Mathematics/Hydrodynamics Refresher

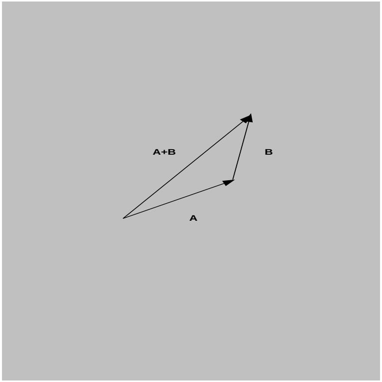
V. Zeitlin

M1 ENS

Vectors: definitions and superposition principle

Vector $\boldsymbol{A}$ is a coordinate-independent (invariant) object having a magnitude $|\boldsymbol{A}|$ and a direction. Alternative notation $\vec{A}$.

Adding/subtracting vectors:



Superposition principle: Linear combination of vectors is a vector

Products of vectors

Scalar product of two vectors:

Projection of one vector onto another:

$$\boldsymbol{A} \cdot \boldsymbol{B} := |\boldsymbol{A}|\,|\boldsymbol{B}| \cos \phi_{AB} \equiv \boldsymbol{B} \cdot \boldsymbol{A},$$

where ϕ_{AB} is an included angle between the two.

Vector product of two vectors:

$$\boldsymbol{A} \wedge \boldsymbol{B} := \hat{\boldsymbol{i}}_{AB}\,|\boldsymbol{A}|\,|\boldsymbol{B}| \sin \phi_{AB} = -\boldsymbol{B} \wedge \boldsymbol{A},$$

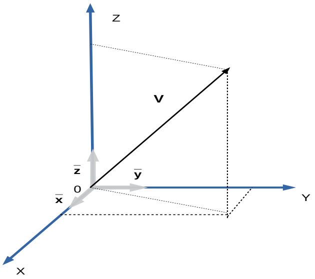
where $\hat{\boldsymbol{i}}_{AB}$ is a unit vector, $|\hat{\boldsymbol{i}}_{AB}| = 1$, perpendicular to both $\boldsymbol{A}$ and $\boldsymbol{B}$, with the orientation of a right-handed screw rotated from $\boldsymbol{A}$ toward $\boldsymbol{B}$.

$\times$ is an alternative notation for $\wedge$.

Distributive properties:

$(\boldsymbol{A}+\boldsymbol{B}) \cdot \boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{C} + \boldsymbol{B} \cdot \boldsymbol{C}$, $(\boldsymbol{A}+\boldsymbol{B}) \wedge \boldsymbol{C} = \boldsymbol{A} \wedge \boldsymbol{C} + \boldsymbol{B} \wedge \boldsymbol{C}$.

Vectors in Cartesian coordinates



Cartesian coordinates: defined by a right triad of mutually orthogonal unit vectors forming a basis:

$$(\hat{\boldsymbol{x}},\, \hat{\boldsymbol{y}},\, \hat{\boldsymbol{z}}) \equiv (\hat{\boldsymbol{x}}_1,\, \hat{\boldsymbol{x}}_2,\, \hat{\boldsymbol{x}}_3),$$

Tensor notation and Kronecker delta

$(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}) \to \hat{\boldsymbol{x}}_i,\ i = 1, 2, 3$. Ortho-normality of the basis:

$$\hat{\boldsymbol{x}}_i \cdot \hat{\boldsymbol{x}}_j = \delta_{ij},$$

where δ_{ij} is Kronecker delta-symbol, an invariant tensor of second rank (3×3 unit diagonal matrix):

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

The components V_i of a vector $\boldsymbol{V}$ are given by its *projections* on the axes $V_i = \boldsymbol{V} \cdot \hat{\boldsymbol{x}}$:

$$\boldsymbol{V} = V_1 \hat{\boldsymbol{x}}_1 + V_2 \hat{\boldsymbol{x}}_2 + V_3 \hat{\boldsymbol{x}}_3 \equiv \sum_{i=1}^{3} V_i \, \hat{\boldsymbol{x}}_i$$

Einstein's convention:
$\sum_{i=1}^{3} A_i \, B_i \equiv A_i \, B_i$ (self-repeating index is "dumb").

