beta}{U}\psi\right) = 0, \tag{5.4}$$

wherre $x' = x - Ut$.

Equation (5.4) can be interpreted as vorticity equation for steady moving configurations, with a new "vorticity" $\nabla^2\psi - \frac{\beta}{U}\psi$. For a given "vorticity" configuration, the streamfunction, and thus velocity, is given by the Green's function, obeying

$$\nabla^2 G(|\mathbf{x} - \mathbf{x}'|) - \frac{\beta}{U}\psi = \delta(|\mathbf{x} - \mathbf{x}'|) \tag{5.5}$$

Depending on the sign of U, this is either $-K_0(\frac{|\mathbf{x}-\mathbf{x}'|}{R})$ (negative U), or $\frac{1}{4}Y_0(\frac{|\mathbf{x}-\mathbf{x}'|}{R})$ (positive U). We here introduced the notation $R = \sqrt{\frac{|U|}{\beta}}$. In both cases, for the vortex dipole of intensity κ situated at $x = 0$, $y = \pm\frac{d}{2}$ we get

$$\psi(x, y) = \kappa \left[G(x, y - \frac{d}{2}) - G(x, y + \frac{d}{2}) \right]. \tag{5.6}$$

The dipole is a solution of (5.3) if the velocity of its displacement, induced by the velocity induced by a partner vortex at the location of a given vortex coincides with U, which gives the following relation:

$$U = \frac{\kappa}{R} G'\left(\frac{d}{R}\right) \to U^{\frac{3}{2}} = \kappa\sqrt{\beta} G'\left(d\sqrt{\frac{\beta}{U}}\right). \tag{5.7}$$

This equation can be solved, qualitatively, by looking for intersection of the curves representing the left- and the right-hand sides. Already this analysis shows a qualitative difference between the cases positive (eastward propagation) and negative (westward propagation) U: there are only two solutions in the first case, and an infinite series of solutions in the second. The structure of solutions, given by modified and ordinary Bessel functions, respectively, is totally different: exponentially decaying in the first case, and having an weakly decaying oscillating tail in the second. The solutions are analogs of the modon solution described in the main text, with an addition of the Rossby-wave tail in the case of westward propagation. Exact solutions can be found by solving (5.7)

This solution was found in (Gryanik 1986)

5.9 Problem 9

Conservation of energy and enstrophy within the triad means that for variations of energy of each component of the triad δE_i, $i = 1, 2, 3$ we get a system of two equations

$$\delta E_1 + \delta E_2 + \delta E_3 = 0, \quad k_1^2 \delta E_1 + k_2^2 \delta E_2 + k_3^2 \delta E_3 = 0. \tag{5.8}$$

If the variation of the median component δE_2 is given, (5.8) becomes an inhomogeneous system of two linear equations for δE_i, $i = 1, 2$. To make things transparent, let us take $a = 2$, $b = 3$. The solution of (5.8) then is $\delta E_1 = -\frac{5}{8}\delta E_2$, $\delta E_3 = -\frac{3}{8}\delta E_2$, that is, more energy goes to smaller k. At the same time, $k_1^2\delta E_1 = -\frac{5}{32}k_2^2\delta E_2$, $k_3^2\delta E_3 = -\frac{27}{32}k_2^2\delta E_2$, that is more enstrophy goes to higher k.

6

Solutions to Problems of Chapter 7

6.1 Problem 1

Erratum To be compatible with the definitions given in the main text, the inequality in the announcement should be inverted: $[h] > 0$

Let us consider the RH conditions for one-dimensional shallow water in the absolute reference frame, that is with $\mathcal{U} = 0$, denote the quantities behind (in front of) the shock as $u_{+(-)}$, $h_{+(-)}$, and suppose that the shock is moving from left to right, that is $u_{+(-)} > 0$ (this means that the fluid goes through the shock from right to left). The thicknesses $h_\pm$ are positive, by definition. The RH conditions become:

$$u_+ h_+ = u_- h_- \tag{6.1}$$

$$u_+^2 h_+ + \frac{g h_+^2}{2} = u_-^2 h_- + \frac{g h_-^2}{2} \tag{6.2}$$

$$u_+^3 h_+ + g h_+^2 u_+ \leq u_-^3 h_- + g h_-^2 u_- \tag{6.3}$$

Substituting $u_+ = \frac{h_-}{h_+} u_-$, as follows from (6.1), in (6.3), and dividing by $h_- u_-$, which is positive, as supposed, we get

