# Rotating Shallow Water on a Tangent Plane

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# Motion in a rotating frame

## Material point in a frame rotating with angular velocity $\boldsymbol{\Omega}$:

$$m\frac{d\boldsymbol{v}}{dt} + 2m\boldsymbol{\Omega}\wedge\boldsymbol{v} + m\boldsymbol{\Omega}\wedge(\boldsymbol{\Omega}\wedge\boldsymbol{x}) = \boldsymbol{F}, \quad \boldsymbol{v} = \frac{d\boldsymbol{x}}{dt} \qquad (1)$$

$m$- mass, $\boldsymbol{x}$-current position of the point, $\boldsymbol{F}$ - sum of forces acting on the point

## Euler equations in the rotating frame +gravity:

Fluid under the influence of gravity: $m \to \rho$, $\frac{d}{dt} \to \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}$, forces: pressure + gravity $\Rightarrow$

$$\frac{\partial\boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} + 2\boldsymbol{\Omega}\wedge\boldsymbol{v} = -\frac{\boldsymbol{\nabla}P}{\rho} + \boldsymbol{g}^* \qquad (2)$$

Effective gravity: gravity + centrifugal acceleration (also potential)

$$\boldsymbol{g}^* = \boldsymbol{g} + \boldsymbol{\Omega}\wedge(\boldsymbol{\Omega}\wedge\boldsymbol{x}) \qquad (3)$$

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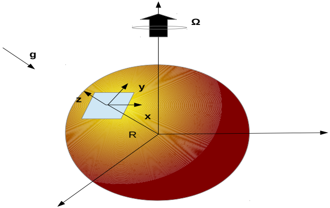
# Tangent plane approximation




$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + f\hat{z} \wedge \boldsymbol{v} = -\frac{\nabla P}{\rho} + \boldsymbol{g} \tag{4}$$

$f$ - plane: $f = const$; $\beta$ - plane: $f = f + \beta y$; $f$ - Coriolis parameter: $f = 2\Omega \sin\phi$, where $\phi$ - latitude

# Hydrostatics. Stratification

The state of rest $\boldsymbol{v} \equiv 0$ is solution of (4) if hydrostatic equilibrium holds:

$$0 = -\frac{\boldsymbol{\nabla} P}{\rho} + \boldsymbol{g}$$

The continuity equation:

$$\frac{d\rho}{dt} + \rho \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$$

is satisfied by time-independent $\rho$ in a state of rest. Statically stable states: $\rho = \rho_0(z)$, $\rho_0'(z) \leq 0 \rightarrow$

$$P = P_0(z) = -\int dz \, g \, \rho_0(z)$$

Dependence of $\rho_0$ on $z$ is called stratification. Surfaces of constant $\rho$: isopycnals.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# Parent equations

Fluid with constant density $\rho_0$ with a free surface $\leftrightarrow$ single isopycnal, simplest stratification. Thin layer $\rightarrow$ columnar motion $\equiv$ horizontal velocity $\boldsymbol{v}_h = \boldsymbol{v}_h(x, y, t)$.
Hydrostatic equations, $\boldsymbol{v} = \boldsymbol{v}_h + \hat{\boldsymbol{z}} w$:

$$\frac{\partial \boldsymbol{v}_h}{\partial t} + \boldsymbol{v}_h \cdot \boldsymbol{\nabla} \boldsymbol{v}_h + f \hat{z} \wedge \boldsymbol{v}_h = -\frac{\boldsymbol{\nabla}_h P}{\rho}, \quad \boldsymbol{\nabla}_h = (\partial_x, \partial_y)$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0, \quad g = -\frac{\partial_z P}{\rho}.$$

# Boundary conditions

## Horizontal boundary conditions:

Periodic, decay, or other

## Vertical kinematic boundary conditions:

- Free surface $z = h(x, y, t)$:

$$w|_{z=h} = \frac{dh}{dt} = \partial_t h + \boldsymbol{v}_h \cdot \boldsymbol{\nabla}_h h.$$

  Meaning: material surface made of fluid parcels.

- Flat bottom:

$$w|_{z=0} = 0.$$

  Meaning: non-penetration through the boundary.

## Vertical dynamic boundary condition:

Continuity of pressure: $P|_{z=h} = P_0 = \text{const.}$


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# Eliminating $P$ and $w$

- Integrating hydrostatic equation:

$$P(x, y, z, t) = -\rho_0\, g\, z + \mathcal{P}(x, y, t),$$

$\mathcal{P}(x, y, t)$ - integration “constant”. Dynamic boundary condition: $\mathcal{P}(x, y, t) = \rho_0 g h(x, y, t) + P_0$.

