Primitive Equations on the tangent plane

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M1 ENS

Oceanic stratification

Typical density profile:
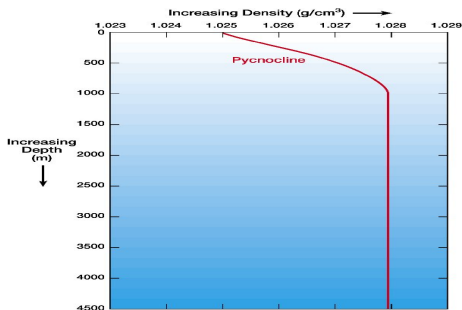
$$\rho(\vec{x}, t) = \rho_0 + \rho_s(z) + \sigma(x, y, z; t), \quad \rho_0 \gg \rho_s \gg \sigma. \quad (1)$$

Hydrostatic approximation for large-scale motions:

$$g\rho + \partial_z P = 0, \ \Rightarrow P = P_0 + P_s(z) + \pi(x, y, z; t), \quad (2)$$

Further approximations.

Boussinesq approximation

Deviations of density from ρ_0 neglected in the horizontal $\rightarrow$

$$\frac{\partial \boldsymbol{v}_h}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}_h + f\hat{\boldsymbol{z}} \wedge \boldsymbol{v}_h = -\frac{\boldsymbol{\nabla}_h \pi}{\rho} \approx -\boldsymbol{\nabla}_h \phi, \quad (3)$$

where $\phi = \frac{\pi}{\rho_0}$ - geopotential.

Incompressibility of water

Continuity equation splits in two:

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0, \quad \boldsymbol{v} = \boldsymbol{v}_h + \hat{\boldsymbol{z}}w. \quad (4)$$

$$\partial_t \rho + \boldsymbol{v} \cdot \boldsymbol{\nabla} \rho = 0. \quad (5)$$

Full set of oceanic PE

$$\frac{\partial \boldsymbol{v}_h}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}_h + f\hat{z} \wedge \boldsymbol{v}_h = -\frac{\boldsymbol{\nabla}_h \pi}{\rho} \equiv -\boldsymbol{\nabla}_h \phi, \quad (6)$$

$$\partial_t \sigma + \boldsymbol{v} \cdot \boldsymbol{\nabla} \sigma + w \rho'_s(z) = 0. \quad (7)$$

$$g\frac{\sigma}{\rho_0} = -\partial_z \phi, \quad \boldsymbol{\nabla}_h \cdot \boldsymbol{v}_h + \partial_z w = 0, \quad (8)$$

Remark
Hydrostatic approximation $\leftrightarrow$ anisotropic scaling proper
for mesoscale motions:

$$W \ll U, \quad H \ll L, \quad \frac{W}{H} \sim \frac{U}{L}$$

where L, H and U, W are horizontal and vertical spatial
and velocity scales, respectively.

Vertical boundary conditions

Most often sufficient for our purposes: rigid lid and flat bottom:

$$w|_{z=0} = w|_{z=H} = 0 \qquad (9)$$

Non-trivial bathymetry : fluid parcels follow the bottom profile

$$w|_{z=b(x,y)} = \frac{db}{dt} = \boldsymbol{v} \cdot \boldsymbol{\nabla} b$$

Free surface: fluid parcels move with the surface:

$$w|_{z=h(x,y;t)} = \frac{dh}{dt} = \frac{\partial h}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} h$$

Atmosphere: pressure coordinates

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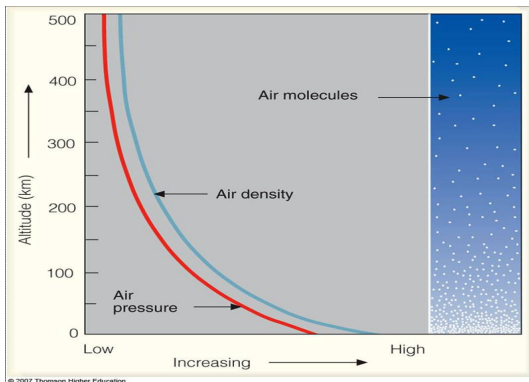waves

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Altitude $\leftrightarrow$ Pressure $\Rightarrow$ vertical coordinate.

