Chapter 1: GFD models: reminder/derivations

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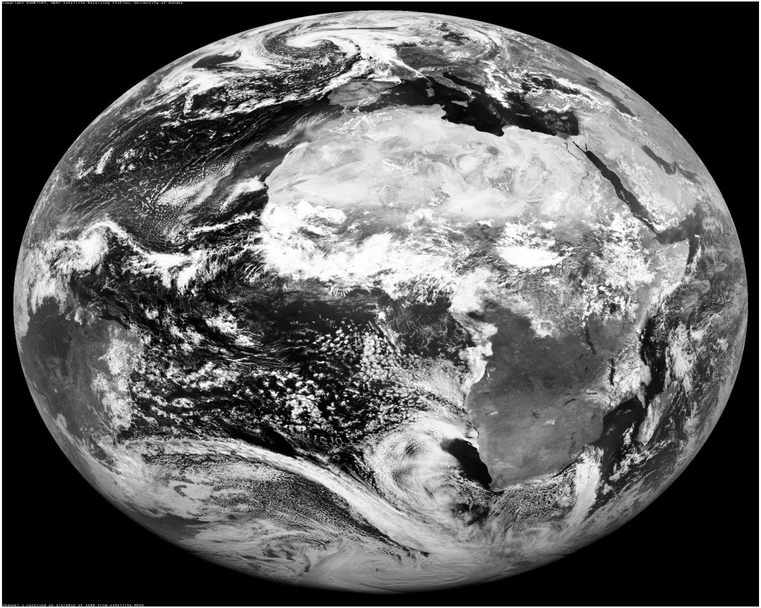
Cours GFD M2 MOCIS

Plan

GFD: space view



GFD: what's that?

Hydrodynamics in all its complexity plus:

- Rotating frame
- Variable density (thermal/stratification) effects
- Spherical geometry (large- and meso-scales)
- Fluid in the complex domains (coasts, topography/bathymetry)
- Multi-phase fluid (water vapor, ice)

But!
Some of these additional effects often allow to simplify the analysis

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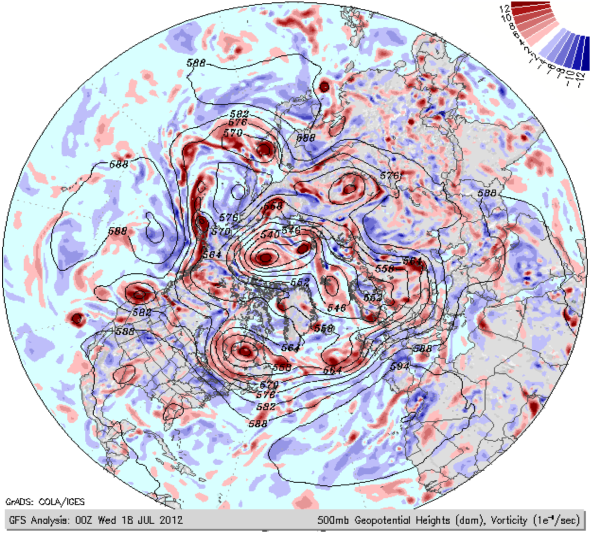
Scales:

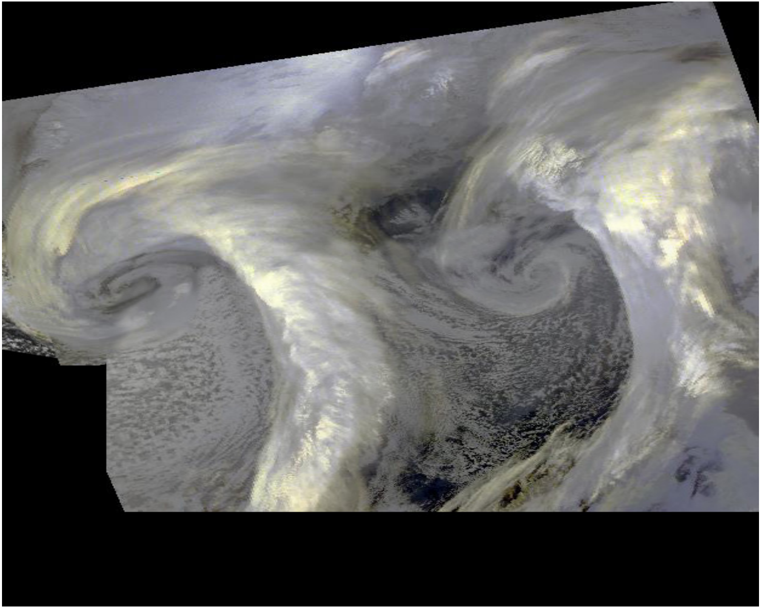
- Large: planetary 10^4 km
- Medium: atmosphere - synoptic, 10^3 km; ocean - meso-scale $10 - 10^2$ km
- Small: atmosphere - meso-scale $1 - 10$ km; ocean - sub-meso scale 1 km
- Very small: meters

Our prime interest: large and medium scales.

Dynamical actors: vortices, atmosphere



Atmospheric vortices for real



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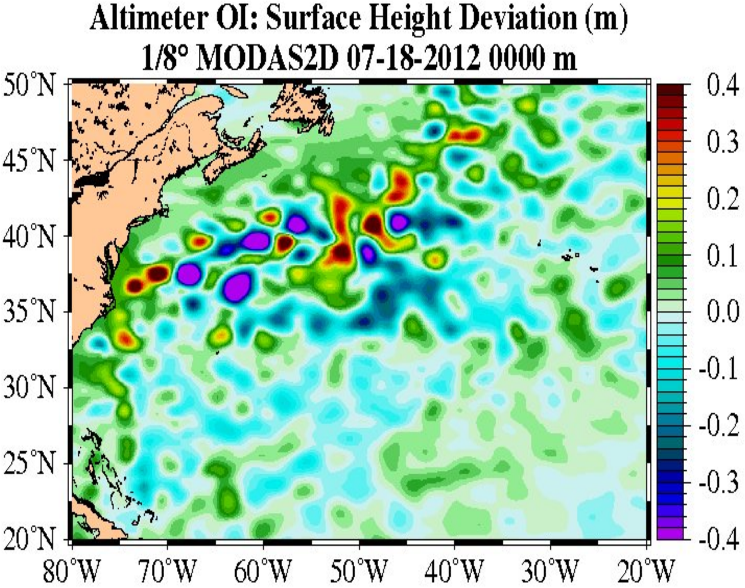

Dynamical actors: waves, atmosphere

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Where the governing equations come from:

- Mechanical system $\Rightarrow$ Newton's 2nd law $\leftrightarrow$ momentum conservation.
- Continuous medium $\Rightarrow$ local mass conservation
- Thermodynamical system $\Rightarrow$ 1st and 2nd laws of thermodynamics, equation of state

Principal difficulty - nonlinearity

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Example of essentially nonlinear process: wave breaking

Mathematical methods and tools

Related mathematical fields

- Linear algebra
- Partial differential equations
- Vector and tensor analyses
- Fourier analysis

Toolbox

- Method of small perturbations. Linearisation. Eigenproblems.
- Method of (time- and space-) averaging
- Asymptotic expansions, multi-scale analysis

Fluid dynamics according to Lagrange:

Description in terms of instantaneous positions of fluid parcels $\vec{X}(\vec{x}, t)$, along their trajectories, where $\vec{x}$ are initial positions (Lagrangian labels).
Newton's 2nd law:

$$\rho(\vec{X}, t)\frac{d^2\vec{X}}{dt^2} = -\vec{\nabla}P(\vec{X}, t). \tag{1}$$

Continuity equation:

$$\rho_i(x)d^3\vec{x} = \rho(\vec{X}, t)d^3\vec{X}, \leftrightarrow \rho_i(x) = \rho(\vec{X}, t)\mathcal{J} \tag{2}$$

where ρ_i is initial distribution of density of the fluide, $\mathcal{J} = \frac{\partial(X,Y,Z)}{\partial(x,y,z)}$ is the Jacobi determinant d(Jacobian). Fluid velocity: $\vec{v}(\vec{X}, t) = \frac{d\vec{X}}{dt} \equiv \dot{\vec{X}}$.