- Integrating continuity equation:

$$\boldsymbol{\nabla}_h \boldsymbol{v}_h(x, y, t) + \partial_z w(x, y, z, t) = 0 \rightarrow$$

$$w = -z\, \boldsymbol{\nabla}_h \boldsymbol{v}_h(x, y, t) + \mathcal{W}(x, y, t).$$

$\mathcal{W}(x, y, t)$ - integration “constant”. Bottom boundary condition: $\mathcal{W}(x, y, t) = 0$.

- Kinematic boundary condition at the surface:

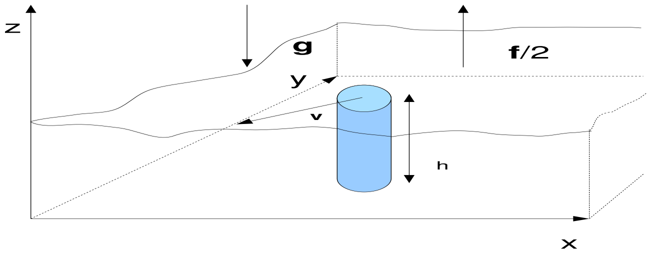
$$w|_{z=h} = -h \boldsymbol{\nabla}_h \boldsymbol{v}_h(x, y, t) = \partial_t h + \boldsymbol{v}_h \cdot \boldsymbol{\nabla}_h h.$$

# Rotating shallow water (Saint-Venant) equations

$$\partial_t \boldsymbol{v}_h + \boldsymbol{v}_h \cdot \nabla \boldsymbol{v}_h + f\hat{\mathbf{z}} \wedge \mathbf{v}_h + g\nabla h = 0\,, \quad (5)$$

$$\partial_t h + \nabla \cdot (\mathbf{v}_h h) = 0\,, \quad (6)$$

Meaning: horizontal motion of columns of fluid of variable depth.





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# Conservation laws and acoustic analogy

## Eulerian conservation laws

Equations (5), (6) express the local conservation of the horizontal momentum and mass.
By direct calculation using (5), (6), for energy density:

$$e = h\frac{\boldsymbol{v}^2}{2} + g\frac{h^2}{2} \qquad (7)$$

we get

$$\partial_t e = -\nabla \cdot \left( \boldsymbol{v} h \left( \frac{\boldsymbol{v}^2}{2} + gh \right) \right) \Rightarrow \qquad (8)$$

total energy, $E = \int dxdy\, e =$ const, for isolated system.

## Acoustic analogy

Equation (6) is a continuity equation for "density" $h$.
Equations. (5) are 2-dimensional Euler equations in a rotating frame for a barotropic fluid with density $h$ and pressure $P = \frac{g\,h^2}{2}$.

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# Vorticity and Potential vorticity

Only the vertical component of relative vorticity counts in RSW:

$$\zeta = v_x - u_y$$

Relative vorticity: vorticity measured in the rotating frame.
Absolute vorticity: vorticity measured in a fixed frame

$$\zeta_a = \zeta + f$$

Planetary vorticity $f$: vorticity due to rotation of the system.
Potential vorticity (PV):

$$q = \frac{\zeta + f}{h}. \tag{9}$$


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## Lagrangian conservation of PV:

$$\frac{dq}{dt} \equiv (\partial_t + \boldsymbol{v} \cdot \nabla)\, q = 0, \tag{10}$$

is obtained by combining equations of vorticity:

$$\frac{d(\zeta + f)}{dt} + (\zeta + f)\nabla \cdot \boldsymbol{v} = 0, \tag{11}$$

and mass conservation:

$$\frac{dh}{dt} + h\nabla \cdot \boldsymbol{v} = 0 \ : \tag{12}$$

$$\frac{d}{dt}\frac{\zeta + f}{h} = \frac{1}{h}\frac{d}{dt}(\zeta + f) - \frac{\zeta + f}{h^2}\frac{d}{dt}h = 0, \tag{13}$$

## Eulerian expression:

Conservation of PV leads to independence of time of any integral:

$$\int dxdy\, h\mathcal{F}(q), \tag{14}$$

over the whole flow, with $\mathcal{F}$ - arbitrary function.

## Qualitative image of the RSW dynamics:

Two-dimensional motion of the fluid columns of variable depth, each preserving its potential vorticity.

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# Spectrum of small perturbations to the state of rest on the $f$-plane

## Method of small perturbations

State of rest $\mathbf{v} = 0, \; h = H_0 = const$ - exact solution. Consider small perturbations $\mathbf{v} = (u, \; v) \, h = H_0 + \eta$, such that $||u||, \; ||v||, \; ||\eta||$ are all $\ll 1$, which allows to neglect the nonlinear terms $\rightarrow$ linearization.