Vector products by Levi-Civita tensor

Formula for the vector product:

$$\boldsymbol{A} \wedge \boldsymbol{B} = \left\| \begin{matrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{matrix} \right\|$$

Tensor notation (with Einstein's convention):

$$(\boldsymbol{A} \wedge \boldsymbol{B})_i = \epsilon_{ijk} A_j B_k ,$$

where

$$\epsilon_{ijk} = \begin{cases} 1, \text{ if } ijk = 123, 231, 312 \\ -1, \text{ if } ijk = 132, 321, 213 \\ 0, \text{ otherwise} \end{cases}$$

Magic identity:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}. \quad (1)$$

Scalar, vector, and tensor fields

Any point in space is given by its radius-vector $\boldsymbol{x} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$.
A field is an object defined at any point of space $(x, y, z) \equiv (x_1, x_2, x_3)$ at any moment of time t, i.e. a function of $\boldsymbol{x}$ and t.
Different types of fields:

- scalar $f(\boldsymbol{x}, t)$,
- vector $\boldsymbol{v}(\boldsymbol{x}, t)$,
- tensor $t_{ij}(\boldsymbol{x}, t)$

The fields are dependent variables, and x, y, z and t - independent variables.
Physical examples: scalar fields - temperature, density, pressure, geopotential, vector fields - velocity, electric and magnetic fields, tensor fields - stresses, gravitational field.

Differential operations on scalar fields

Partial derivatives:

$$\frac{\partial f}{\partial x} := \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x},$$

and similar for other independent variables. Differential operator nabla:

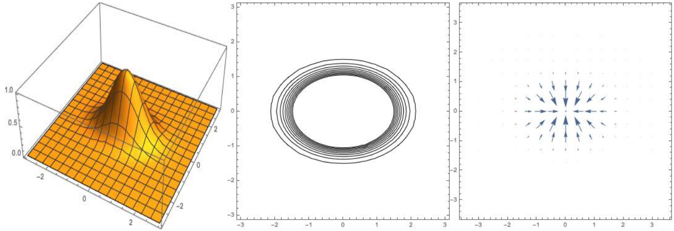
$$\boldsymbol{\nabla} := \hat{\boldsymbol{x}} \frac{\partial}{\partial x} + \hat{\boldsymbol{y}} \frac{\partial}{\partial y} + \hat{\boldsymbol{z}} \frac{\partial}{\partial z}$$

Gradient of a scalar field: the vector field

$$\mathrm{grad}\, f \equiv \boldsymbol{\nabla} f = \hat{\boldsymbol{x}} \frac{\partial f}{\partial x} + \hat{\boldsymbol{y}} \frac{\partial f}{\partial y} + \hat{\boldsymbol{z}} \frac{\partial f}{\partial z}$$

Heuristic meaning: a vector giving direction and rate of fastest increase of the function f.

Visualizing gradient in 2D



From left to right: 2D relief, its contour map, and its gradient. Graphics by Mathematica©

Differential operations with vectors

- Scalar product: divergence

$$\operatorname{div} \boldsymbol{v} \equiv \nabla \cdot \boldsymbol{v}(\boldsymbol{x}) = \frac{\partial v_i}{\partial x_i}$$

- Vector product: curl

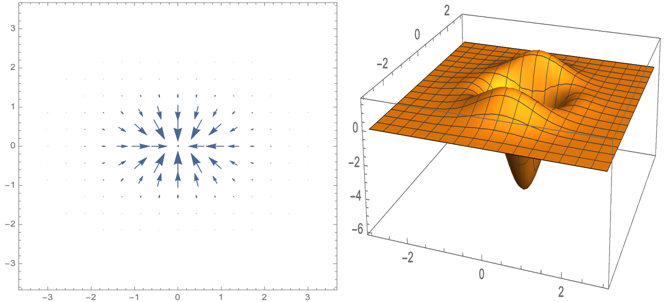
$$\operatorname{curl} \boldsymbol{v} \equiv \nabla \wedge \boldsymbol{v}(\boldsymbol{x}); \quad (\operatorname{curl} \boldsymbol{v})_i = \epsilon_{ijk} \frac{\partial v_k}{\partial x_j}$$