Thermodynamics of the dry atmosphere

Equation of state - ideal gas:

$$P = \rho R T, \ \ c_{P,V} = T \left(\frac{\partial s}{\partial T} \right)_{P,V} = const, \ \ c_p - c_v = R. \tag{10}$$

Entropy:

$$s = c_p \ln T - R \ln P + const. \tag{11}$$

Adiabatic process:

$$s = const \ \Rightarrow c_p \frac{dT}{T} - R \frac{dP}{P} = 0, \Rightarrow T = T_s \left(\frac{P}{P_s} \right)^{\frac{R}{c_p}}. \tag{12}$$

Potential temperature :

$$\theta = T \left(\frac{P_s}{P} \right)^{\frac{R}{c_p}}, \ s = c_p \ln \theta + const. \tag{13}$$

Geopotential and hydrostatics

Geopotential variation: work to lift a unit mass against gravity: $\delta\phi = g\delta z$.
$z = z(p)$ becomes a thermodynamical variable.
Hydrostatic approximation:

$$\delta\phi = -\frac{RT}{P}\delta P \Rightarrow \qquad (14)$$

$$\frac{\partial\phi}{\partial P} = -\frac{RT}{P} = -\frac{1}{\rho}. \qquad (15)$$

Useful relation for small variations ρ, P, θ with respect to background ρ_0, P_0, θ_0:

$$\theta = \theta_0 \left[\frac{\left(1 - \frac{R}{c_p}\right) P}{P_0} - \frac{\rho}{\rho_0} \right] \qquad (16)$$

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Elimination of ρ in Euler equations

"Triangular" relation :

$$\left(\frac{\partial P}{\partial x}\right)_z \left(\frac{\partial x}{\partial z}\right)_P \left(\frac{\partial z}{\partial P}\right)_x = -1 \Rightarrow \qquad (17)$$

$$\left(\frac{\partial P}{\partial x}\right)_z = -\left(\frac{\partial P}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_P = \rho \left(\frac{\partial \phi}{\partial x}\right)_P. \qquad (18)$$

Incompressibility in pressure coordinates

Lagrangian volume element in pressure coordinates:

$$\rho dx dy dz = -\frac{1}{g} dx dy dP \qquad (19)$$

Mass conservation $\Rightarrow$ Volume conservation in P.

Adiabatic primitive equations

Equations of motion

$$div(\boldsymbol{v}) = \boldsymbol{\nabla}_h \cdot \boldsymbol{v}_h + \partial_p \omega = 0, \ \ \omega = \frac{dP}{dt}. \tag{20}$$

$$\frac{\partial \boldsymbol{v}_h}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}_h + f\hat{\boldsymbol{z}} \wedge \boldsymbol{v}_h = -\boldsymbol{\nabla}_h \phi. \tag{21}$$

$$\partial_t \theta + \boldsymbol{v} \cdot \boldsymbol{\nabla} \theta = 0. \tag{22}$$

$$\frac{\partial \phi}{\partial P} = -\frac{RT}{P} = -\frac{R}{P} \left(\frac{P}{P_s} \right)^{\frac{R}{c_p}} \theta. \tag{23}$$

Boundary conditions

Bottom: ground $\equiv$ free surface in terms of pressure, geopotential fixed.
Top: rigid lid $\equiv$ fixed value of pressure, e.g. tropopause.

Boussinesq approximation for atmosphere

Varying background density in atmosphere: $\rho_0 = \rho_0(z)$.
Boussinesq approximation in x, y, z coordinates, with
$\rho = \rho_0(z) + \tilde{\rho}$, $P = P_0(z) + \tilde{p}$, $\theta = \theta_0(z) + \tilde{\theta}$, $(\tilde{...})$ omitted
below:

$$\frac{\partial \boldsymbol{v}_h}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}_h + f\hat{\boldsymbol{z}} \wedge \boldsymbol{v}_h = -\boldsymbol{\nabla}_h \phi, \qquad (24)$$

with geopotential $\phi = \frac{p}{\rho_0}$. Hydrostatics:

$$-\frac{\partial \phi}{\partial z} - \frac{p}{\rho_0^2}\frac{\partial \rho_0}{\partial z} - g\frac{\rho}{\rho_0} = 0. \qquad (25)$$