## Linearized RSW equations :

Linearized equations in the approximation $f = const$:

$$
\begin{aligned}
u_t - fv + g\eta_x &= 0, \\
v_t + fu + g\eta_y &= 0, \qquad (15) \\
\eta_t + H_0(u_x + v_y) &= 0,
\end{aligned}
$$

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## Method of Fourier

Solutions - harmonic waves:

$$(u, v, \eta) = (u_0, v_0, \eta_0)e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})} + \text{c.c.}, \tag{16}$$

where $\omega$ and $\boldsymbol{k}$ are frequency and wavenumber, respectively $\Rightarrow$
algebraic system for $(u_0, v_0, \eta_0)$:

$$\begin{pmatrix} i\omega & -f & -igk_x \\ f & i\omega & -igk_y \\ -iH_0k_x & -iH_0k_y & i\omega \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \\ \eta_0 \end{pmatrix} = 0, \tag{17}$$

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# Dispersion equation

## Condition of solvability:

$$\det \begin{pmatrix} i\omega & -f & -igk_x \\ f & i\omega & -igk_y \\ -iH_0k_x & -iH_0k_y & i\omega \end{pmatrix} = 0, \qquad (18)$$

which gives:

$$\omega\left(\omega^2 - gH_0\boldsymbol{k}^2 - f^2\right) = 0. \qquad (19)$$

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# Physical meaning of solutions

3 roots of the equation correspond to

- Stationary solutions $\omega = 0 \leftrightarrow$ linearized PV-conservation equation:

$$\partial_t \left( \frac{\partial_x v - \partial_y u}{H_0} - \frac{f\,\eta}{H_0^2} \right) = 0$$

- Propagative waves with the dispersion relation:

$$\omega^2 - gH_0 \boldsymbol{k}^2 - f^2 = 0. \qquad (20)$$

  Inertia-gravity waves.

Dispersion relation (20) is isotropic. No-rotation limit:

$$\omega = \pm\sqrt{gH_0}|\boldsymbol{k}| \rightarrow$$

acoustic waves with "speed of sound" $c = \sqrt{gH_0}$.


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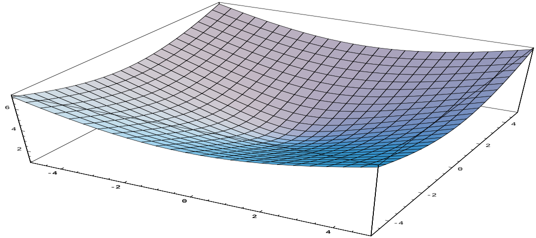
# Dispersion relation




Dispersion relation for inertia-gravity waves. $c = \sqrt{gH_0} = 1$, $f = 1$ , the part with $\omega < 0$ is not presented.

# Horizontal motion equations

$$\frac{\partial \boldsymbol{v}_h}{\partial t} + \boldsymbol{v}_h \cdot \boldsymbol{\nabla}_h \boldsymbol{v}_h + f\hat{z} \wedge \boldsymbol{v}_h = -g\boldsymbol{\nabla}_h h. \tag{21}$$

$$f = f_0(1 + \beta y), \quad H = H_0 + \eta \tag{22}$$

## “Vortex” scaling

- Velocity $\boldsymbol{v}_h = (u, v), \;\; u, v \sim U$
- Unique horizontal spatial scale $L$,
- Vertical scale $H_0 << L$,
- Time-scale: tirn-over time $T \sim L/U$.

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# Characteristic parameters

Intrinsic scale of the system: deformation (Rossby) radius:

$$R_d = \frac{\sqrt{gH_0}}{f_0} \tag{23}$$

- Rossby number: $Ro = \frac{U}{f_0 L}$,
- Burger number: $Bu = \frac{R_d^2}{L^2}$,
- Typical amplitude of the free surface elevations = non-linearity parametre: $\lambda = \Delta H / H_0$, where $\Delta H$ is the typical value of $\eta$,
- Non-dimensional meridional gradientof $f$: $\tilde{\beta} \sim \beta L$

# Non-dimensional RSW equations

$$Ro\,(\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\mathbf{v}) + (1+\tilde{\beta}y)\hat{\mathbf{z}}\wedge\mathbf{v} = -\frac{\lambda Bu}{Ro}\nabla\eta\,, \qquad (24)$$

$$\lambda\partial_t\eta + \nabla\cdot(\mathbf{v}(1+\lambda\eta)) = 0\,. \qquad (25)$$

## Geostrophic equilibrium

Equilibrium between the force of Coriolis and pressure force $\rightarrow$ geostrophic wind:

$$f\hat{\mathbf{z}}\wedge\mathbf{v}_g = -g\nabla h \qquad (26)$$


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# Quasi-geostrophic approximation

## Conditions of realization of the geostrophic equilibrium:

- $Ro \to 0$,
- $\lambda\, Bu \sim Ro$,
- $\tilde{\beta} \to 0$.