Fluid dynamics according to Euler:

Description in terms of instantaneous values of the velosity, density and pressure fields at the fixed point of space: $\vec{v}(\vec{x},t)$, $\rho(\vec{x},t)$, $P(\vec{x},t)$. Duality: $\vec{X} \leftrightarrow \vec{x}$
Newton's 2nd law:

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v}\right) = -\vec{\nabla}P. \tag{3}$$

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0. \tag{4}$$

Lagrangian derivative:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}. \tag{5}$$

Proposition

Lagrangian and Eulerian continuity equations are equivalent

Proof.

$$\frac{d}{dt}(\rho\mathcal{J}) = \frac{d\rho}{dt}\mathcal{J} + \rho\frac{d\mathcal{J}}{dt} = \frac{d\rho_i}{dt} = 0, \tag{6}$$

$$\begin{aligned}\frac{d\mathcal{J}}{dt} &= \frac{\partial(\dot{X},Y,Z)}{\partial(x,y,z)} + \frac{\partial(X,\dot{Y},Z)}{\partial(x,y,z)} + \frac{\partial(X,Y,\dot{Z})}{\partial(x,y,z)} \\ &= \left(\frac{\partial(\dot{X},Y,Z)}{\partial(X,Y,Z)} + \dots\right)\mathcal{J} = \left((\frac{\partial\dot{X}}{\partial X} + \frac{\partial\dot{Y}}{\partial Y} + \frac{\partial\dot{Z}}{\partial Z}\right)\mathcal{J} \Rightarrow\end{aligned}$$

$$\frac{d\rho}{dt} + \rho\vec{\nabla}\cdot\vec{v} = 0 \leftrightarrow \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0. \tag{7}$$

□

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□

Closure of the system: equation of state

General equation of state

$$P = P(\rho, s), \tag{8}$$

wher s - entropy per unit mass;

- Barotropic fluid:

$$P = P(\rho) \leftrightarrow s = \text{const}, \tag{9}$$

- Baroclinic fluid:

$$P = P(\rho, s), \Rightarrow \tag{10}$$

Equation for s neccessary. Perfect fluid:

$$\frac{ds}{dt} = \frac{\partial s}{\partial t} + \vec{v} \cdot \vec{\nabla} s = 0. \tag{11}$$

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Particular case of the barotropic fluid - incompressible fluid:

Volume conservation:

$$\mathcal{J} = 1 \leftrightarrow \vec{\nabla} \cdot \vec{v} = 0 \Rightarrow . \qquad (12)$$

pressure is not independente variable.

1. If in addition, $\rho = const$:

$$\vec{\nabla} \cdot \left(\vec{v} \cdot \vec{\nabla} \vec{v} \right) = -\frac{1}{\rho} \vec{\nabla}^2 P. \qquad (13)$$

2. Otherwise

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla} \rho = 0. \qquad (14)$$

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$$\vec{\nabla} \cdot \left(\vec{v} \cdot \vec{\nabla} \vec{v} \right) = -\vec{\nabla} \cdot \left(\frac{\vec{\nabla} P}{\rho} \right). \qquad (15)$$

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Thermodynamics: reminder

1st principle, "dry" thermodynamics

$$\delta\epsilon = T\delta s - P\delta v, \tag{16}$$

where ϵ - internal energy per unit mass, $v = \frac{1}{\rho}$ - volume per unit mass.
Enthalpy per unit mass: $h = \epsilon + Pv$:

$$\delta h = T\delta s + v\delta P. \tag{17}$$

Energy density of the fluid:

$$e = \frac{\rho\vec{v}^2}{2} + \rho\epsilon. \tag{18}$$

Local conservation of energy:

$$\frac{\partial e}{\partial t} + \vec{\nabla}\cdot\left[\rho\vec{v}\left(\frac{\vec{v}^2}{2} + h\right)\right] = 0. \tag{19}$$

Kelvin theorem

Circulation:

$$\gamma = \int_{\Gamma} \vec{v} \cdot d\vec{l} = \int_{S_{\Gamma}} \left(\vec{\nabla} \wedge \vec{v}\right) \cdot d\vec{s}, \qquad (21)$$

where Γ - arbitrary contour, S_{Γ} - surface with the boundary Γ.

Kelvin theorem

- Barotropic fluid

$$\frac{d\gamma}{dt} = 0, \qquad (22)$$

- Baroclinic fluid

$$\frac{d\gamma}{dt} = -\int_{\Gamma} \frac{\vec{\nabla} P}{\rho} \cdot d\vec{l}, \qquad (23)$$

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Exercise

- Prove energy conservation and Kelvin theorem for the barotropic fluid
- Same for the baroclinic fluid
- Write down, with demonstration, the Euler equations for the incompressible fluid in cylindrical coordinates

Dissipative phenomena: molecular fluxes

Effects of dissipation: correction of the macroscopic fluxes of:

- momentum
- mass
- internal energy (heat)

by the corresponding molecular fluxes, calculated from the flux - gradient relations:

$$\vec{f}_A = -k_A \vec{\nabla} A, \tag{24}$$

A - a thermodynamical variable, $\vec{f}_A$ - corresponding molecular flux.

Viscosity

Tensor notation

$$\vec{x} \to x_i, \quad \vec{v} \to v_i, \quad \vec{\nabla} \to \partial_i, \; i = 1, 2, 3. \qquad (25)$$

Einstein's convention: repeating indices - summation from 1 to 3.

Conservation of the momentum:

$$\partial_t(\rho v_i) + \partial_k \pi_{ik} = 0, \quad \pi_{ik} = \rho v_i v_k + P\delta_{ik}, \quad \delta_{ik} = \mathrm{diag}(1, 1, 1). \qquad (26)$$

Viscous tensions - (density of) the molecular flux of the momentum:

$$\sigma_{ik} = \nu\rho(\partial_i v_k + \partial_k v_i) \Rightarrow \partial_t(\rho v_i) + \partial_k(\pi_{ik} - \sigma_{ik}) = 0, \qquad (27)$$

Incompressible case: Navier -Stokes (NS) equation

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} = -\frac{\vec{\nabla} P}{\rho} + \nu \vec{\nabla}^2 \vec{v}, \; \vec{\nabla} \cdot \vec{v} = 0. \tag{28}$$

Reynolds' number

Dimensionless form of the NS equation:

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} = -\frac{\vec{\nabla} P}{\rho} + \frac{1}{Re} \vec{\nabla}^2 \vec{v}, \tag{29}$$

$Re = UL/\nu$, U, L -typical velocity- and length-scales.