Equation of state (ideal gas) + (16) $\rightarrow$

$$-\frac{\partial \phi}{\partial z} + b = 0, \qquad (26)$$

$b = g\frac{\theta}{\theta_0}$ - buoyancy, $\frac{\partial b}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} b = 0$ for adiabatic
motions.
Continuity equation $\rightarrow$ anelastic equation:

$$\boldsymbol{\nabla} \cdot (\rho_0(z)\boldsymbol{v}) = 0.$$

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Eulerian conservation (ocanic PE)

- Horizontal momentum (modulo Coriolis force), with density $\rho_0 \boldsymbol{v}_h$
- Mass, with density ρ
- Energy, with density

$$e = \rho_0 \frac{\boldsymbol{v}_h^2}{2} + \rho g Z, \quad \rho = \rho_s + \sigma, \qquad (27)$$

where Z is Lagrangian position of the fluid element.

Proof of energy conservation

$$
\begin{aligned}
\frac{d\,e}{dt} &= \rho_0 \boldsymbol{v}_h \cdot \frac{d\,\boldsymbol{v}_h}{dt} + g\frac{d\,\rho}{dt}\,Z + g\rho\frac{d\,Z}{dt} \\
&= -\rho_0 \boldsymbol{v}_h \cdot \boldsymbol{\nabla}_h \phi - \rho_0 \frac{\partial \phi}{\partial z} w \equiv -\boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{v} \phi)\,, (28)
\end{aligned}
$$

where hydrostatic relation between geopotential ϕ and density was used. Recalling that $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$, this gives:

$$\frac{\partial e}{\partial t} = -\boldsymbol{\nabla} \cdot (\boldsymbol{v}(\rho_0 \phi - e))\,.$$

Absolute vorticity in PE

Absolute vorticity:

$$\zeta_a = \zeta + \hat{\mathbf{z}}f, \quad \nabla \cdot \zeta_a = 0, \tag{29}$$

where relative vorticity under PE scaling is:

$$\zeta = -\partial_z v\hat{\mathbf{x}} + \partial_z u\hat{\mathbf{y}} + (\partial_x v - \partial_y u)\hat{\mathbf{z}} \tag{30}$$

Application of $\nabla\wedge$ to PE + "hydrodynamic identity":

$$\mathbf{v} \cdot \nabla \mathbf{v} = \frac{1}{2}\nabla \mathbf{v}^2 - \mathbf{v} \wedge (\nabla \wedge \mathbf{v}) \tag{31}$$

$\rightarrow$ equation for ζ_a:

$$\begin{aligned}
\frac{d\zeta_a}{dt} &= \zeta_a \cdot \nabla \mathbf{v} + \frac{g}{\rho_0}\hat{\mathbf{z}} \wedge \nabla\sigma \rightarrow \\
\frac{\partial \zeta_a}{\partial t} &= \nabla \wedge (\mathbf{v} \wedge \zeta_a) + \frac{g}{\rho_0}\hat{\mathbf{z}} \wedge \nabla\sigma.
\end{aligned} \tag{32}$$

Conservation of potential vorticity

$$q := \boldsymbol{\zeta}_a \cdot \boldsymbol{\nabla}\rho, \quad \rho = \rho_s(z) + \sigma, \quad \frac{dq}{dt} = 0. \tag{33}$$

Using identities

$$\boldsymbol{\nabla}A \cdot (\boldsymbol{\nabla} \wedge \boldsymbol{B}) = -\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}A \wedge \boldsymbol{B}), \tag{34}$$

$$\boldsymbol{A} \wedge (\boldsymbol{B} \wedge \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B}), \tag{35}$$

and $\boldsymbol{\nabla} \cdot \boldsymbol{v} = \boldsymbol{\nabla} \cdot \boldsymbol{\zeta}_a = 0$:

$$
\begin{aligned}
\partial_t(\boldsymbol{\zeta}_a \cdot \boldsymbol{\nabla}\rho) &= (\partial_t \boldsymbol{\zeta}_a) \cdot \boldsymbol{\nabla}\rho + \boldsymbol{\zeta}_a \cdot \boldsymbol{\nabla}(\partial_t\rho) \\
&= \boldsymbol{\nabla}\rho \cdot (\boldsymbol{\nabla} \wedge (\boldsymbol{v} \wedge \boldsymbol{\zeta}_a)) - \boldsymbol{\zeta}_a \cdot \boldsymbol{\nabla}(\boldsymbol{v} \cdot \boldsymbol{\nabla}\rho) \\
&= -\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\rho \wedge (\boldsymbol{v} \wedge \boldsymbol{\zeta}_a)) - \boldsymbol{\zeta}_a \cdot \boldsymbol{\nabla}(\boldsymbol{v} \cdot \boldsymbol{\nabla}\rho) \\
&= -\boldsymbol{\nabla} \cdot (\boldsymbol{v}(\boldsymbol{\zeta}_a \cdot \boldsymbol{\nabla}\rho)) + \boldsymbol{\nabla} \cdot (\boldsymbol{\zeta}_a(\boldsymbol{v} \cdot \boldsymbol{\nabla}\rho)) \\
&\quad - \boldsymbol{\zeta}_a \cdot \boldsymbol{\nabla}(\boldsymbol{v} \cdot \nabla\rho) = -\boldsymbol{v} \cdot \boldsymbol{\nabla}(\boldsymbol{\zeta}_a \cdot \boldsymbol{\nabla}\rho). \tag{36}
\end{aligned}
$$

Spectrum of small perturbations - PE model

Linearised equations:

Perturbations about the state of rest: $\boldsymbol{v} = 0$ with constant stratification on the f- plane. Linearised equations:

$$
\begin{aligned}
u_t - fv + \phi_x &= 0, \\
v_t + fu + \phi_y &= 0, \\
\phi_z + \frac{g}{\rho_0}\sigma = 0, \ \ \sigma_t + w\rho'_s &= 0, \\
u_x + v_y + w_z &= 0,
\end{aligned}
\tag{37}
$$

$$
\tag{38}
$$

where u, v w - three components of velocity perturbation, ϕ - geopotential perturbation, σ - perturbation of the profile of background density ρ_s, with $\rho'_s = const$.

Elimination of σ and w:

Elimination of σ:

$$\phi_{zt} + wN^2 = 0, \tag{39}$$

where $N^2 = -\frac{g\rho'_s}{\rho_0}$, N - Brunt - Väisälä frequency

Elimination of w:

$$
\begin{array}{rcl}
u_t - fv + \phi_x & = & 0, \\
v_t + fu + \phi_y & = & 0, \\
u_x + v_y - N^{-2}\phi_{zzt} & = & 0,
\end{array}
$$

$$\tag{40}$$
$$\tag{41}$$

Method of Fourier
Solutions - harmonic waves:

$$(u, v, \phi) = (u_0, v_0, \phi_0)e^{i(\omega t - \boldsymbol{k} \cdot \boldsymbol{x})} + \text{c.c.}, \qquad (42)$$

where ω et $\boldsymbol{k} = (k_x, k_y, k_z)$ are frequency and wavenumber, respectively.
Algebraic system for (u_0, v_0, ϕ_0):

$$\begin{pmatrix} i\omega & -f & -ik_x \\ f & i\omega & -ik_y \\ -ik_x & -ik_y & i\frac{\omega}{N^2}k_z^2 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \\ \eta_0 \end{pmatrix} = 0, \qquad (43)$$

Dispersion equation

Condition of solvability:

$$\det \begin{pmatrix} i\omega & -f & -ik_x \\ f & i\omega & -ik_y \\ -ik_x & -ik_y & i\frac{\omega}{N^2}k_z^2 \end{pmatrix} = 0, \qquad (44)$$

which gives:

$$\omega \left(\omega^2 - \left(N^2 \frac{k_x^2 + k_y^2}{k_z^2} + f^2 \right) \right) = 0. \qquad (45)$$

Physical meaning of solutions

Three roots of this equation correspond to

- Stationary solutions $\omega = 0$
- Propagative waves with dispersion relation:

$$\omega^2 = N^2 \frac{k_x^2 + k_y^2}{k_z^2} + f^2 \tag{46}$$

Internal inertia-gravity waves: IGW.