## Quasi- geostrophy (QG):

$$Ro \equiv \epsilon \ll 1, \lambda \sim Ro, \Rightarrow Bu \sim 1, \Rightarrow L \sim R_d,\ \tilde{\beta} \sim Ro \quad (27)$$

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# Potential vorticity in QG approximation

Non-dimensional potential vorticity:

$$
\begin{aligned}
q &= \frac{f_0}{H_0} \frac{\epsilon(v_x - u_y) + (1 + \epsilon y)}{1 + \epsilon\eta} \\
&= \frac{f_0}{H_0} \left(\epsilon(v_x - u_y) + (1 + \epsilon y)\right)(1 - \epsilon\eta + \ldots) \\
&= \frac{f_0}{H_0} \left[1 + \epsilon\left(v_x - u_y + y - \eta\right) + \mathcal{O}\left(\epsilon^2\right)\right]. \qquad (28)
\end{aligned}
$$

Non-dimensional geostrophic wind:

$$
v = \eta_x \quad u = -\eta_y \Rightarrow v_x - u_y = \nabla^2\eta \qquad (29)
$$

Advection by the geostrophic wind:

$$
\partial_t \ldots + u\partial_x \ldots + v\partial_y \ldots \to \partial_t \ldots + \mathcal{J}(\eta, \ldots) \qquad (30)
$$

$\mathcal{J}(A, B) = A_x B_y - A_y B_x$ - Jacobian.


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# QG equation

$$\partial_t \left( \nabla^2 \eta - \eta \right) + \mathcal{J}(\eta, \nabla^2 \eta - \eta) + \partial_x \eta = 0. \quad (31)$$

Physical meaning: conservation of (non-dimensional) geostrophic PV $q_G = \nabla^2 \eta - \eta + y$. Restitution of dimensions:

$$\nabla^2 \eta - \eta \to \nabla^2 \eta - \frac{1}{R_d^2} \eta, \; \partial_x \eta \to \beta \partial_x \eta. \quad (32)$$

Formal linearization:

$$\partial_t \eta - \nabla^2 \partial_t \eta - \partial_x \eta = 0. \quad (33)$$

Wave solutions: $\eta \propto exp^{i(kx+ly-\omega t)} \to$ dispersion relation:

$$\omega = -\frac{k}{k^2 + l^2 + 1}. \quad (34)$$

Rossby waves: strongly dispersive, with anisotropic dispersion $\leftrightarrow$ vorticity waves.


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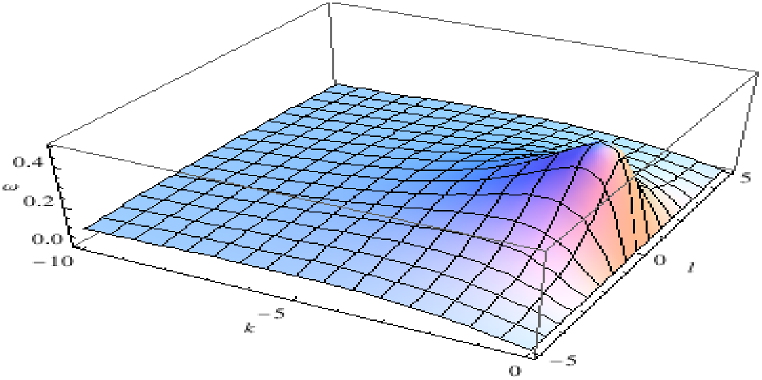
# Dispersion diagram for Rossby waves

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# Rossby waves over a mean flow

Zonal flow: $(u, v) = (U(y), 0)$. Corresponding geopotential anomaly:

$$\eta = \eta_0(y) = -\int^y dy' U(y') \Rightarrow \nabla^2 \eta = -U'(y). \quad (35)$$

Linearization of (31) about $\eta_0$, $\eta \to \eta_0 + \eta$:

$$(\partial_t + U(y)\partial_x)\left(\nabla^2 \eta - \eta\right) + \partial_x \eta(-U''(y) + U(y)) + \partial_x \eta = 0. \Rightarrow \quad (36)$$

PV gradient of the mean flow $(-U''(y) + U(y))$ plays the same rôle as $\beta$ (last term in (36)).
If $U = const$, equation (36) has constant coefficients $\rightarrow$ Fourier -transform$\rightarrow$ dispersion relation for short waves with $k^2 + l^2 \gg 1$:

$$\omega = Uk - \frac{k}{k^2 + l^2} \quad (37)$$

absolute frequency . $\omega = -\frac{k}{k^2+l^2}$ - intrinsic frequency .


