Chapter 2: Slow vortex dynamics.

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Equations of horizontal motion

$$\frac{\partial \vec{v}_h}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v}_h + f \hat{z} \wedge \vec{v}_h = -\vec{\nabla}_h \Phi. \quad (1)$$

$$f = f_0(1 + \beta y), \quad \Phi = \Phi_0 + \phi = g(H_0 + h) \quad (2)$$

h - geopotential (perturbation) height.

Scaling for vortex motions

- Velocity $\vec{v}_h = (u, v)$, $u, v \sim U$, $w \sim W << U$
- Unique horizontal spatial scale L,
- Vertical scale $H << L$,
- Time-scale : turn-over time $T \sim L/U$.

Characteristic parameters

Intrinsic scale of the system : deformation (Rossby) radius :

$$R_d = \frac{\sqrt{gH_0}}{f_0} \tag{3}$$

- Rossby number : $Ro = \frac{U}{f_0 L}$,
- Burger number : $Bu = \frac{R_d^2}{L^2}$,
- Characteristic non-linearity : $\lambda = \Delta H / H_0$, where ΔH is the typical value of h,
- Dimensionless gradient of f : $\tilde{\beta} \sim \beta L$

Non-dimensional equations of horizontal motion

$$Ro\,(\partial_t \mathbf{v}_h + \mathbf{v}\cdot\nabla\mathbf{v}_h) + (1+\tilde{\beta}y)\hat{\mathbf{z}}\wedge\mathbf{v}_h = -\frac{\lambda Bu}{Ro}\nabla_h h\,, \quad (4)$$

Geostrophic equilibrium

Equilibrium between the Coriolis force and the pressure force $\rightarrow$ geostrophic wind :

$$\hat{\mathbf{z}}\wedge\mathbf{v}_g = -\nabla h \quad (5)$$

Conditions of realisation :

- $Ro \rightarrow 0$,
- $\lambda\ Bu \sim Ro$,
- $\tilde{\beta} \rightarrow 0$.

Non-dimensional RSW equations

$$Ro\,(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) + (1+\tilde{\beta}y)\hat{\mathbf{z}}\wedge\mathbf{v} = -\frac{\lambda Bu}{Ro}\nabla\eta\,, \quad (6)$$

$$\lambda\partial_t\eta + \nabla\cdot(\mathbf{v}(1+\lambda\eta)) = 0\,. \quad (7)$$

Regimes close to geostrophy : $Ro \equiv \epsilon \ll 1$

- Quasi-geostrophic(QG) : weak non-linearity :

$$\lambda \sim Ro, \Rightarrow Bu \sim 1, \Rightarrow L \sim R_d,\ \tilde{\beta} \sim Ro \quad (8)$$

- Frontal geostrophic (FG) : strong non-linearity :

$$\lambda \sim 1, \Rightarrow Bu \sim Ro, \Rightarrow L \gg R_d,\ \tilde{\beta} \sim Ro \quad (9)$$

QG regime

$$\epsilon\left(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}\right) + (1+\epsilon y)\hat{\mathbf{z}}\wedge\mathbf{v} = -\nabla\eta\,, \tag{10}$$

$$\epsilon\partial_t\eta + \nabla\cdot\left(\mathbf{v}(1+\epsilon\eta)\right) = 0\,. \tag{11}$$

Asymptotic expansions :

$$\mathbf{v} = \mathbf{v}^{(0)} + \epsilon\mathbf{v}^{(1)} + \epsilon^2\mathbf{v}^{(2)} + \ldots \tag{12}$$

Order by order in *Ro*

Order ϵ^0

Geostrophic wind :

$$u^{(0)} = -\partial_y \eta, \quad v^{(0)} = \partial_x \eta \quad \Rightarrow \quad \partial_x u^{(0)} + \partial_y v^{(0)} = 0, \tag{13}$$

$$\frac{d^{(0)}}{dt} \cdots = \partial_t \ldots + u^{(0)} \partial_x \ldots + v^{(0)} \partial_y \ldots \equiv \partial_t \cdots + \mathcal{J}(\eta, \ldots). \tag{14}$$

$$\mathcal{J}(A, B) \equiv \partial_x A \partial_y B - \partial_y A \partial_x B. \tag{15}$$

