Waves in the presence of coasts and at the Equator

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Special places: coasts and Equator

What changes near the coasts?

A very idealized coast: a straight wall.
Change: boundary conditions $\leftrightarrow$ normal velocity vanishes, in the absence of dissipation.
Consequence: homogeneity in the cross-coast direction is broken $\Rightarrow$ Fourier transformation compromised.

What changes at the equator?

Change: Coriolis parameter in the tangent plane at the Equator (*equatorial beta-plane*) has no constant part

$$f = \beta y$$

Consequence: coordinate - dependent coefficients in the equations of motion $\Rightarrow$ Fourier transformation impossible.

Conclusion: analysis of the linearized equations to be revisited. Below: linear wave analysis using RSW model.

Linearised RSW with a lateral boundary

Setup: non-dissipative 1-layer RSW equations in a half-plane with a rectilinear meridional boundary at $x = 0$. Linearised non-dimensional RSW equations:

$$\begin{aligned} u_t - v + \eta_x &= 0, \\ v_t + u + \eta_y &= 0, \\ \eta_t + u_x + v_y &= 0 \end{aligned} \tag{1}$$

Rectlinear meridional west coast: b.c.: $u|_{x=0} = 0$.
Inhomogeneity in x, but Fourier-transform in y, t possible:

$$(u, v, \eta) = (\bar{u}_0(x), \bar{v}_0(x), \bar{h}_0(x))e^{i(ly-\omega t)} \Rightarrow$$

$$\begin{aligned} -i\omega\bar{u}_0 - \bar{v}_0 + \bar{h}_0' &= 0, \\ -i\omega\bar{v}_0 + \bar{u}_0 + il\bar{h}_0 &= 0, \\ -i\omega\bar{h}_0 + il\bar{v}_0 + \bar{u}_0' &= 0, \end{aligned} \tag{2}$$

Reduction to a single equation ($\omega \neq 1$)

$$\bar{h}_0'' + (\omega^2 - 1 - l^2)\bar{h}_0 = 0, \tag{3}$$

while

$$\bar{u}_0 = i\frac{l\bar{h}_0 - \omega\bar{h}_0'}{\omega^2 - 1}, \tag{4}$$

and hence the b. c. is:

$$l\bar{h}_0 - \omega\bar{h}_0'\big|_{x=0} = 0. \tag{5}$$

Solutions of two different types:

- **Free** inertia-gravity waves:

$$\omega^2 - 1 - l^2 \equiv k^2 > 0, \tag{6}$$

$$\bar{h}_0 \propto e^{\pm ikx}, \quad \omega^2 = 1 + k^2 + l^2. \tag{7}$$

- **Trapped** at the boundary waves:

$$\omega^2 - 1 - l^2 \equiv -\kappa^2 < 0, \tag{8}$$

$$\bar{h}_0 \propto e^{-\kappa x}. \tag{9}$$

The second type of solution is exponentially growing for $x < 0$, this is why it was discarded on the whole plane.

Trapped solutions - Kelvin waves

Kelvin waves are dispersionless. Boundary condition $\rightarrow$

$$l\bar{h}_0 - \omega \bar{h}_0'\big|_{x=0} = 0 \Rightarrow \kappa = -\frac{l}{\omega},$$
$$\Rightarrow \omega^2 - 1 - l^2 + \frac{l^2}{\omega^2} = 0, \Rightarrow \omega^2 = l^2 \; (\omega \neq 1), \quad (10)$$

and

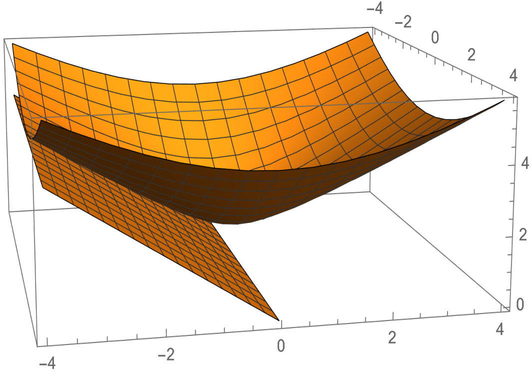
$$\kappa > 0 \Rightarrow \omega = -l, \quad \eta \propto e^{-x}. \quad (11)$$

Any packet of Kelvin waves:

$$(u, v, h) = (0, K(y+t), -K(y+t))e^{-x}, \quad (12)$$

where K - an arbitrary function, is a solution of linearised RSW equations. Kelvin waves are traveling along the boundary leaving it on their right. Normal to the boundary component of the velocity is absent, and the along-boundary velocity and height anomaly are in quadrature.

Dispersion diagram for RSW with a meridional boundary



Dispersion relation for internal-gravity and coastal Kelvin waves in the RSW model. Upper curved surface: inertia-gravity waves, lower plane: Kelvin waves.