Remarque: typical Re for synoptic motions $\to \infty$

Diffusivity, thermal conductivity

Molecular fluxes of mass and heat:

$$-D\vec{\nabla}\rho, \qquad -\kappa\vec{\nabla}T \tag{30}$$

Corrected continuity equation:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = D\vec{\nabla}^2\rho. \tag{31}$$

Equation of heat/temperature

$$\frac{\partial T}{\partial t} + \vec{v}\cdot\vec{\nabla}T = \chi\vec{\nabla}^2 T. \tag{32}$$

Motion in a frame rotating with angular velocity $\vec{\Omega}$

Material point in the rotating frame:

$$m\frac{d\vec{v}}{dt} + 2m\vec{\Omega} \wedge \vec{v} + m\vec{\Omega} \wedge \left(\vec{\Omega} \wedge \vec{x}\right) = \vec{F}, \quad \vec{v} = \frac{d\vec{x}}{dt} \qquad (33)$$

m- mass, $\vec{x}$-current position of the point, $\vec{F}$ - sum of forces acting on the point

Euler equations in the rotating frame +gravity:

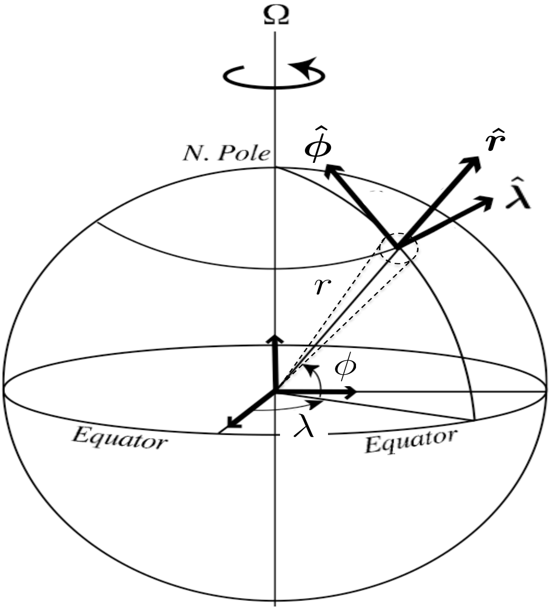
Fluid under the influence of gravity: $m \to \rho$, $\frac{d}{dt} \to \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}$, forces: pressure + gravity $\Rightarrow$

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla}\vec{v} + 2\vec{\Omega} \wedge \vec{v} = -\frac{\vec{\nabla}P}{\rho} + \vec{g}^* \qquad (34)$$

Effective gravity: gravity + centrifugal acceleration (also potential)

$$\vec{g}^* = \vec{g} - \vec{\Omega} \wedge \left(\vec{\Omega} \wedge \vec{x}\right) \qquad (35)$$

Spherical coordinates



Euler and continuity equations

$$\frac{dv_r}{dt} - \frac{v_\lambda^2 + v_\phi^2}{r} - 2\Omega\cos\phi v_\lambda + g^* = -\frac{1}{\rho}\partial_r P,$$

$$\frac{dv_\lambda}{dt} + \frac{v_r v_\lambda - v_\phi v_\lambda \tan\phi}{r} + 2\Omega\left(-\sin\phi v_\phi + \cos\phi v_r\right) = -\frac{1}{\rho r\cos\phi}\partial_\lambda P,$$

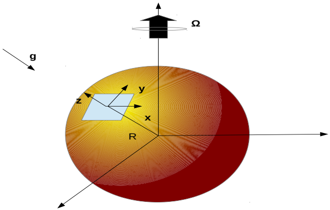
$$\frac{dv_\phi}{dt} + \frac{v_r v_\phi + v_\lambda^2\tan\phi}{r} + 2\Omega\sin\phi v_\lambda = -\frac{1}{\rho r}\partial_\phi P,$$

$$\frac{d\rho}{dt} + \rho\left[\frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\cos\phi}\left(\frac{\partial(\cos\phi v_\phi)}{\partial\phi} + \frac{\partial v_\lambda}{\partial\lambda}\right)\right],$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_r\partial_r + \frac{v_\phi}{r}\partial_\phi + \frac{v_\lambda}{r\cos\phi}\partial_\lambda$$

Traditional approx.: green + red $\rightarrow$ out, $r \rightarrow R = \text{const}$
Non-traditional approx: green $\rightarrow$ out.

Tangent plane approximation



$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} + f \hat{z} \wedge \vec{v} = -\frac{\vec{\nabla} P}{\rho} + \vec{g} \qquad (36)$$

f - plane: $f = const$; β - plane: $f = f + \beta y$; f - Coriolis parameter: $f = 2\Omega \sin \phi = 2\Omega \sin \theta$

Hydrostatics. Stratification

The state of rest $\vec{v} \equiv 0$ is solution of (36) if hydrostatic equilibrium holds:

$$0 = -\frac{\vec{\nabla} P}{\rho} + \vec{g}$$

The continuity equation with time-independent ρ

$$\frac{d\rho}{dt} + \rho \vec{\nabla} \cdot \vec{v} = 0$$

is satisfied in a state of rest.
Statically stable states: $\rho = \rho_0(z)$, $\rho_0'(z) \leq 0 \rightarrow$

$$P = P_0(z) = -\int dz\, g\, \rho_0(z)$$

Dependence of ρ_0 on z is called stratification.

Exercises

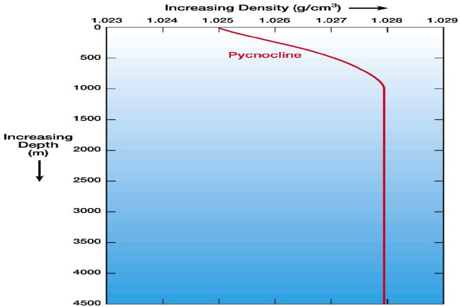
Deduce Euler and continuity equations in spherical coordinates.

Determine conditions of validity of the tangent plane approximation.

By considering displacements of fluid parcels from their positions in the state of rest, demonstrate qualitatively that stratifications with $\rho_0'(z) > 0$ are unstable;

Oceanic stratification

Typical density profile:



$$\rho(\vec{x}, t) = \rho_0 + \rho_s(z) + \sigma(x, y, z; t), \quad \rho_0 \gg \rho_s \gg \sigma. \qquad (37)$$

Hydrostatic approximation for large-scale motions:

$$g\rho + \partial_z P = 0, \Rightarrow P = P_0 + P_s(z) + \pi(x, y, z; t), \qquad (38)$$

Approximations. Non-dissipative equations of motion

Boussinesq approximation

Deviations of density from ρ_0 neglected in the horizontal $\rightarrow$

$$\frac{\partial \vec{v}_h}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v}_h + f \hat{z} \wedge \vec{v}_h = -\frac{\vec{\nabla}_h \pi}{\rho} \approx -\vec{\nabla}_h \phi, \tag{39}$$

where $\phi = \frac{\pi}{\rho_0}$ - geopotential.