$$g(h_+ - h_-) \leq \left(1 - \frac{h_-^2}{h_+^2} u_-^2\right) \tag{6.4}$$

Substituting $u_+ = \frac{h_-}{h_+} u_-$ in (6.2) gives the expression of u_-^2 in terms of $h_\pm$:

$$u_-^2 = g \frac{h_+ + h_-}{2} \frac{h_+}{h_-} \tag{6.5}$$

Substituting this expression in (6.4) gives quadratic inequality for h_+

$$g(h_+ - h_-) \leq g \frac{(h_+ + h_-)^2 (h_+ - h_-)}{2 h_+ h_-}, \tag{6.6}$$

which can be only satisfied for $h_+ > h_-$.

6.2 Problem 2

The demonstration follows by using definitions and straightforward algebra.

6.3 Problem 3

Adding viscous dissipation to the right-hand sides of the momentum equations in (7.1) in the main text, that is, respectively $\nu\nabla^2 u$ and $\nu\nabla^2 v$, and then differentiating the v - equation with respect to x, we get the vorticity equation in the presence of viscosity:

$$\frac{d}{dt}(v_x + f) + (v_x + f)\,u_x = \nu\nabla^2(v_x + f)\,. \tag{6.7}$$

combining this equation with the mass-conservation equation written in the form

$$\frac{d}{dt}h + hu_x = 0, \tag{6.8}$$

we get

$$\frac{d}{dt}\frac{v_x + f}{h} = \nu\frac{\nabla^2(v_x + f)}{h}. \tag{6.9}$$

6.4 Problem 4

The 2-layer 1d RSW model with a rigid lid read :

$$\begin{aligned}
\partial_t u_1 + u_1\partial_x u_1 + \rho_1^{-1}\partial_x\pi_1 &= 0\ ,\\
\partial_t u_2 + u_2\partial_x u_2 + \rho_2^{-1}\partial_x\pi_1 + g'\partial_x\eta &= 0\ ,\\
\partial_t(H_1 - \eta) + \partial_x((H_1 - \eta)u_1) &= 0\ ,\\
\partial_t(H_2 + \eta) + \partial_x((H_2 + \eta)u_2) &= 0\ ,
\end{aligned}$$

Their linearisation about a solution with constant pressure and constant velocities in the layers $u_{1,2} = U_{1,2}$ gives

$$\begin{aligned}
D_1 u_1 + \rho_1^{-1}\partial_x\pi_1 &= 0\ ,\\
D_2 u_2 + \rho_2^{-1}\partial_x\pi_1 + g'\partial_x\eta &= 0\ ,\\
-D_1\eta + H_1\partial_x u_1 &= 0\ ,\\
D_2\eta + H_2\partial_x u_2 &= 0\ ,
\end{aligned}$$

where $D_{1,2} = \partial_t + U_{1,2}\partial_x$. By eliminating subsequently $u_{1,2}$ and then π_1, with the help of extra differentiations, and recalling the definition of reduced gravity, we arrive to a single equation for η:

$$\frac{\rho_1}{H_1}D_1^2\eta + \frac{\rho_2}{H_2}D_2^2\eta - g\Delta\rho\partial_{xx}^2\eta = 0. \tag{6.12}$$

After Fourier - transformation $\eta = \bar{\eta}e^{i(\omega t - kx)}$ (6.12) gives a quadratic dispersion relation with the discriminant

$$D = g\Delta\rho\left(\frac{\rho_1}{H_1} + \frac{\rho_2}{H_2}\right) - (U_1 - U_2)^2\,\frac{\rho_1}{H_1}\frac{\rho_2}{H_2}, \tag{6.13}$$

which, if negative, corresponds to instability. This gives the required condition of KH instability.

In the presence of rotation, we can not work in the one-dimensional framework any more, as the background flow should be in geostrophic equilibrium, which necessitates across-flow gradients of pressure. The linearised about the flow with layer-wise constant component u of velocity, as above, equations on the f-plane read in this case:

$$\begin{aligned} D_1 u_1 - f v_1 + \rho_1^{-1}\partial_x \pi_1 =& 0 \ , \\ D_1 v_1 + f u_1 + \rho_1^{-1}\partial_y \pi_1 =& 0 D_2 u_2 - f v_2 + \rho_2^{-1}\partial_x \pi_1 + g'\partial_x \eta = \qquad 0 \ , \\ D_2 v_2 + f u_2 \rho_2^{-1}\partial_y \pi_1 + g'\partial_y \eta =& 0 - D_1 \eta + H_1(\partial_x u_1 + \partial_y v_1) = \qquad 0 \ , \\ D_2 \eta + H_2(\partial_x u_2 + \partial_y v_2) =& 0 \ . \end{aligned}$$