Modification of inertia-gravity waves: reflexion

Boundary condition ⇒ "free" wave is a sum of incident and reflected waves:

$$(u, v, h) = (u_i, v_i, h_i) + (u_r, v_r, h_r)$$

$$\begin{aligned}(u_i, v_i, h_i) &= A_i \left(\frac{k\omega + il}{\omega^2 - 1}, \frac{l\omega - ik}{\omega^2 - 1}, 1 \right) e^{i(kx+ly-\omega t)} + \text{c.c.},\\ (u_r, v_r, h_r) &= A_r \left(\frac{-k\omega + il}{\omega^2 - 1}, \frac{l\omega + ik}{\omega^2 - 1}, 1 \right) e^{i(-kx+ly-\omega t)} + \text{c.c.}.\end{aligned}$$

Boundary condition:

$$u_i + u_r|_{x=0} = 0, \Rightarrow A_r = A_i \frac{k\omega + il}{k\omega - il}, \ \omega^2 = 1 + k^2 + l^2. \quad (13)$$

→ analog of Snell's law in optics.

RSW model on the equatorial β- plane

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \beta y \hat{\mathbf{z}} \wedge \mathbf{v} + g \nabla h = 0 \,. \tag{14}$$

$$\partial_t h + \nabla \cdot (\mathbf{v} h) = 0 \,, \tag{15}$$

Boundary conditions: decay at $y \to \pm\infty \leftrightarrow$ waveguide.
Characteristic scales:

- Spatial scale - equatorial deformation radius:
  $L \sim \left(\frac{\sqrt{gH}}{\beta} \right)^{\frac{1}{2}}$
- Time-scale - $T \sim (\beta L)^{-1}$
- Velocity scale - $U \sim \sqrt{gH}$;

Non-dimensional linearized system:

$$
\begin{aligned}
u_t - yv + h_x &= 0, && (16)\\
v_t + yu + h_y &= 0, && (17)\\
h_t + u_x + v_y &= 0. && (18)
\end{aligned}
$$

Useful change of variables:

$$f = \frac{1}{2}(u + h); \quad g = \frac{1}{2}(u - h). \qquad (19)$$

Equations (**??**) - (**??**) are simplified:

$$
\begin{aligned}
f_t + f_x + \frac{1}{2}(v_y - yv) &= 0, && (20)\\
g_t - g_x - \frac{1}{2}(v_y + yv) &= 0, && (21)\\
v_t + y(f + g) + (f - g)_y &= 0. && (22)
\end{aligned}
$$

Kelvin waves

Particular solution with $v \equiv 0 \Rightarrow$:

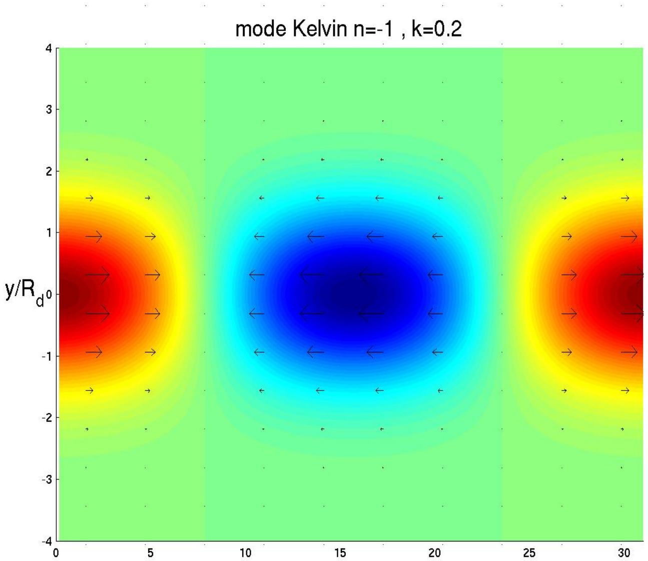
$$f_t + f_x = 0,\ g_t - g_x = 0,\ \Rightarrow f = F(x-t,y),\ g = G(x+t,y). \tag{23}$$

$$y(f+g) + (f-g)_y = 0,\ \Rightarrow F \propto e^{-\frac{y^2}{2}},\ G \propto e^{+\frac{y^2}{2}} \tag{24}$$

B.C. at $y \pm \infty\ \Rightarrow G \equiv 0 \Rightarrow$

$$u = F_0(x-t)e^{-\frac{y^2}{2}};\ \ h = F_0(x-t)e^{-\frac{y^2}{2}};\ \ v = 0. \tag{25}$$

Velocity and pressure distribution in a Kelvin wave



Yanai waves

Particular solution with $g = 0$, $f \neq 0$, $v \neq 0$. From (**??**) - (**??**) we get:

$$f_t + f_x + \frac{1}{2}(v_y - yv) = 0, \quad (26)$$
$$v_y + yv = 0, \quad (27)$$
$$v_t + yf + f_y = 0, \quad (28)$$