Incompressibility of water

Continuity equation splits in two:

$$\vec{\nabla} \cdot \vec{v} = 0, \quad \vec{v} = \vec{v}_h + \hat{\boldsymbol{z}} w. \tag{40}$$

$$\partial_t \rho + \vec{v} \cdot \vec{\nabla} \rho = 0. \tag{41}$$

Vertical boundary conditions

Most often sufficient for our purposes: rigid lid and flat bottom:

$$w|_{z=0} = w|_{z=H} = 0 \qquad (42)$$

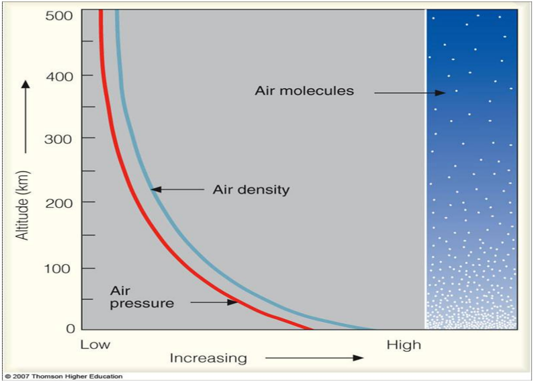
Non-trivial bathymetry : fluid parcels follow the bottom profile

$$w|_{z=b(x,y)} = \frac{db}{dt} = \vec{v} \cdot \vec{\nabla} b$$

Free surface: fluid parcels move with the surface:

$$w|_{z=h(x,y;t)} = \frac{dh}{dt} = \frac{\partial h}{\partial t} + \vec{v} \cdot \vec{\nabla} h$$

Atmosphere: pressure coordinates



Altitude $\leftrightarrow$ Pressure $\Rightarrow$ vertical coordinate.

Thermodynamics of the dry atmosphere

Equation of state - ideal gas:

$$P = \rho R T, \;\; c_{P,V} = T\left(\frac{\partial s}{\partial T}\right)_{P,V} = const, \;\; c_p - c_v = R. \tag{43}$$

Entropy:

$$s = c_p \ln T - R \ln P + const. \tag{44}$$

Adiabatic process:

$$s = \mathrm{const} \;\Rightarrow c_p \frac{dT}{T} - R\frac{dP}{P} = 0, \Rightarrow T = T_s \left(\frac{P}{P_s}\right)^{\frac{R}{c_p}}. \tag{45}$$

Potential temperature :

$$\theta = T\left(\frac{P_s}{P}\right)^{\frac{R}{c_p}}, \, s = c_p \ln\theta + \mathrm{const}. \tag{46}$$

Geopotential and hydrostatics

Geopotential variation: work to lift a unit mass against gravity: $\delta\phi = g\delta z$.
$z = z(p)$ becomes a thermodynamical variable.
Hydrostatic approximation:

$$\delta\phi = -\frac{RT}{P}\delta P \Rightarrow \tag{47}$$

$$\frac{\partial\phi}{\partial P} = -\frac{RT}{P} = -\frac{1}{\rho}. \tag{48}$$

Useful relation for small variations ρ, P, θ with respect to background ρ_0, P_0, θ_0:

$$\theta = \theta_0 \left[\frac{\left(1 - \frac{R}{c_p}\right) P}{P_0} - \frac{\rho}{\rho_0} \right] \tag{49}$$

Elimination of ρ in Euler equations

"Triangular" relation :

$$\left(\frac{\partial P}{\partial x}\right)_z \left(\frac{\partial x}{\partial z}\right)_P \left(\frac{\partial z}{\partial P}\right)_x = -1 \;\Rightarrow \tag{50}$$

$$\left(\frac{\partial P}{\partial x}\right)_z = -\left(\frac{\partial P}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_P = \rho \left(\frac{\partial \phi}{\partial x}\right)_P . \tag{51}$$

Incompressibility in pressure coordinates

Lagrangian volume element in pressure coordinates:

$$\rho dxdydz = -\frac{1}{g} dxdydP \tag{52}$$

Mass conservation $\Rightarrow$ Volume conservation in P.

Adiabatic primitive equations

Equations of motion

$$div(\vec{v}) = \vec{\nabla}_h \cdot \vec{v}_h + \partial_p \omega = 0, \;\; \omega = \frac{dP}{dt}. \tag{53}$$

$$\frac{\partial \vec{v}_h}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v}_h + f \hat{\boldsymbol{z}} \wedge \vec{v}_h = -\vec{\nabla}_h \phi. \tag{54}$$

$$\partial_t \theta + \vec{v} \cdot \vec{\nabla} \theta = 0. \tag{55}$$

$$\frac{\partial \phi}{\partial P} = -\frac{RT}{P} = -\frac{R}{P} \left(\frac{P}{P_s} \right)^{\frac{R}{c_p}} \theta. \tag{56}$$

Boundary conditions

Bottom: ground $\equiv$ free surface in terms of pressure, geopotential fixed.
Top: rigid lid $\equiv$ fixed value of pressure, e.g. tropopause.

"Pseudo-height" coordinate

New vertical coordinate:

$$\bar{z} = z_0 \left(1 - \left(\frac{P}{P_s} \right)^{\frac{R}{c_p}} \right) \equiv z_0 \left(1 - \left(\frac{P}{P_s} \right)^{\frac{\gamma-1}{\gamma}} \right), \qquad (57)$$

$$z_0 = \frac{\gamma}{\gamma - 1} \frac{P_s}{g \rho_s} \approx 28\text{km}. \qquad (58)$$

Pseudo- density:

$$r: \quad r d\bar{z} = \rho dz = -\frac{1}{g} dP. \qquad (59)$$

Mass conservation:

$$dxdydP = -gr(\bar{z})dxdyd\bar{z} \Rightarrow \tag{60}$$

$$r\left(\vec{\nabla}_h \cdot \vec{v}_h + \frac{\partial \bar{w}}{\partial \bar{z}}\right) + \bar{w}\frac{\partial r}{\partial \bar{z}} = 0, \quad \vec{v} = (\vec{v}_h, \bar{w} = \dot{\bar{z}}). \tag{61}$$

Approximation $\bar{z} \ll z_0$:

$$\vec{\nabla}_h \cdot \vec{v}_h + \frac{\partial \bar{w}}{\partial \bar{z}} = -\frac{\bar{w}}{r}\frac{\partial r}{\partial \bar{z}} = \frac{\bar{w}}{(\gamma - 1)z_0\left(1 - \frac{\bar{z}}{z_0}\right)} \approx 0. \tag{62}$$

Equations of motion

$$\frac{\partial \vec{v}_h}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v}_h + f \hat{z} \wedge \vec{v}_h = -\vec{\nabla}_h \phi, \tag{63}$$

$$-g \frac{\theta}{\theta_0} + \frac{\partial \phi}{\partial \bar{z}} = 0, \tag{64}$$

$$\frac{\partial \theta}{\partial t} + \vec{v} \cdot \vec{\nabla} \theta = 0; \quad \vec{\nabla} \cdot \vec{v} = 0. \tag{65}$$

Identical to oceanic equations with $\sigma \to -\theta$.