These equations can be processed by introducing relative vorticity $\zeta = \partial_x v_{1,2} - \partial_y u_{1,2}$, and divergence $\chi = \partial_y v_{1,2} + \partial_x u_{1,2}$, cross differentiating momentum equations to get the equations for vorticity $D_{1,2}\zeta_{1,2} + f\chi_{1,2} = 0$, allowing to express divergence in terms of vorticity, which allows, in turn, to express vorticity in each layer in terms of η from the mass conservation equations. Finally, as in the case without rotation, the Laplacian of the barotropic pressure, entering the remaining divergence equations layer-wise, can be eliminated, which leads to the equation for η, which is similar to (6.12), but contains a correction due to rotation:

$$\frac{\rho_1}{H_1} D_1^2 \eta + \frac{\rho_2}{H_2} D_2^2 \eta - f^2 \left(\frac{\rho_1}{H_1} + \frac{\rho_2}{H_2} \right) - g\Delta\rho \partial_{xx}^2 \eta = 0. \qquad (6.15)$$

The discriminant of the dispersion relation resulting after Fourier-transformation is

$$D = k^2 \left[g\Delta\rho \left(\frac{\rho_1}{H_1} + \frac{\rho_2}{H_2} \right) - (U_1 - U_2)^2 \frac{\rho_1}{H_1} \frac{\rho_2}{H_2} \right] + f^2 \left(\frac{\rho_1}{H_1} + \frac{\rho_2}{H_2} \right)^2 . \qquad (6.16)$$

As is easy to see, the new term increases the threshold for the instability, by an amount which is negligible at high k (short-wave limit), and is substantial for small k (long-wave limit). Hence, rotation tends to stabilise the flow, especially for long-wave perturbations.

6.5 Problem 5

We start with 2-layer 1.5d RSW model with a rigid lid:

$$\begin{aligned} \partial_t u_1 + u_1 \partial_x u_1 - f v_1 + \rho_1^{-1}\partial_x \pi_1 =& 0 \ , \\ \partial_t v_1 + u_1(f + \partial_x v_1) =& 0 \ , \\ \partial_t u_2 + u_2 \partial_x u_2 - f v_2 + \rho_2^{-1}\partial_x \pi_1 + g'\partial_x \eta =& 0 \ , \\ \partial_t v_2 + u_2(f + \partial_x v_2) =& 0 \ , \\ \partial_t (H_1 - \eta) + \partial_x((H_1 - \eta)u_1) =& 0 \ , \\ \partial_t (H_2 + \eta) + \partial_x((H_2 + \eta)u_2) =& 0 \ , \end{aligned}$$

where $(u_1,\ v_1)$, $(u_2,\ v_2)$ are velocities in the upper/lower layer; $\pi_2 = \pi_1 + g(\rho_1 h_1 + \rho_2 h_2)$, η - interface displacement, H_1 and H_2 are the heights of two layers at rest, g' is reduced gravity: $g' = g(\rho_2 - \rho_1)/\rho_2$. We will be using the full layers' heights $h_{1,2} = H_{1,2} \mp \eta$ as dynamical variables.

By eliminating the barotropic pressure and imposing condition of zero overall transverse momentum, we get 4 equations for 4 variables u_2, h_2, v_2, v_1:

$$\frac{du_2}{dt} - fv_2 + \frac{\rho_1}{\rho_2 h_1 + \rho_1 h_2}\Big(f(h_1 v_1 + h_2 v_2) - \partial_x\left(h_1 u_1^2 + h_2 u_2^2\right) + \frac{g\Delta\rho}{\rho_1} h_1 \partial_x h_2\Big) = 0,$$

$$\frac{dh_2}{dt} + h_2 \partial_x u_2 = 0, \quad \frac{dv_2}{dt} + fu_2 = 0,$$

$$\frac{dv_1}{dt} + (u_1 - u_2)\partial_x v_1 + fu_1 = 0,$$

$$u_1 = \frac{h_2 u_2}{h_2 - H}, \quad h_1 = H - h_2.$$

Here $\frac{d}{dt} = \partial_t + u_2 \partial_x$ is Lagrangian derivative, as usual.