Order ϵ^1

$$u^{(1)} = -\frac{d^{(0)}}{dt} v^{(0)} - y u^{(0)}, \quad v^{(1)} = \frac{d^{(0)}}{dt} u^{(0)} - y v^{(0)}, \Rightarrow \tag{16}$$

$$\partial_x u^{(1)} + \partial_y v^{(1)} = -\frac{d^{(0)}}{dt} \vec{\nabla}^2 \eta - v^{(0)}, \Rightarrow \tag{17}$$

$$\frac{d^{(0)}}{dt} \left(\eta - \vec{\nabla}^2 \eta \right) - \partial_x \eta = 0 \leftrightarrow \frac{d^{(0)}}{dt} \left(\eta - \vec{\nabla}^2 \eta - y \right) = 0. \tag{18}$$

With restored dimensions

$$\frac{d^{(0)}}{dt} \left(\frac{f_0^2}{g H_0} \left(\frac{gh}{f_0} \right) - \vec{\nabla}^2 \left(\frac{gh}{f_0} \right) - f_0 (1 + \beta y) \right) = 0.$$

Non-dimensional QG equation on the β- plane

$$\partial_t \eta - \vec{\nabla}^2 \partial_t \eta - \mathcal{J}(\eta, \vec{\nabla}\eta) - \partial_x \eta = 0. \tag{19}$$

Physical meaning : conservation of PV.
Formal linearisation :

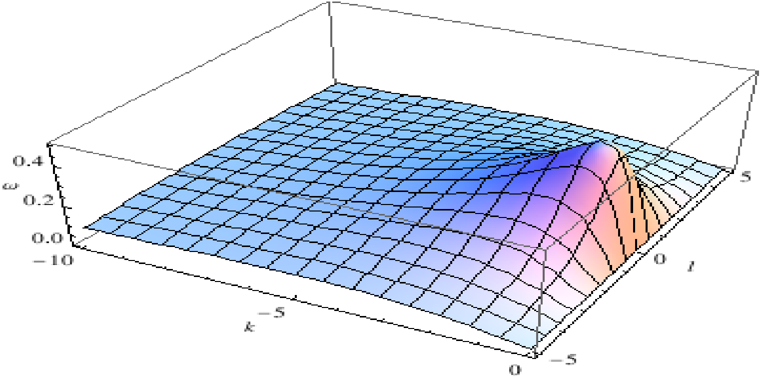
$$\partial_t \eta - \vec{\nabla}^2 \partial_t \eta - \partial_x \eta = 0. \tag{20}$$

Wave solutions $\eta \propto exp^{i(kx+ly-\omega t)} \rightarrow$ dispersion relation :

$$\omega = -\frac{k}{k^2 + l^2 + 1}. \tag{21}$$

Rossby waves - strongly dispersive ; anisotropic dispersion.

Dispersion diagram for Rossby waves



Exercise

- Obtain the QG equation directly from the conservation of PV,
- Calculate the phase and the group velocities of the Rossby waves and analyse them

RSW model with 2 layers with a rigid lid.

Equations of horizontal motion layerwise

$$\partial_t \mathbf{v}_i + \mathbf{v}_i \cdot \nabla \mathbf{v}_i + f\hat{\mathbf{z}} \wedge \mathbf{v}_i + \frac{1}{\rho_i}\nabla \pi_i = 0\,,\, i = 1, 2; \qquad (22)$$

Conservation of mass layerwise

$$\partial_t (H_i - (-1)^{i+1}\eta) + \nabla \cdot \left(\mathbf{v}_i (H_i - (-1)^{i+1}\eta)\right) = 0\,,\, i = 1, 2; \qquad (23)$$

$H_i, i = 1, 2$ - non-perturbed thicknesses of the layers, $H_1 + H_2 = H$, η - position of the interface, $\rho_2 > \rho_1$.

Dynamical boundary condition at the interface

$$(\rho_2 - \rho_1)g\eta = \pi_2 - \pi_1\,. \qquad (24)$$

Conservation laws

Conservation of PV layerwise

$$(\partial_t + \mathbf{v}_i \cdot \nabla)\, q_i = 0, \quad q_i = \frac{\zeta_i + f}{H_i - (-1)^{i+1}\eta}, \tag{25}$$

where $\zeta_i = \hat{\mathbf{z}} \cdot \nabla \wedge \mathbf{v}_i$ relative vorticity in the layer i.

Conservation of energy

$$E = \int dxdy \left(\sum_{i=1,2} \rho_i (H_i - (-1)^{i+1}\eta) \frac{\mathbf{v}_i^2}{2} + (\rho_2 - \rho_1) g \frac{\eta^2}{2} \right) \tag{26}$$

Here $\int dxdy\, (\rho_2 - \rho_1) g \frac{\eta^2}{2}$ is available potential energy.