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# Rossby waves over mean flow on the $f$-plane

Equation (36) on the $f$-plane, in the limit $R_d \to \infty$:

$$\nabla^2 \eta_t + U(y)\nabla^2 \eta_x - \eta_x U''(y) = 0. \qquad (38)$$

Partial Fourier-transform: $\eta(x, y, t) \to \hat{\eta}(y) e^{ik(x-ct)} \Rightarrow$

$$\hat{\eta}''(y) - \left[k^2 + \frac{U''(y)}{U(y) - c}\right] \hat{\eta}(y) = 0. \qquad (39)$$

Boundary conditions: free-slip in the zonal channel $y_1 \le y \le y_2$

$$v|_{y=y_{1,2}} = \eta_x|_{y=y_{1,2}} = 0, \Rightarrow \hat{\eta}|_{y=y_{1,2}} = 0$$

# Integral estimate

## Integration over $y$

$$\int_{y_1}^{y_2} dy \left[\hat{\eta}^*(y)\left(\hat{\eta}''(y) - \left[k^2 + \frac{U''(y)}{U(y)-c}\right]\hat{\eta}(y)\right)\right] = 0 \tag{40}$$

Integration by parts + boundary conditions:

$$\int_{y_1}^{y_2} dy \left(\hat{\eta}^{*\prime}(y)\hat{\eta}'(y) + \left[k^2 + \frac{U''(y)}{U(y)-c}\right]\hat{\eta}^*(y)\hat{\eta}(y)\right) = 0 \tag{41}$$

## Imaginary part:

Only in phase velocity $\Rightarrow$

$$c_i \int_{y_1}^{y_2} dy \, \frac{U''(y)}{|U(y)-c|^2}\hat{\eta}^*(y)\hat{\eta}(y) = 0$$


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# Rayleigh criterion of barotropic instability

$$\int_{y_1}^{y_2} dy \, \frac{U''(y)}{|U(y) - c|^2} \hat{\eta}^*(y)\hat{\eta}(y) = 0, \quad \text{if} \;\; c_i \neq 0 \Rightarrow$$

In the absence of critical levels ($U(y) - c \neq 0$), if the flow is unstable, then $U(y)$ has inflexion point
$\exists y_0 : U''(y_0) = 0$.


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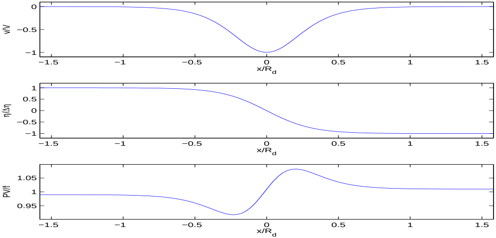
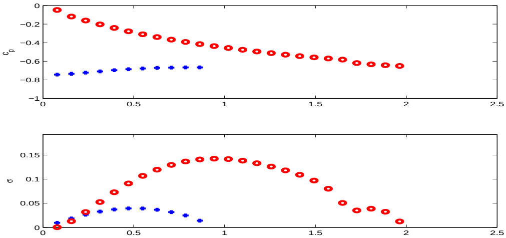
# Example: barotropic instability of a meridional jet on the $f$-plane in RSW




Jet in geostrophic equilibrium. From top to bottom: meridional velocity, geopotential, and PV.

# Dispersion relation and growth rate




Phase velocity (top) and growth rate (bottom) of two most unstable modes.

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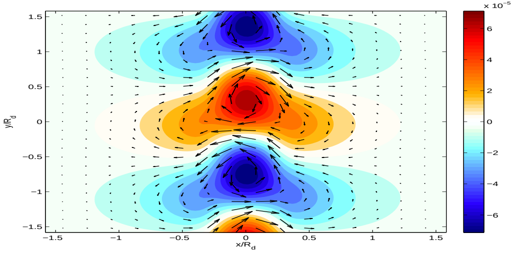
# The most unstable mode




Pressure and velocity anomalies of the most unstable mode.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# Nonlinear evolution of the instability (relative vrticity)