Isentropic coordinates

Montgomery potential

$$\psi = \phi - \frac{\theta}{\theta_0} g \bar{z}, \Rightarrow \tag{66}$$

$$\begin{aligned} d\psi &= d\phi - g\frac{\theta}{\theta_0}\, d\bar{z} - g\bar{z}\, d\frac{\theta}{\theta_0} \\ &= \vec{\nabla}_h \phi \cdot d\vec{x}_h + \partial_{\bar{z}}\phi d\bar{z} - \frac{\theta}{\theta_0} g\, d\bar{z} - g\bar{z}\, d\frac{\theta}{\theta_0} \\ &= \vec{\nabla}_h \phi \cdot d\vec{x}_h - g\bar{z} d\frac{\theta}{\theta_0} \end{aligned} \tag{67}$$

Therefore:

$$\left(\vec{\nabla}_h \psi\right)_\theta = \left(\vec{\nabla}_h \phi\right)_{\bar{z}};\ \partial_\theta \psi = -g\bar{z}/\theta_0 \tag{68}$$

and $\bar{z}$ is a new dependent variable.

Velocity and continuity equations in isentropic coordinates

Velocity

$$\vec{v} \to \left(\vec{v}_h, \tilde{w} = \frac{d\theta}{dt} \right) \Rightarrow \tag{69}$$

$\tilde{w} \equiv 0$ for adiabatic processes

Mass conservation

$$dxdyd\bar{z} = \frac{\partial \bar{z}}{\partial \theta} dxdyd\theta = const \to \tag{70}$$

$$\partial_t \left(\frac{\partial \bar{z}}{\partial \theta} \right) + \vec{\nabla}_h \cdot \left(\frac{\partial \bar{z}}{\partial \theta} \vec{v}_h \right) = 0. \tag{71}$$

Complete equations, adiabatic motions

$$\frac{\partial \vec{v}_h}{\partial t} + \vec{v}_h \cdot \vec{\nabla}_h \vec{v}_h + f\hat{z} \wedge \vec{v}_h = -\vec{\nabla}_h \psi, \qquad (72)$$

$$+\frac{g\bar{z}}{\theta_0} + \frac{\partial \psi}{\partial \theta} = 0, \qquad (73)$$

$$\partial_t \left(\frac{\partial \bar{z}}{\partial \theta} \right) + \vec{\nabla}_h \cdot \left(\frac{\partial \bar{z}}{\partial \theta} \vec{v}_h \right) = 0. \qquad (74)$$

Boussinesq approximation for atmosphere

Background density ρ_0: Ocean - $\rho_0 = \text{const}$, Atmosphere: $\rho_0 = \rho_0(z)$; Boussinesq approximation in x, y, z coordinates, with $\rho = \rho_0(z) + \tilde{\rho}$, $P = P_0(z) + \tilde{p}$, $\theta = \theta_0(z) + \tilde{\theta}$, $(\tilde{\ldots})$ omitted below:

$$\frac{\partial \vec{v}_h}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v}_h + f \hat{\boldsymbol{z}} \wedge \vec{v}_h = -\vec{\nabla}_h \phi, \tag{75}$$

with geopotential $\phi = \frac{p}{\rho_0}$. Vertical motion (non-hydrostatic):

$$\frac{\partial w}{\partial t} + \vec{v} \cdot \vec{\nabla} w = -\frac{\partial \phi}{\partial z} - \frac{p}{\rho_0^2} \frac{\partial \rho_0}{\partial z} - g \frac{\rho}{\rho_0} \tag{76}$$

Equation of state (ideal gas) + (49) $\rightarrow$

$$\frac{\partial w}{\partial t} + \vec{v} \cdot \vec{\nabla} w = -\frac{\partial \phi}{\partial z} + b \tag{77}$$

$b = g \frac{\theta}{\theta_0}$ - buoyancy, $\frac{\partial b}{\partial t} + \vec{v} \cdot \vec{\nabla} b = 0$ for adiabatic motions. Continuity equation $\rightarrow$ anelastic equation:

$$\vec{\nabla} \cdot (\rho_0(z) \vec{v}) = 0.$$

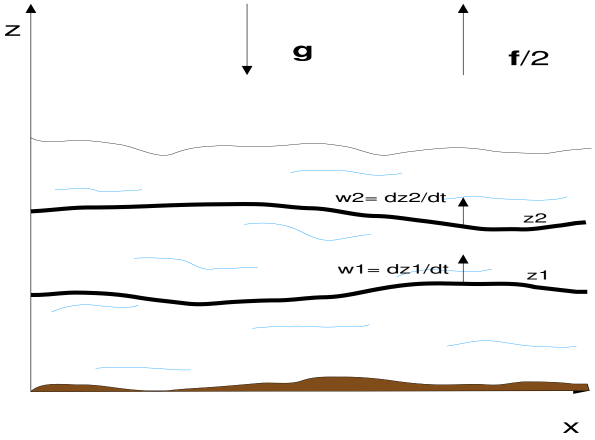
Conservative form of equations of motion

Full Euler equations + continuity equation $\rightarrow$

$$\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_y(\rho v u) + \partial_z(\rho w u) - f\rho v = -\partial_x p, \quad (78)$$

$$\partial_t(\rho v) + \partial_x(\rho u v) + \partial_y(\rho v^2) + \partial_z(\rho w v) + f\rho u = -\partial_y p, \quad (79)$$

Material surfaces



Vertical integration

Integration between two material surfaces $z_{1,2}$.

By definition of material surface:

$$w|_{z_i} = \frac{dz_i}{dt} = \partial_t z_i + u\partial_x z_i + v\partial_y z_i, \quad i = 1, 2. \tag{80}$$

Leibniz formula:

$$\int_{z_1}^{z_2} dz \partial_x F = \partial_x \int_{z_1}^{z_2} dz F - \partial_x z_2 \left. F \right|_{z_2} + \partial_x z_1 \left. F \right|_{z_1} \tag{81}$$

Integrated equations

Using (80) and (81) we obtain:

$$\partial_t \int_{z_1}^{z_2} dz\rho u + \partial_x \int_{z_1}^{z_2} dz\rho u^2 + \partial_y \int_{z_1}^{z_2} dz\rho uv$$
$$- \quad f \int_{z_1}^{z_2} dz\rho v = -\partial_x \int_{z_1}^{z_2} dzp - \partial_x z_1 \; p|_{z_1} + \partial_x z_2 \; p|_{z_2} .$$

$$\partial_t \int_{z_1}^{z_2} dz\rho v + \partial_x \int_{z_1}^{z_2} dz\rho uv + \partial_y \int_{z_1}^{z_2} dz\rho v^2$$
$$+ \quad f \int_{z_1}^{z_2} dz\rho u = -\partial_y \int_{z_1}^{z_2} dzp - \partial_y z_1 \; p|_{z_1} + \partial_y z_2 \; p|_{z_2} .$$

Continuity equation:

$$\partial_t \int_{z_1}^{z_2} dz\rho + \partial_x \int_{z_1}^{z_2} dz\rho u + \partial_y \int_{z_1}^{z_2} dz\rho v = 0. \quad (82)$$

Integrated density:

$$\mu = \int_{z_1}^{z_2} dz\rho = -\frac{1}{g}\left(p|_{z_2} - p|_{z_1} \right), \quad (83)$$