- Tensor product:

$$\nabla \otimes \boldsymbol{v}(\boldsymbol{x}); \quad (\nabla \otimes \boldsymbol{v})_{ij} = \frac{\partial v_i}{\partial x_j}$$

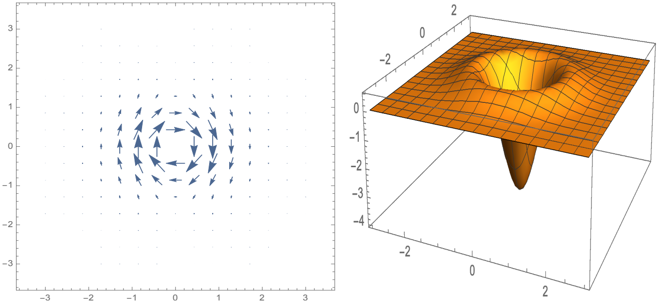
For any $\boldsymbol{v}$, f : $\operatorname{div} \operatorname{curl} \boldsymbol{v} \equiv 0$, $\operatorname{curl} \operatorname{grad} f \equiv 0$,
$\operatorname{div} \operatorname{grad} f = \nabla^2 f$, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ - Laplacian.

Visualizing divergence in 2D



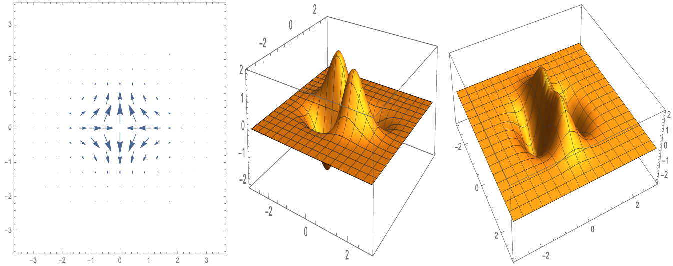
From left to right: vector field $\boldsymbol{v}(x,y) = (v_1(x,y), v_2(x,y))$, and its divergence $\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}$. The curl $\hat{\boldsymbol{z}}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$ of this field is identically zero. (The field is a gradient of the previous example.) Graphics by Mathematica©

Visualizing curl in 2D



From left to right: vector field $\boldsymbol{v}(x, y) = (v_1(x, y), v_2(x, y))$, and its curl $\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$. The divergence $\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}$ of this field is identically zero, so the field is a curl of another vector field. Graphics by Mathematica©

Strain field with non-zero curl and divergence



From left to right: vector field, and its curl and divergence.
Graphics by Mathematica©

Useful identities

$$\nabla \wedge (\nabla \wedge \boldsymbol{v}) = \nabla(\nabla \cdot \boldsymbol{v}) - \nabla^2 \boldsymbol{v}, \tag{2}$$

$$\boldsymbol{v} \wedge (\nabla \wedge \boldsymbol{v}) = \nabla \left(\frac{\boldsymbol{v}^2}{2} \right) - (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v}, \tag{3}$$

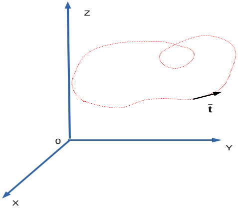
$$\nabla f \cdot (\nabla \wedge \boldsymbol{v}) = -\nabla \cdot (\nabla f \wedge \boldsymbol{v}). \tag{4}$$

Proofs: using tensor representation $(\nabla \wedge \boldsymbol{v})_i = \epsilon_{ijk}\partial_j v_k$, with shorthand notation $\frac{\partial}{\partial x_i} \equiv \partial_i$, exploiting the antisymmetry of ϵ_{ijk}, using that $\delta_{ij} v_j = v_i$, and applying the magic formula (1).

Example: proof of (2).

$$\epsilon_{ijk}\partial_j \epsilon_{klm}\partial_l v_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j \partial_l v_m = \partial_i \partial_j v_j - \partial_j \partial_j v_i.$$

Integration of a field along a (closed) 1D contour

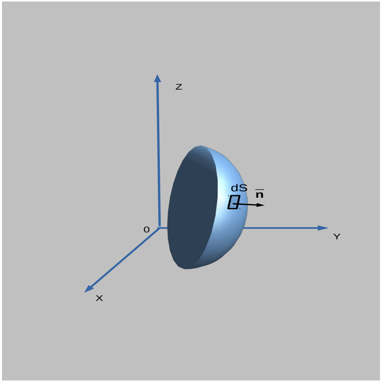


Summation of the values of the field at the points of the contour times oriented line element $d\boldsymbol{l} = \hat{\boldsymbol{t}}\, dl$:

$$\oint d\boldsymbol{l}\,(\ldots),$$

where $\hat{\boldsymbol{t}}$ is unit tangent vector, and dl is a length element along the contour. Positive orientation: anti-clockwise.