Characteristic scales

- Typical horizontal velocity : U
- Typical horizontal scale : L
- Time-scale : $T \sim L/U$ - turn-over time
- Pressure scale layerwise : $P_i \sim \rho_i U L f_0$
- Typical vertical scale : H ; $D_i = \frac{H_i}{H}$ - non-dimensional unperturbed thicknesses, $i = 1, 2$.

Parameters

- Rossby number : $Ro = \frac{U}{f_0 L} \equiv \epsilon$
- Typical dimensionless deviation of the interface : λ
- Dimensionless gradient of the Coriolis parameter : $\tilde{\beta}$
- Aspect ratio : $d = \frac{H_1}{H_2}$
- Stratification parameter : $N = 2\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}$
- Burger number : $Bu = \frac{R_d^2}{L^2}$, $R_d^2 = \frac{NgH}{f_0^2}$

Baroclinic deformation radius : $R_d^2 = \frac{g' H}{f_0}$, where g' - reduced gravity $g' = gN$.
Weak stratification limit (often used in the oceanic context) $N \to 0$, and disappears everywhere except g'.

Non-dimensional equations

$$\epsilon \frac{d_i}{dt}\mathbf{v}_i + (1+\tilde{\beta}y)\hat{z}\wedge\mathbf{v}_i = -\vec{\nabla}\pi_i, \quad i = 1, 2. \tag{27}$$

$$\begin{aligned} -\lambda\frac{d_1}{dt}\eta + (D_1 - \lambda\eta)\vec{\nabla}\cdot\mathbf{v}_1 &= 0 \\ \lambda\frac{d_2}{dt}\eta + (D_2 + \lambda\eta)\vec{\nabla}\cdot\mathbf{v}_2 &= 0 \end{aligned} \tag{28}$$

$$\pi_2 - \pi_1 + \frac{N}{2}(\pi_2 + \pi_1) = \frac{\lambda Bu}{2\epsilon}\eta. \tag{29}$$

Here

$$\frac{d_i}{dt} = \partial_t + \mathbf{v}_i\cdot\nabla \tag{30}$$

and the same notation is kept for non-dimensional variables.

QG regime

$$\lambda \sim \tilde{\beta} \sim \epsilon \ll 1, \quad \Rightarrow \quad L \sim R_d \tag{31}$$

Asymptotic expansion in ϵ

Equations of the horizontal motion are the same as in 1-layer model $\Rightarrow$

$$\begin{aligned} u_i &= u_i^{(0)} - \epsilon \left[\partial_t v_i^{(0)} + \mathcal{J}(\pi_i, v_i^{(0)}) + y u_i^{(0)} \right] + \dots \\ v_i &= v_i^{(0)} + \epsilon \left[\partial_t u_i^{(0)} + \mathcal{J}(\pi_i, u_i^{(0)}) - y v_i^{(0)} \right] + \dots \end{aligned} \tag{32}$$

Geostrophic wind layerwise

$$u_i^{(0)} = -\partial_y \pi_i, \quad v_i^{(0)} = \partial_x \pi_i \quad i = 1, 2. \tag{33}$$

Divergence of velocity

Same as in 1-layer model layerwise :

$$\partial_x u_i^{(1)} + \partial_y v_i^{(1)} = -\left[\partial_t \vec{\nabla}^2 \pi_i + \mathcal{J}(\pi_i, \vec{\nabla}^2 \pi_i) + \partial_x \pi\right] \tag{34}$$

2-layer QG model

Mass conservation equations layerwise

$$\partial_t \eta + \mathcal{J}(\pi_i, \eta) - (-1)^i D_i \left[\partial_t \vec{\nabla}^2 \pi_i + \mathcal{J}(\pi_i, \vec{\nabla}^2 \pi_i) + \partial_x \pi \right] = 0, \quad i = 1, 2. \quad (35)$$

are rewritten as equations for the pressures in the layers $\leftrightarrow$ 2-layer quasi-geostrophic equations :

$$\frac{d_i^{(0)}}{dt} \left[\vec{\nabla}^2 \pi_i - (-1)^i D_i^{-1} \eta + y \right] = 0 \,, \; i = 1, 2. \quad (36)$$

where

$$\frac{d_i^{(0)}}{dt} (\ldots) := \partial_t (\ldots) + J(\pi_i, \ldots) \,, \; i = 1, 2 \quad (37)$$

Standard limit : weak stratification : $\rho_2 \to \rho_1 \Rightarrow \eta = \pi_2 - \pi_1$

Baroclinic and barotropic components in the limit of weak stratification

$\eta = \pi_2 - \pi_1$ - baroclinic ; $\Pi = D_1\pi_1 + D_2\pi_2$ - barotropic.