Density-weighted vertical average:

$$\langle F \rangle = \frac{1}{\mu} \int_{z_1}^{z_2} dz\rho F. \quad (84)$$

Equations for the averages:

$$\begin{aligned}\partial_t\left(\mu\langle u\rangle\right) &+ \partial_x\left(\mu\langle u^2\rangle\right)+\partial_y\left(\mu\langle uv\rangle\right)-f\mu\langle v\rangle = \\ &- \partial_x\int_{z_1}^{z_2} dzp-\partial_x z_1\ p\big|_{z_1}+\partial_x z_2\ p\big|_{z_2}, \quad (85)\end{aligned}$$

$$\begin{aligned}\partial_t\left(\mu\langle v\rangle\right) &+ \partial_x\left(\mu\langle uv\rangle\right)+\partial_y\left(\mu\langle v^2\rangle\right)+f\mu\langle u\rangle = \\ &- \partial_y\int_{z_1}^{z_2} dzp-\partial_y z_1\ p\big|_{z_1}+\partial_y z_2\ p\big|_{z_2}, \quad (86)\end{aligned}$$

$$\partial_t\mu+\partial_x\left(\mu\langle u\rangle\right)+\partial_y\left(\mu\langle v\rangle\right)=0. \quad (87)$$

Hydrostatics and mean-field approximation:

Hydrostatic pressure:

Pressure inside the layer (z_1, z_2) in terms of pressure at the lower surface and vertical position:

$$p(x, y, z, t) = -g \int_{z_1}^{z} dz' \rho(x, y, z', t) + p|_{z_1} . \qquad (88)$$

Closure hypothesis: mean-field $\equiv$ columnar motion

Weak variations in the vertical, correlations decoupled:

$$\langle uv \rangle \approx \langle u \rangle \langle v \rangle, \ \langle u^2 \rangle \approx \langle u \rangle \langle u \rangle, \ \langle v^2 \rangle \approx \langle v \rangle \langle v \rangle. \qquad (89)$$

Approximate equations

Mean density:

Mean density $\bar{\rho}$:

$$\bar{\rho} = \frac{1}{(z_2 - z_1)} \int_{z_1}^{z_2} dz \rho, \quad \mu = \bar{\rho}(z_2 - z_1). \tag{90}$$

Pressure in terms of $\bar{\rho}$:

$$p(x, y, z, t) \approx -g\bar{\rho}(z - z_1) + p|_{z_1} . \tag{91}$$

Hypothesis: $\bar{\rho} = \text{const}$ in what follows.

Omitting the brackets we obtain for the averages from (85), (86), (89), (91), with the help of (87), (90):

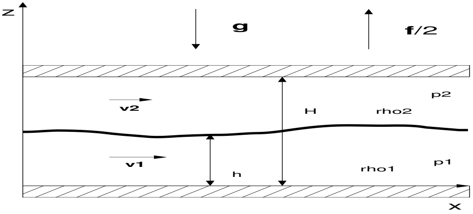
$$\bar{\rho}(z_2 - z_1)(\partial_t \mathbf{v}_h + \mathbf{v}_h \cdot \nabla_h \mathbf{v}_h + f\hat{\mathbf{z}} \wedge \mathbf{v}_h) = \\ -\nabla_h \left(-g\bar{\rho}\frac{(z_2 - z_1)^2}{2} + (z_2 - z_1) \left. p\right|_{z_1} \right) \\ -\nabla_h z_1 \left. p\right|_{z_1} + \nabla_h z_2 \left. p\right|_{z_2} . \tag{92}$$

Any variable in this equation is a fonction only of horizontal coordinates and time. Alternative notation: $\vec{v}_h \equiv \mathbf{v}_h$.

Rotating shallow water (RSW), 2 layers

2- layer configuration, rigid lid

Application of general equations (92) to the fluid between the bottom $z_1 = 0$ and the top $z_3 = H$ planes. Choose a material surface $z = z_2(x, y, t) \equiv h(x, y, t)$ in the fluid interior, $\vec{\nabla}_h \to \vec{\nabla}$, $\vec{v}_h \to \mathbf{v}$. Vertical boundaries - material surfaces. Generalisation to non-trivial topography: $z_1 = b(x, y)$.



Equations of motion

$\mathbf{v}_{1(2)}, \bar{\rho}_{1(2)}$ - velocity and density in the inferior (superior) layer.

$$\partial_t \mathbf{v}_2 + \mathbf{v}_2 \cdot \nabla \mathbf{v}_2 + f\hat{\mathbf{z}} \wedge \mathbf{v}_2 = -\frac{1}{\bar{\rho}_2} \nabla \left. p \right|_H \tag{93}$$

$$\partial_t \mathbf{v}_1 + \mathbf{v}_1 \cdot \nabla \mathbf{v}_1 + f\hat{\mathbf{z}} \wedge \mathbf{v}_1 = -\frac{1}{\bar{\rho}_1} \nabla \left. p \right|_H - g\frac{\bar{\rho}_1 - \bar{\rho}_2}{\bar{\rho}_1} \nabla h, \tag{94}$$

$$\partial_t h + \nabla \cdot (\mathbf{v}_1 h) = 0\,, \tag{95}$$

$$\partial_t (H - h) + \nabla \cdot (\mathbf{v}_2 (H - h)) = 0\,, \tag{96}$$

Re-interpretation of the equations

2-dimensional Euler equations in each layer with dynamical boundary condition at the interface:

$$p_1 = (\bar{\rho}_1 - \bar{\rho}_2) g h + p_2 \,, \tag{97}$$

Reduced gravity

Remark: g enter equations uniquely in combination $g\frac{\bar{\rho}_1 - \bar{\rho}_2}{\bar{\rho}_1}$ - reduced gravity

Exercise

Deduce the equations (93) - (96).

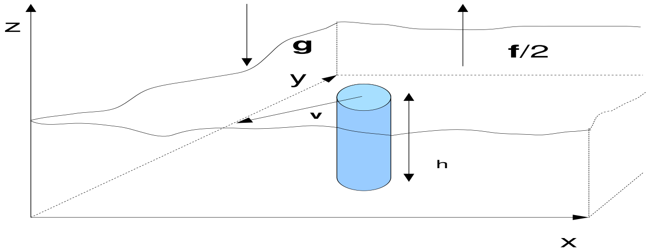
1-layer rotating shallow water model (Saint-Venant)

In the limit $\bar{\rho}_2 \to 0$:

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + f \hat{\mathbf{z}} \wedge \mathbf{v} + g \nabla h = 0 \,, \tag{98}$$

$$\partial_t h + \nabla \cdot (\mathbf{v} h) = 0 \,, \tag{99}$$

In the presence of non-trivial topography $h \to h - b(x, y)$ in the second equation.



Conservation laws, RSW model

Energy

By construction, equations (98), (99) express the local conservation of the horizontal momentum and mass. Energy density:

$$e = h\frac{\mathbf{v}^2}{2} + g\frac{h^2}{2} \tag{100}$$

obeys the conservation equation:

$$\partial_t e + \nabla \cdot \left(\mathbf{v} h \left(\frac{\mathbf{v}^2}{2} + gh \right) \right) = 0, \tag{101}$$

and total energy, $E = \int dxdy\, e$, is constant for an isolated system.