Further simplifications are achieved by using the mass Lagrangian variable a: $h_2 = \partial a/\partial x$):

$$\frac{du_2}{dt} - fv_2 + \frac{\rho_1}{\rho_2 h_1 + \rho_1 h_2}\Big(f(h_1 v_1 + h_2 v_2) - h_2 \frac{\partial}{\partial a}\left(h_1 u_1^2 + h_2 u_2^2\right) + \frac{g\Delta\rho}{\rho_1} h_1 h_2 \partial_a h_2\Big) = 0,$$

$$\frac{dh_2}{dt} + h_2 \partial_a u_2 = 0,$$

$$\frac{dv_2}{dt} + fu_2 = 0,$$

$$\frac{dv_1}{dt} + (u_1 - u_2) h_2 \partial_a v_1 + fu_1 = 0.$$

These equations can be rewritten in matrix form:

$$\frac{d}{dt}\begin{pmatrix} v_2 \\ v_1 \\ h_2 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & T & 0 & 0 \\ 0 & 0 & 0 & h_2^2 \\ 0 & 0 & M & 2N \end{pmatrix} \frac{\partial}{\partial a}\begin{pmatrix} v_2 \\ v_1 \\ h_2 \\ u_2 \end{pmatrix} = \begin{pmatrix} -fu_2 \\ -fu_1 \\ 0 \\ fh_1 \frac{\rho_2 v_2 - \rho_1 v_1}{\rho_2 h_1 + \rho_1 h_2} \end{pmatrix},$$

where we introduce the notation:

$$T = (u_1 - u_2) h_2 = \frac{H u_2}{H - h_2} h_2,$$

$$M = \frac{\rho_1 h_2}{\rho_2 h_1 + \rho_1 h_2}\left(g\frac{\Delta\rho}{\rho_1}(H - h_2) - \frac{H^2 u_2^2}{(H - h_2)^2}\right),$$

$$N = -\frac{\rho_1 h_2}{\rho_2 h_1 + \rho_1 h_2}\frac{H h_2}{H - h_2} u_2,$$

and are, thus, quasi-linear. For nontrivial eigenvalues (the trivial ones are 0 and T), we get:

$$\det\begin{pmatrix} -\lambda & h_2^2 \\ M & 2N - \lambda \end{pmatrix} = \lambda^2 - 2N\lambda - h_2^2 M = 0 \tag{6.18}$$

with the solution

$$\lambda_\pm = N \pm \sqrt{N^2 + M h_2^2}. \tag{6.19}$$

The discriminant of this equation is

$$D = \frac{\rho_1 \rho_2 h_1 h_2^3}{(\rho_2 h_1 + \rho_1 h_2)^2}\left\{ g\Delta\rho \left(\frac{h_1}{\rho_1} + \frac{h_2}{\rho_2}\right) - \frac{H^2 u_2^2}{h_1^2}\right\}. \tag{6.20}$$

The eigenvalues (6.18) are real and, hence, the system is hyperbolic when D is positive. On the contrary, if

$$(u_2 - u_1)^2 = \frac{H^2 u_2^2}{h_1^2} > g\Delta\rho\left(\frac{h_1}{\rho_1} + \frac{h_2}{\rho_2}\right) \tag{6.21}$$

then $D < 0$ and the system loses hyperbolicity. The former condition may be rewritten as a condition on u_2

$$u_2^2 > g\Delta\rho\left(\frac{h_1}{\rho_1} + \frac{h_2}{\rho_2}\right)\frac{h_1^2}{H^2}.$$

We, thus, see that, unlike its one-layer counterpart, the 2-layer RSW changes type if the vertical shear of the transverse velocity is too strong. as $h_1 = H - h_2$, the condition of KH instability of the previous Problem is sufficent for hyperbolicity loss.

References

Zeitlin, V., 2010, "Lagrangian dynamics of fronts, vortices and waves: understanding the (semi-) geostrophic adjustment", *Lecture Notes in Physics*, 805, 109–137.

Benilov, E.S., and Reznik, G.M., 1996, "The complete classification of large-amplitude geostrophic flows in a two-layer fluid", *Geophys. Astrophys. Fluid Dyn.*, **82**, 1–22.

Gryanik, V. M. , 1986, "The singular geostrophic vortices on the beta-plane as a model for synoptic vortices, *Okeanologia*, 26, 174–179 (in Russian)