Integration of a field over a 2D surface

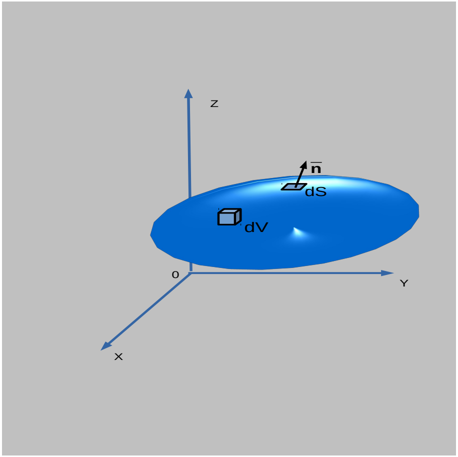


Summation of the values of the field at the points of the surface times oriented surface element $d\boldsymbol{s} = \hat{\boldsymbol{n}}\,ds$:

$$\int\int d\boldsymbol{s}\,(\ldots) \equiv \int_S d\boldsymbol{s}\,(\ldots),$$

where $\hat{\boldsymbol{n}}$ is unit normal vector. Positive orientation for closed surfaces: outwards.

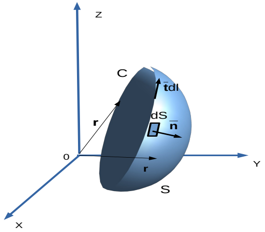
Integration of a field over a 3D volume



Summation of the values of the field at the points in the volume times volume element dV.

$$\int\int\int dV\,(\ldots) \equiv \int_V dV\,(\ldots).$$

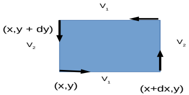
Linking contour and surface integrations: Stokes theorem



$$\oint_C d\boldsymbol{l} \cdot \boldsymbol{v}(\boldsymbol{x}) = \int_{S_C} d\boldsymbol{s} \cdot (\boldsymbol{\nabla} \wedge \boldsymbol{v}(\boldsymbol{x})). \qquad (5)$$

Left-hand side: circulation of the vector field over the contour C. Right-hand side: curl of $\boldsymbol{v}$ integrated over any surface S_C having the contour C as a base.

Stokes theorem: the idea of proof



Circulation of the vector $\boldsymbol{v} = v_1 \hat{\boldsymbol{x}} + v_2 \hat{\boldsymbol{y}}$ over an elementary contour, with $dx \to 0$, $dy \to 0$, using first-order Taylor expansions:

$$v_1(x, y)dx + v_2(x + dx, y)dy - v_1(x, y + dy)dx - v_2(x, y)dy = \frac{\partial v_2}{\partial x} dx\, dy - \frac{\partial v_1}{\partial y} dx\, dy,$$

with a z-component of curl$\boldsymbol{v}$ multiplied by the z-oriented surface element arising in the right-hand side.

Linking surface and volume integrations: Gauss theorem

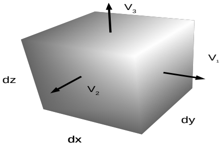
$$\oint_{S_V} d\boldsymbol{s} \cdot \boldsymbol{v}(\boldsymbol{x}) = \int_V dV\, \boldsymbol{\nabla} \cdot \boldsymbol{v}(\boldsymbol{x}). \tag{6}$$

Left-hand side: flux of the vector field through the surface S_V which is a boundary of the volume V. Right-hand side: volume integral of the divergence of the field.