Potential vorticity, RSW model

Specific Lagrangian conservation law: potential vorticity q (PV), constructed from the relative vorticity (vertical component) $\zeta = v_x - u_y$, Coriolis le parametre f, and fluid depth h.

$$q = \frac{\zeta + f}{h}. \tag{102}$$

here $\zeta + f$ -absolute vorticity , f - planetary vorticity.

Lagrangian conservation:

$$\frac{dq}{dt} \equiv (\partial_t + \mathbf{v} \cdot \nabla)\, q = 0, \tag{103}$$

is obtained by combining equations of vorticity:

$$\frac{d(\zeta + f)}{dt} + (\zeta + f)\nabla \cdot \mathbf{v} = 0, \tag{104}$$

and continuity

$$\frac{dh}{dt} + h\nabla \cdot \mathbf{v} = 0\ : \tag{105}$$

$$\frac{d}{dt}\frac{\zeta + f}{h} = \frac{1}{h}\frac{d}{dt}(\zeta + f) - \frac{\zeta + f}{h^2}\frac{d}{dt}h = 0, \tag{106}$$

Eulerian expression:

Conservation of PV leads to independence of time of any integral:

$$\int dxdy\, h\mathcal{F}(q), \qquad (107)$$

over the whole flow, with $\mathcal{F}$ - arbitrary function.

Qualitative image of the RSW dynamics:

Two-dimensional motion of the fluid columns of variable depth, each preserving its potential vorticity.

Spectrum of small perturbations - RSW model

Linearized equations :

Perturbations about state of rest: $\mathbf{v} = 0,\ h = H_0 = const$.
Linéarised equations in the approximation $f = f_0 = const$:

$$\begin{aligned} u_t - fv + g\eta_x &= 0, \\ v_t + fu + g\eta_y &= 0, \\ \eta_t + H_0(u_x + v_y) &= 0, \end{aligned} \tag{108}$$

where u, v - 2 components of the velocity perturbation, η - perturbation of the interface.

Method of Fourier

Solutions - harmonic waves:

$$(u, v, \eta) = (u_0, v_0, \eta_0) e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} + \text{c.c.}, \qquad (109)$$

where ω and $\mathbf{k}$ are frequency and wavenumber, respectively $\Rightarrow$
algebraic system for (u_0, v_0, η_0):

$$\begin{pmatrix} i\omega & -f & -igk_x \\ f & i\omega & -igk_y \\ -iH_0 k_x & -iH_0 k_y & i\omega \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \\ \eta_0 \end{pmatrix} = 0, \qquad (110)$$

Dispersion equation

Condition of solvability:

$$\det \begin{pmatrix} i\omega & -f & -igk_x \\ f & i\omega & -igk_y \\ -iH_0 k_x & -iH_0 k_y & i\omega \end{pmatrix} = 0, \qquad (111)$$

which gives:

$$\omega \left(\omega^2 - gH_0 \mathbf{k}^2 - f^2\right) = 0. \qquad (112)$$

Physical meaning of solutions

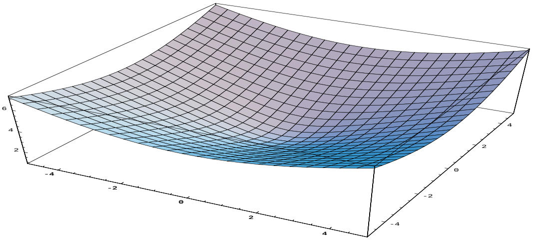
3 roots of the equation correspond to

- Stationary solutions $\omega = 0$
- Propagative waves with the dispersion relation:

$$\omega^2 - gH_0\mathbf{k}^2 - f^2 = 0 \tag{113}$$

inertia-gravity waves.

Dispersion relation



Dispersion relation for inertia-gravity waves. $c = \sqrt{gH_0} = 1$, $f = 1$, the part with $\omega < 0$ is not presented.

Exercise

1. Demonstrate (104),
2. Obtain the polarisation relations, i.e. the relations between u_0, v_0, η_0 for inertia - gravity waves,
3. Calculate phase and group velocity of inertia - gravity waves,
4. Demonstrate that inertia-gravity waves bear no PV anomaly (PV anomaly: $q - f/H_0$),
5. Determine the spectrum of small perturbations in the 2-layer RSW model,
6. Demonstrate that PV of each layer in multi-layer RSW is

$$q_i = \frac{\zeta_i + f}{h_i}, \; i = 1, 2, \frac{d_i q_i}{dt} = 0. \qquad (114)$$

Primitive equations, ocean

$$\frac{\partial \vec{v}_h}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v}_h + f \hat{z} \wedge \vec{v}_h = -\frac{\vec{\nabla}_h \pi}{\rho} \equiv -\vec{\nabla}_h \phi, \qquad (115)$$

$$\partial_t \sigma + \vec{v} \cdot \vec{\nabla} \sigma + w \rho_s'(z) = 0. \qquad (116)$$

$$g \frac{\sigma}{\rho_0} = -\partial_z \phi, \quad \vec{\nabla}_h \cdot \vec{v}_h + \partial_z w = 0, \qquad (117)$$

Remark

Hydrostatic approximation $\leftrightarrow$ scaling for mesoscale motions:

$$W \ll U, \quad H \ll L, \quad \frac{W}{H} \sim \frac{U}{L}$$

where L, H and U, W are horizontal and vertical spatial and velocity scales, respectively.

Absolute vorticity in PE model

Absolute vorticity:

$$\vec{\zeta}_a = \vec{\zeta} + \hat{\mathbf{z}} f, \quad \vec{\nabla} \cdot \vec{\zeta}_a = 0, \tag{118}$$

where relative vorticity under the PE scaling:

$$\vec{\zeta} = -\partial_z v \hat{\mathbf{x}} + \partial_z u \hat{\mathbf{y}} + (\partial_x v - \partial_y u)\hat{\mathbf{z}} \tag{119}$$

Application of $\vec{\nabla}\wedge$ to PE + "hydrodynamic identity":

$$\vec{v} \cdot \vec{\nabla} \vec{v} = \frac{1}{2} \vec{\nabla} \vec{v}^2 - \vec{v} \wedge (\vec{\nabla} \wedge \vec{v}) \tag{120}$$

$\rightarrow$ evolution equation for ζ_a:

$$\frac{d\vec{\zeta}_a}{dt} = \vec{\zeta}_a \cdot \vec{\nabla} \vec{v} + \frac{g}{\rho_0} \hat{\mathbf{z}} \wedge \vec{\nabla} \sigma. \tag{121}$$

Lagrangian conservation of potential vorticity

$$\frac{dq}{dt} = 0, \quad q = \vec{\zeta}_a \cdot \vec{\nabla}\rho, \quad \rho = \rho_0 + \rho_s(z) + \sigma. \tag{122}$$