Remark: at each fixed k_z - dispersion relation of RSW with $\sqrt{gH_0} \to \frac{N}{|k_z|}$

Non-hydrostatic Boussinesq equations

$$g\frac{\sigma}{\rho_0} = -\phi_z \rightarrow \frac{dw}{dt} + g\frac{\sigma}{\rho_0} = -\phi_z. \qquad (47)$$

Elimination of $b = -g\frac{\sigma}{\rho_0}$ and w:

$$b = \phi_z + w_t, \quad -\left(\partial_{tt} + N^2\right)(u_x + v_y) + \phi_{zzt} = 0 \Rightarrow \quad (48)$$

$$u_t - fv = -\phi_x, \qquad (49)$$

$$v_t + fu = -\phi_y, \qquad (50)$$

$$\left(\partial_{tt} + N^2\right)(u_x + v_y) - \phi_{zzt} = 0, \qquad (51)$$

Dispersion relation:

$$\omega\left[\omega^2 - \left(N^2\frac{k_x^2 + k_y^2}{k_x^2 + k_y^2 + k_z^2} + f^2\frac{k_z^2}{k_x^2 + k_y^2 + k_z^2}\right)\right] = 0 \qquad (52)$$

Typically in the atmosphere and ocean

$$N^2 > f^2 \Rightarrow f^2 \leq \omega^2 \leq N^2 \qquad (53)$$

PE, oceanic case

Characteristic scales

- ► Typical horizontal velocity: U
- ► Typical horizontal scale: L
- ► Time-scale: $T \sim L/U$ -turn-over time
- ► Typical vertical scale: H
- ► Typical vertical velocity: W $\frac{W}{H} \sim \lambda \frac{U}{L}$ -to confirm *aposteriori*
- ► Presssure scale: $\rho_0 gH$

Parameters

- Rossby number: $Ro = \frac{U}{f_0 L} \equiv \epsilon$
- Typical dimensionless deviation of the isopycnal surfaces: λ
- Dimensionless gradient of the Coriolis parametre: $\tilde{\beta}$
- Stratification parameter:
  $S = \dfrac{\text{variable part of density}}{\text{constant part of density}}$
- Burger number: $Bu = \frac{R_d^2}{L^2}$, where R_d baroclinic deformation radius with reduced gravity $g' = S g$.

Pressure and density related via hydrostatics:

$$
\begin{aligned}
\rho &= \rho_0 \left[1 + S \left(\rho_s(z) + \lambda \sigma(x, y, z; t) \right) \right], \Rightarrow \\
P &= \rho_0 g H \left[(1 - z) + S \left(p_s(z) + \lambda \pi(x, y, z; t) \right) \right] \quad (54)
\end{aligned}
$$

Non-dimensional equations

$$\epsilon \frac{d}{dt} \boldsymbol{v}_h + (1 + \tilde{\beta} y) \hat{z} \wedge \boldsymbol{v}_h = -\boldsymbol{\nabla}_h \pi. \tag{55}$$

$$\frac{d}{dt} \sigma + \rho_s' w = 0, \quad \partial_z \pi + \sigma = 0. \tag{56}$$

$$\boldsymbol{\nabla}_h \cdot \boldsymbol{v}_h + \lambda \partial_z w = 0; \tag{57}$$

where

$$\frac{d}{dt} = \partial_t + \boldsymbol{v}_h \cdot \nabla_h + \lambda w \partial_z \tag{58}$$

Boundary conditions - rigid lid/flat bottom, for simplicity:

$$w|_{z=0,1} = 0. \tag{59}$$

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QG regime

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$$\lambda \sim \tilde{\beta} \sim \epsilon \ll 1, \quad \Rightarrow \quad L \sim R_d, \frac{W}{H} = \lambda \frac{U}{L}. \tag{60}$$

Leading order in small parameters

Geostrophic + hydrostatic equilibria:

$$u = -\partial_y \pi, \; v = \partial_x \pi, \; \sigma = -\partial_z \pi \Rightarrow \partial_z v = -\partial_x \sigma, \; \partial_z u = +\partial_y \sigma \tag{61}$$

Horizontal density gradient $\leftrightarrow$ vertical shear of the horizontal velocity. Atmosphere: $\sigma \to -\theta$.
Leading-order vertical velocity:

$$w = -\frac{1}{\rho_s'(z)} \frac{d_g \sigma}{dt} = \frac{1}{\rho_s'(z)} \frac{d_g \partial_z \pi}{dt}, \tag{62}$$

where $\frac{d_g}{dt} = \partial_t + \partial_x \pi \partial_y - \partial_y \pi \partial_x$ - geostrophic advection.