Important. The theorem is also valid for the scalar field:

$$\oint_{S_V} d\boldsymbol{s} \cdot f(\boldsymbol{x}) = \int_V dV\, \boldsymbol{\nabla} f(\boldsymbol{x}). \tag{7}$$

Gauss theorem: the idea of proof



Flux of the vector $\boldsymbol{v} = v_1\hat{\boldsymbol{x}} + v_2\hat{\boldsymbol{y}} + v_3\hat{\boldsymbol{z}}$ over a surface of an elementary volume, taking into account the opposite orientation of the oriented surface elements:

$$
\begin{aligned}
&[v_1(x+dx,y,z) - v_1(x,y,z)]\,dydz+\\
&[v_2(x,y+dy,z) - v_2(x,y,z)]\,dxdz+\\
&[v_3(x,y,z+dz) - v_3(x,y,z)]\,dxdy = \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) dx\,dy\,dz
\end{aligned}
$$

Fourier series for periodic functions

Consider $f(x) = f(x + 2\pi)$, a periodic smooth function on the interval $[0, 2\pi]$. Fourier series:

$$f(x) = \sum_{n=0}^{\infty} \left[a_n \cos(n\,x) + b_n \sin(n\,x) \right].$$

The expansion is unique du to ortogonality of the basis functions:

$$\int_0^{2\pi} dx\ \cos(n\,x)\cos(m\,x) = \int_0^{2\pi} dx\ \sin(n\,x)\sin(m\,x) = \pi\delta_{nm},$$

$$\int_0^{2\pi} dx\ \sin(n\,x)\cos(m\,x) \equiv 0.$$

The coefficients of expansion, thus, are uniquely defined:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} dx\, f(x)\, \cos(n\,x), \quad b_n = \frac{1}{\pi}\int_0^{2\pi} dx\, f(x)\, \sin(n\,x)$$

Complex exponential form

$$e^{inx} = \cos(n\,x) + i\sin(n\,x) \Rightarrow$$

$$\cos(n\,x) = \frac{e^{inx} + e^{-inx}}{2}, \ \sin(n\,x) = \frac{e^{inx} - e^{-inx}}{2i}$$

Hence

$$f(x) = \sum_{n=0}^{\infty} \frac{(a_n - ib_n)}{2} e^{inx} + c.c \equiv \sum_{-\infty}^{\infty} A_n e^{inx}, \ A_n^* = A_{-n}$$

Orthogonality:

$$\int_0^{2\pi} dx\, e^{inx} e^{-imx} = 2\pi\delta_{nm}$$

Expression for coefficients

$$A_n = \frac{1}{2\pi}\int_0^{2\pi} dx\, f(x)\, e^{-inx}$$

Fourier integral

Fourier series on arbitrary interval L: $\sin(nx)$, $\cos(nx) \rightarrow \sin(\frac{2\pi}{L}nx)$, $\cos(\frac{2\pi}{L}nx)$, $\int_0^{2\pi} dx \rightarrow \int_0^L dx$, normalization $\frac{1}{\pi} \rightarrow \frac{1}{L}$. In the limit $L \rightarrow \infty$: $\sum_{-\infty}^{\infty} \rightarrow \int_{-\infty}^{\infty}$.
Fourier-transformation and its inverse:

$$f(x) = \int_{-\infty}^{\infty} dk\, F(k)\, e^{ikx}, \quad F(k) = \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx}.$$

Based on orthogonality:

$$\int_{-\infty}^{\infty} dx\, e^{ikx} e^{-ilx} = \delta(k-l),$$

where $\delta(x)$ - Dirac's delta-function, continuous analog of Kronecker's δ_{nm}, with properties:

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1, \quad \int_{-\infty}^{\infty} dy\, \delta(x-y)\, F(y) = F(x).$$

Multiple variables and differentiation

$$f(x,y,z) = \int_{-\infty}^{\infty} dk\, dl\, dm\, F(k,l,m)\, e^{i(kx+ly+mz)},$$

$$F(k,l,m) = \int_{-\infty}^{\infty} dx\, dy\, dz\, f(x,y,z)\, e^{-i(kx+ly+mz)}.$$

Physical space $(x,y,z) \longrightarrow (k,l,m)$, Fourier space.
Radius-vector $\boldsymbol{x} \to \boldsymbol{k}$, "wavevector",

$$f(\boldsymbol{x}) = \int_{-\infty}^{\infty} d\boldsymbol{k}\, F(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

Main advantage: differentiation in physical space $\to$ multiplication by the corresponding component of the wavevector in Fourier space $\frac{\partial}{\partial x} \to ik$:

$$\frac{\partial}{\partial x} f(\boldsymbol{x}) = \int_{-\infty}^{\infty} d\boldsymbol{k}\, ik\, F(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}},$$

and similarly for other variables.