Solution by separation of variables:

$$v = v_0(x,t)\phi_0(y), \quad f = F_1(x,t)\phi_1(y), \quad (29)$$

where

$$\phi_n(y) = \frac{H_n(y)e^{-\frac{y^2}{2}}}{\sqrt{2^n n! \sqrt{\pi}}}, \quad (30)$$

and H_n - Hermite polynomials:

$$H_0 = 1, \;\; H_1 = 2y, \;\; H_2 = 4y^2 - 2, \;\; \ldots . \quad (31)$$

Equations for v_0 and F_1:

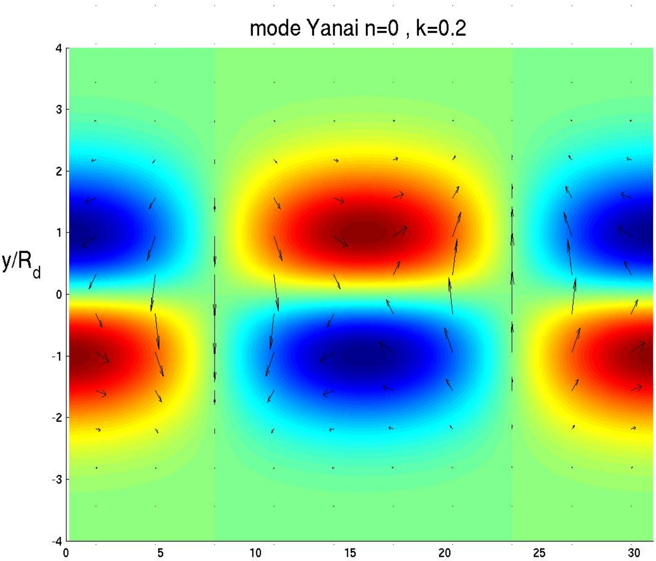
$$F_{1_t} + F_{1_x} - \frac{1}{\sqrt{2}} v_0 = 0, \quad v_{0_t} + \sqrt{2} F_1 = 0. \tag{32}$$

Dispersion relation:

Fourier-transformation $\propto e^{i(\omega t - kx)}$ $\rightarrow$ algebraic system for amplitudes. Solvability condition $\rightarrow$

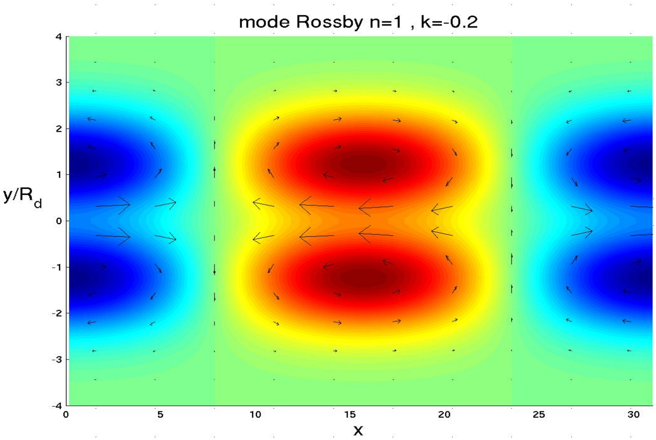
$$\omega = \frac{k}{2} \pm \sqrt{\frac{k^2}{4} + 1}, \tag{33}$$

Velocity and pressure distribution in a Yanai wave; eastward propagation



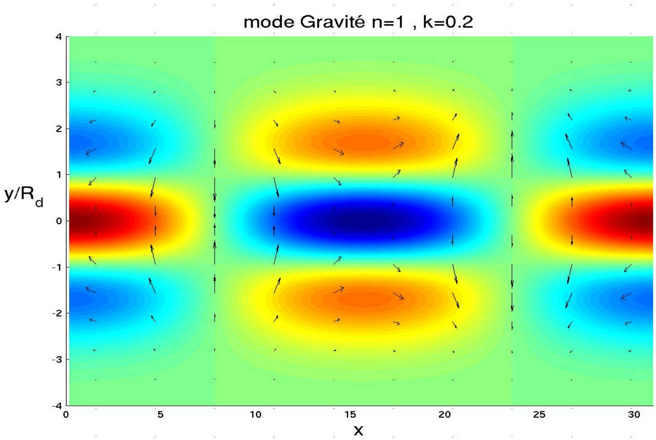
Rossby wave, propagation uniquely westward

As usual at the beta-plane Rossby waves exist, too (technically more difficult to demonstrate):



Inertia-gravity wave, eastward propagation

Gravity always present ⇒ inertia-gravity waves, too (technically more difficult to demonstrate):



Dispersion diagram for equatorial waves