PV in QG approximation

QG PV

$$q = (\zeta + \hat{\boldsymbol{z}}f) \cdot \left(\rho_s'(z)\hat{\boldsymbol{z}} + \boldsymbol{\nabla}\sigma\right) \propto$$

$$(\epsilon\zeta + \hat{\boldsymbol{z}}(1 + \epsilon y)) \cdot \left((\rho_s'(z) + \epsilon\partial_z\sigma)\hat{\boldsymbol{z}} + \epsilon\frac{H}{L}\boldsymbol{\nabla}_h\sigma\right)$$

Using (61), and $H \ll L$:

$$q \propto \rho_s' + \epsilon\left[(\boldsymbol{\nabla}_h^2\pi + y)\rho_s' - \partial_{zz}^2\pi\right] + \mathcal{O}(\epsilon^2)$$

QG advection

$$\frac{d}{dt} = \partial_t + \boldsymbol{v}_h\cdot\boldsymbol{\nabla}_h + \epsilon w\partial_z = \frac{d_g}{dt} + \epsilon\left(\frac{1}{\rho_s'(z)}\frac{d_g\partial_z\pi}{dt}\partial_z + \boldsymbol{v}_h^a\cdot\boldsymbol{\nabla}_h\right) + \mathcal{O}(\epsilon^2),$$

where $\boldsymbol{v}_h^a$ is ageostrophic horizontal velocity.

QG with continuous stratification

QG PV conservation

$$
\begin{aligned}
\frac{d_g q}{dt} &\propto \frac{d_g}{dt}\left[(\boldsymbol{\nabla}_h^2 \pi + y)\rho'_s - \partial_{zz}^2 \pi\right] + \frac{\rho''_s}{\rho'_s(z)}\frac{d_g \partial_z \pi}{dt} \\
&= \rho'_s \frac{d_g}{dt}\left[(\boldsymbol{\nabla}_h^2 \pi + y) - \partial_z\left(\frac{1}{\rho'_s}\partial_z \pi\right)\right] = 0. \quad (63)
\end{aligned}
$$

Equation (63): quasi-geostrophic approximation to primitive equations.

Boundary conditions
Evolution equations (dynamics!)

$$
w|_{z=0,1} = 0 \Rightarrow \left.\frac{d_g}{dt}\partial_z \pi\right|_{z=0,1} = 0. \quad (64)
$$

Baroclinic Rossby waves

Formal linearisation

$$\partial_t \left[\boldsymbol{\nabla}_h^2 \pi - \partial_z \left(\frac{1}{\rho_s'(z)} \partial_z \pi \right) \right] + \partial_x \pi = 0, \quad \partial_{tz}^2 \pi \Big|_{z=0,1} = 0. \tag{65}$$

Separation of variables

$$\pi(x, y, z; t) = p(x, y; t) S(z) \Rightarrow \tag{66}$$

$$\begin{aligned}
\partial_t \boldsymbol{\nabla}_h^2 p(x, y; t) S(z) - \partial_t p(x, y; t) \left[\frac{1}{\rho_s'(z)} S'(z) \right]' & + \\
\partial_x p(x, y; t) S(z) & = 0 \Rightarrow
\end{aligned}$$

Equations in z and in x, y, t:

-

$$\frac{1}{S(z)}\left[\frac{1}{\rho'_s(z)}S'(z)\right]' = \kappa^2 \qquad (67)$$

-

$$\partial_t\boldsymbol{\nabla}_h^2 p(x, y; t) - \kappa^2\partial_t p(x, y; t) + \partial_x p(x, y; t) = 0, \quad (68)$$

κ - separation constant

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Vertical modes
Sturm - Liouville problem:

$$\left[\frac{1}{\rho'_s(z)} S'(z) \right]' - \kappa^2 S(z) = 0, \quad S'(z)\big|_{z=0,1} = 0 \quad (69)$$

Eigenfunctions $S_n(z)$ and eigenvalues κ_n, $n = 0, 1, 2, ...$

Example: linear stratification $\rho_s = -N^2 z$

$$S''(z) + (N\kappa)^2 S(z) = 0, \quad S_n \propto \cos(\pi n z), \; \kappa_n = \frac{\pi n}{N}. \quad (70)$$

Horizontal motion
Wave solutions: $p(x, y; t) \propto e^{i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)} \rightarrow$

$$\omega = -\frac{k_x}{\boldsymbol{k}^2 + \kappa_n^2} - \text{Rossby waves.} \quad (71)$$

$n \nearrow$ (stronger vertical shear) $\Rightarrow c_{phase} \searrow$.