Equations of motion

Eulerian description: in terms of fluid velocity field $\mathbf{v}(\mathbf{x}, t)$, and scalar density and pressure fields $\rho(\mathbf{x}, t)$, $P(\mathbf{x}, t)$, defined at each point $\mathbf{x}$ of the volume occupied by the fluid at any time t.

Euler equations

Local conservation of momentum in the presence of forcing $\mathbf{F}$:

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} \right) = -\nabla P + \mathbf{F}, \qquad (8)$$

Continuity equation

Local conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0. \qquad (9)$$

Equation of state: baroclinic fuid

Fluid: thermodynamical system $\Rightarrow$ equation of state relating P and ρ and closing the system (8), (9) (4 equations for 5 dependent variables).
General equation of state:

$$P = P(\rho, s), \tag{10}$$

$s(\mathbf{x}, t)$ is entropy per unit mass $\Rightarrow$ evolution equation for s required. Perfect fluid:

$$\frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \nabla s = 0. \tag{11}$$

Equation of state: barotropic fluid

$$P = P(\rho) \leftrightarrow s = \text{const}, \tag{12}$$

sufficient to close the system (8), (9).
Particular case: incompressible fluid. Conservation of volume per unit mass $\Rightarrow$ zero divergence:

$$\nabla \cdot \boldsymbol{v} = 0, \; \Rightarrow \tag{13}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot \nabla \rho = 0, \;\; \text{and} \;\; \nabla \cdot (\boldsymbol{v} \cdot \nabla \boldsymbol{v}) = -\nabla \cdot \left(\frac{\nabla P}{\rho} \right) \; \Rightarrow \tag{14}$$

Pressure entirely determined by density and velocity distributions.

Lagrangian view of the fluid: momentum balance

Fluid $\equiv$ ensemble of fluid parcels with time-dependent positions $\mathbf{X}(\mathbf{x}_0, t)$, $\mathbf{X}(\mathbf{x}_0, 0) = \mathbf{x}$.
Euler - Lagrange duality: continuity of the fluid $\Rightarrow$ any point in the flow $\mathbf{x}$ is, at the same time, a position of some fluid parcel $\Rightarrow$ Eulerian velocity at the point $\mathbf{v}(\mathbf{x}) =$ velocity of the parcel $\mathbf{v}(\mathbf{X}, t) = \frac{d\mathbf{X}}{dt} \equiv \dot{\mathbf{X}}$. Lagrangian (material) derivative in Eulerian terms by chain differentiation:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial \boldsymbol{x}}{\partial t} \cdot \boldsymbol{\nabla} \equiv \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla. \tag{15}$$

$\Rightarrow$ Newton's second law for the parcel

$$\rho(\mathbf{X}, t) \frac{d^2\mathbf{X}}{dt^2} = -\boldsymbol{\nabla}_{\mathbf{X}} P(\mathbf{X}, t) + \mathbf{F}, \tag{16}$$

$\Leftrightarrow$ Euler equation (8).

Lagrangian view of the fluid: mass balance

Mass conservation in Lagrangian terms:

$$\rho_i(\mathbf{x})d^3\mathbf{x} = \rho(\mathbf{X},t)d^3\mathbf{X}, \leftrightarrow \rho_i(\mathbf{x}) = \rho(\mathbf{X},t)\mathcal{J} \tag{17}$$

where ρ_i is the initial distribution of density, and $d^3\mathbf{x}$ and $d^3\mathbf{X}$ are initial and current elementary volumes. The Jacobi determinant (Jacobian) in this formula is defined as the determinant:

$$\mathcal{J} = \begin{vmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} & \frac{\partial X}{\partial z} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} & \frac{\partial Y}{\partial z} \\ \frac{\partial Z}{\partial x} & \frac{\partial Z}{\partial y} & \frac{\partial Z}{\partial z} \end{vmatrix} = \frac{\partial(X,Y,Z)}{\partial(x,y,z)}$$

Incompressibility in Lagrangian terms: $\mathcal{J} = 1$. Taking Lagrangian time-derivative of this relation, we obtain the incompressibility condition of zero velocity divergence in Eulerian terms. Advection of entropy (11) $\Leftrightarrow$ conservation of entropy by each fluid parcel $\dot{s} = 0$.