Using vector identities:

$$\vec{\nabla}A \cdot \left(\vec{\nabla} \wedge \vec{B}\right) = -\vec{\nabla} \cdot \left(\vec{\nabla}A \wedge \vec{B}\right), \tag{123}$$

$$\vec{A} \wedge \left(\vec{B} \wedge \vec{C}\right) = \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right), \tag{124}$$

and $\vec{\nabla} \cdot \vec{v} = \vec{\nabla} \cdot \vec{\zeta}_a = 0$:

$$\begin{aligned}
\partial_t \left(\vec{\zeta}_a \cdot \vec{\nabla}\rho\right) &= (\partial_t \vec{\zeta}_a) \cdot \vec{\nabla}\rho + \vec{\zeta}_a \cdot \vec{\nabla}(\partial_t \rho) \\
&= \vec{\nabla}\rho \cdot \left(\vec{\nabla} \wedge \left(\vec{v} \wedge \vec{\zeta}_a\right)\right) - \vec{\zeta}_a \cdot \vec{\nabla}\left(\vec{v} \cdot \vec{\nabla}\rho\right) \\
&= -\vec{\nabla} \cdot \left(\vec{\nabla}\rho \wedge \left(\vec{v} \wedge \vec{\zeta}_a\right)\right) - \vec{\zeta}_a \cdot \vec{\nabla}\left(\vec{v} \cdot \nabla\rho\right) \\
&= -\vec{\nabla} \cdot \left(\vec{v}\left(\vec{\zeta}_a \cdot \vec{\nabla}\rho\right)\right) + \vec{\nabla} \cdot \left(\vec{\zeta}_a\left(\vec{v} \cdot \vec{\nabla}\rho\right)\right) \\
&- \vec{\zeta}_a \cdot \vec{\nabla}\left(\vec{v} \cdot \nabla\rho\right) = -\vec{v} \cdot \vec{\nabla}\left(\vec{\zeta}_a \cdot \vec{\nabla}\rho\right).
\end{aligned} \tag{125}$$

Spectrum of small perturbations - PE model

Linearised equations:

Perturbations about the state of rest: $\vec{v} = 0$ with constant stratification on the f- plane. Linearised equations:

$$\begin{aligned} u_t - fv + \phi_x &= 0, \\ v_t + fu + \phi_y &= 0, \qquad (126) \\ \phi_z + \frac{g}{\rho_0}\sigma = 0, \;\; \sigma_t + w\rho_s' &= 0, \\ u_x + v_y + w_z &= 0, \qquad (127) \end{aligned}$$

where u, v w - three components of velocity perturbation, ϕ - geopotential perturbation, σ - perturbation of the profile of background density ρ_s, with $\rho_s' = const$.

Elimination of σ and w:

Elimination of σ:

$$\phi_{zt} + wN^2 = 0, \tag{128}$$

where $N^2 = -\frac{g\rho_s'}{\rho_0}$ - Brunt - Väisälä frequency

Elimination of w:

$$\begin{aligned} u_t - fv + \phi_x &= 0, \\ v_t + fu + \phi_y &= 0, \qquad (129) \\ u_x + v_y - N^{-2}\phi_{zzt} &= 0, \qquad (130) \end{aligned}$$

Method of Fourier

Solutions - harmonic waves:

$$(u, v, \phi) = (u_0, v_0, \phi_0)e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} + \text{c.c.}, \qquad (131)$$

where ω et $\mathbf{k} = (k_x,\ k_y,\ k_z)$ are frequency and wavenumber, respectively.

Algebraic system for (u_0, v_0, ϕ_0):

$$\begin{pmatrix} i\omega & -f & -ik_x \\ f & i\omega & -ik_y \\ -ik_x & -ik_y & i\frac{\omega}{N^2}k_z^2 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \\ \eta_0 \end{pmatrix} = 0, \qquad (132)$$

Dispersion equation

Condition of solvability:

$$\det \begin{pmatrix} i\omega & -f & -ik_x \\ f & i\omega & -ik_y \\ -ik_x & -ik_y & i\frac{\omega}{N^2}k_z^2 \end{pmatrix} = 0, \qquad (133)$$

which gives:

$$\omega \left(\omega^2 - \left(N^2 \frac{k_x^2 + k_y^2}{k_z^2} + f^2 \right) \right) = 0. \qquad (134)$$

Physical meaning of solutions

Three roots of this equation correspond to

- Stationary solutions $\omega = 0$
- Propagative waves with dispersion relation:

$$\omega^2 = N^2 \frac{k_x^2 + k_y^2}{k_z^2} + f^2 \tag{135}$$

Internal inertia-gravity waves: IGW.

Remark: at each fixed k_z - dispersion relation of RSW with $\sqrt{gH_0} \to \frac{N}{|k_z|}$

Exercise

1. Demonstrate (121)
2. Demonstrate that PV in the PE in isopycnal coordinates:

$$q = \frac{\partial_x v - \partial_y u + f}{\partial_\theta z} \tag{136}$$

is conserved: $\frac{dq}{dt} = 0$.

3. Demonstrate the Eulerian conservation of energy in the PE, with energy density defined as:

$$e = \rho_0 \frac{u^2 + v^2}{2} + \rho g z, \tag{137}$$

where $\rho = \rho_s + \sigma$, z - (Lagrangian) position of the elementary volumeof fluid.

4. Establish polarisation relations and calculate phase and groupe velocities of the IGW.

Euler equations for an incompressible fluid in the rotating frame without hydrostatic hypothesis:

$$\partial_t \vec{v} + \vec{v} \cdot \vec{\nabla} \vec{v} + f \hat{\mathbf{z}} \wedge \vec{v} = -\frac{1}{\rho_0} \vec{\nabla} P \quad \vec{\nabla} \cdot \vec{v} = 0. \qquad (138)$$

Linearisation ($\rho_0 = 1$):

$$\begin{aligned} u_t - fv + P_x &= 0 \\ v_t + fu + P_y &= 0 \\ w_t + P_z = 0, \;\; u_x + v_y + w_z &= 0 \end{aligned} \qquad (139)$$

Solution: inertial (gyroscopic) waves with dispersion relation:

$$\omega^2 = f^2 \frac{k_z^2}{k_x^2 + k_y^2 + k_z^2} \qquad (140)$$

$\rightarrow$ sub-inertial.

Non-hydrostatic Boussinesq equations

$$g\frac{\rho}{\rho_0} = -\phi_z \to \frac{dw}{dt} + g\frac{\rho}{\rho_0} = -\phi_z. \qquad (141)$$

Elimination of $b = -g\frac{\rho}{\rho_0}$ and w:

$$b = \phi_z + w_t, \quad -\left(\partial_{tt} + N^2\right)(u_x + v_y) + \phi_{zzt} = 0 \Rightarrow \qquad (142)$$

$$\begin{aligned} u_t - fv &= -\phi_x, & (143)\\ v_t + fu &= -\phi_y, & (144)\\ \left(\partial_{tt} + N^2\right)(u_x + v_y) - \phi_{zzt} &= 0, & (145) \end{aligned}$$

Dispersion relation:

$$\omega\left[\omega^2 - \left(N^2\frac{k^2+l^2}{k^2+l^2+m^2} + f^2\frac{m^2}{k^2+l^2+m^2}\right)\right] = 0 \qquad (146)$$

Typically in the atmosphere and ocean

$$N^2 > f^2 \Rightarrow f^2 \le \omega^2 \le N^2 \qquad (147)$$

- Two dynamical entities: waves and vortices
- Vortices: slow motions related to Lagrangian conservation of PV; zero frequency in linear approximation.
- Waves: fast motions
- Frequencies of wave and vortices are separated by a spectral gap in hydrostatic approximation.

GFD: vortices, waves, and topography