Eady model

QG with constant stratification $N =$ const on the f-plane

$$\frac{d^{(0)}}{dt}\left(\partial_x^2\pi + \partial_y^2\pi + \frac{1}{N^2}\partial_z^2\pi\right) = 0, \ \frac{d^{(0)}}{dt}\partial_z\pi\bigg|_{z=0,1} = 0 \tag{72}$$

Thermal wind

Exact solution: $\vec{v} = U_0(z)\hat{x}, \ U_0 = z$.

Linearisation: $\pi = -U_0(z)y + \phi(x, y, z; t), \ ||\phi|| \ll 1$:

$$(\partial_t + U_0(z)\partial_x)\left(\partial_x^2\phi + \partial_y^2\phi + \frac{1}{N^2}\partial_z^2\phi\right) = 0$$

$$[(\partial_t + U_0(z)\partial_x)(-y\partial_z U_0(z) + \partial_z\phi) - \partial_x\phi\partial_z U_0(z)]|_{z=0,1} = 0.$$

Solution by separation of variables

Fourier modes
B.C. in y-direction: zonal channel $-1 \leq y \leq 1 \Rightarrow$
$\partial_x \phi|_{y=\pm 1} = 0$.

$$\phi = A(z) \cos l_n y \, e^{ik(x-ct)}, \quad l_n = (n + \frac{1}{2})\pi, n = 0, 1, 2, \ldots$$

Equations and b.c.:

$$(z - c)\left(A''(z) - \mu^2 A(z)\right) = 0, \quad \mu^2 = (k^2 + l_n^2)N^2, \quad (73)$$

$$cA'(0) + A(0) = 0, \quad (c - 1)A'(1) + A(1) = 0. \quad (74)$$

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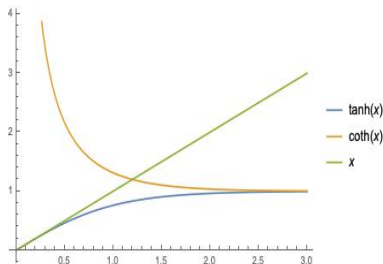Baroclinic Rossby waves

Baroclinic instability

Solution
Non-singular solutions $\Leftrightarrow$ absence of critical layers z_c:
$c = U_0(z_c) \Rightarrow A''(z) - \mu^2 A(z) = 0$.
General solution: $A(z) = a \cosh \mu z + b \sinh \mu z$:

$$
\begin{aligned}
a + c\mu b &= 0, \\
a\left[(c-1)\mu \sinh \mu + \cosh \mu\right] + & \\
b\left[(c-1)\mu \cosh \mu + \sinh \mu\right] &= 0. \quad (75)
\end{aligned}
$$

Dispersion relation:

$$
c^2 - c + \frac{\coth \mu}{\mu} - \frac{1}{\mu^2} = 0 \Rightarrow \quad (76)
$$

$$
c = \frac{1}{2} \pm \left(\frac{1}{4} + \frac{1}{\mu^2} - \frac{\coth \mu}{\mu}\right)^{\frac{1}{2}} \quad (77)
$$

Analysis of dispersion relation

Identity: $\coth\mu = \frac{1}{2}(\tanh\frac{\mu}{2} + \coth\frac{\mu}{2})$:

$$c = \frac{1}{2} \pm \frac{1}{\mu}\left[\left(\frac{\mu}{2} - \coth\frac{\mu}{2}\right)\left(\frac{\mu}{2} - \tanh\frac{\mu}{2}\right)\right]^{\frac{1}{2}}. \qquad (78)$$

$\forall x \ \tanh x \le x \Rightarrow$ instability at
$\coth\frac{\mu}{2} > \frac{\mu}{2} \Rightarrow \mu < \mu_c \approx 2.4 \Rightarrow$ instability of long waves.