1st principle of thermodynamics

Reversible processes in one-phase systems:

$$\delta\epsilon = T\delta s - P\delta v, \tag{18}$$

ϵ - internal energy per unit mass, $v = \frac{1}{\rho}$ - specific volume.Enthalpy per unit mass: $h = \epsilon + Pv \Rightarrow$

$$\delta h = T\delta s + v\delta P. \tag{19}$$

Energy density: sum of kinetic and internal parts:

$$e = \frac{\rho \mathbf{v}^2}{2} + \rho\epsilon. \tag{20}$$

Local conservation of energy :

$$\frac{\partial e}{\partial t} + \boldsymbol{\nabla} \cdot \left[\rho \mathbf{v} \left(\frac{\mathbf{v}^2}{2} + h\right)\right] = 0. \tag{21}$$

Barotropic fluid:

$$\delta h = \frac{\delta P}{\rho} \quad \Rightarrow \frac{\boldsymbol{\nabla} P}{\rho} = \boldsymbol{\nabla} h. \tag{22}$$

Kelvin theorem

Circulation of velocity around a contour Γ consisting of fluid parcels, and moving with the fluid:

$$\gamma = \int_{\Gamma} \mathbf{v} \cdot d\mathbf{l} = \int_{S_{\Gamma}} (\nabla \wedge \mathbf{v}) \cdot d\mathbf{l}, \tag{23}$$

Kelvin theorem states that

- for barotropic fluids

$$\frac{d\gamma}{dt} = 0, \tag{24}$$

- for baroclinic fluids

$$\frac{d\gamma}{dt} = -\int_{\Gamma} \frac{\nabla P}{\rho} \cdot d\mathbf{l}. \tag{25}$$

Proof: direct calculation of the time-derivative of the circulation using the equations of motion, and the Lagrangian nature of Γ.

Perfect vs real fluids

Perfect fluid approximation: macroscopic fluxes of mass, momentum and energy. Real fluids: corrections to these fluxes due to molecular transport. Simplest way to include them: flux-gradient relations following from Le Chatelier principle: molecular fluxes tend to restore the thermodynamical equilibrium. For any thermodynamical variable A

$$\mathbf{f}_A = -k_A \nabla A,$$

where $\mathbf{f}_A$ is related molecular flux, and k_A is molecular transport coefficient.

Viscosity, diffusivity, and thermal conductivity

- Viscosity corrections to the Euler equation in the incompressible case, giving the Navier - Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla P}{\rho} + \nu \nabla^2 \mathbf{v}, \ \nabla \cdot \mathbf{v} = 0. \qquad (26)$$

- Diffusivity corrections to the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = D \nabla^2 \rho. \qquad (27)$$

- Thermal conductivity corrections to the heat/temperature advection giving the heat equation

$$\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \chi \nabla^2 T. \qquad (28)$$

ν, D, χ are kinematic viscosity, diffusivity, and thermo-conductivity, the molecular transport coefficients for momentum, mass, and energy, respectively, all with dimension $\left[\frac{L^2}{T}\right]$

Dimensional/scale analysis. Reynolds number

Molecular transport coefficients: dimensional, value varies with changes if units. Only *non-dimensional parameters* are relevant. Typical space and velocity scales in the incompressible fluid flow: L, U. Time-scale $T = L/U$. Pressure scale: ρU^2.
Scaled NS equation:

$$\frac{U^2}{L}\left(\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}P\right) = \frac{U\,\nu}{L^2}\boldsymbol{\nabla}^2\boldsymbol{v} \rightarrow \tag{29}$$

Non-dimensional NS equation

$$\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\boldsymbol{\nabla}P + \frac{1}{Re}\boldsymbol{\nabla}^2\boldsymbol{v} \tag{30}$$

$Re = \frac{U\,L}{\nu}$ - Reynolds number, the true measure of viscosity. Similar, Pecklet number for diffusivity.