- $\eta = 0$ - columnar motion, velocity the same in both layers
- $\Pi = 0$ - sheared motion, velocity opposite in the layers

Formal linearisation

$$\begin{aligned} \partial_t \left[\nabla^2 \pi_1 + D_1^{-1}(\pi_2 - \pi_1)\right] + \partial_x \pi_1 &= 0 \\ \partial_t \left[\nabla^2 \pi_2 - D_2^{-1}(\pi_2 - \pi_1)\right] + \partial_x \pi_2 &= 0 \end{aligned} \quad (38)$$

Solutions-waves : $\pi_i = A_i e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$.
Condition of solvability :

$$\det \begin{pmatrix} \omega(\mathbf{k}^2 + D_1^{-1}) + k_x & -\omega D_1^{-1} \\ -\omega D_2^{-1} & \omega(\mathbf{k}^2 + D_2^{-1}) + k_x \end{pmatrix} = 0. \quad (39)$$

Dispersion relation :

$$\omega = -\frac{k_x}{2\mathbf{k}^2(\mathbf{k}^2 + D_1^{-1} + D_2^{-1})}\left[(2\mathbf{k}^2 + D_1^{-1} + D_2^{-1}) \pm (D_1^{-1} + D_2^{-1})\right] \quad (40)$$

- Barotropic mode : $\omega_{bt} = -\frac{k_x}{\mathbf{k}^2}$, the faster one.
- Baroclinic mode : $\omega_{bc} = -\frac{k_x}{(\mathbf{k}^2 + D_1^{-1} + D_2^{-1})}$, the slower one.

Baroclinic instability

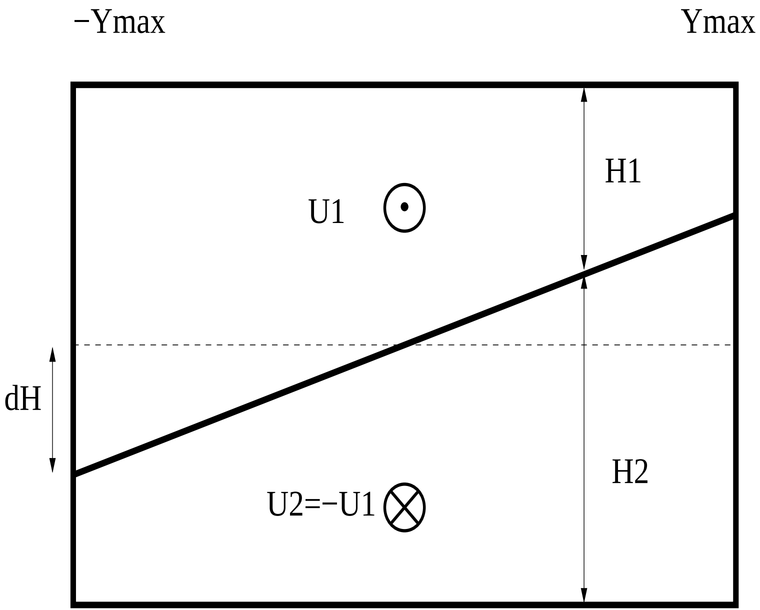
Phillips model

2-layer QG model, $\frac{\rho_2-\rho_1}{\rho_2+\rho_1} \to 0 \Rightarrow \eta = \pi_2 - \pi_1$:

$$\frac{d_i^{(0)}}{dt}\left[\nabla^2\pi_i - (-1)^i D_i^{-1}\eta + y\right] = 0\,,\ i = 1, 2. \tag{41}$$

Background flow

Solution : $U_i = -\partial_y \pi_i$, $i = 1, 2$ $\quad U_1 \neq U_2$ - vertical shear $\Rightarrow$ Inclined interface : $\eta = \pi_2 - \pi_1 = (U_1 - U_2)y \to$ available potential energy.


−Ymax
Ymax
H1
U1
dH
H2
U2=−U1

Linearisation

$$\pi_i = -U_i y + \phi_i, \quad ||\phi|| \ll 1. \Rightarrow$$

$$(\partial_t + U_i\partial_x)\left[\vec{\nabla}^2\phi_i - (-1)^i D_i^{-1}(\phi_2 - \phi_1)\right] + \left[1 - (-1)^i D_i^{-1}(U_1 - U_2)\right]\partial_x\phi_i = 0. \qquad (42)$$

Wave solutions (Fourier transformation) : $\phi_i = A_i e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$.

Notation : $c = \omega/k_x$, $U_1 - U_2 = \Delta U$, $F_i = D_i^{-1}$.

$$\begin{aligned} A_1\left[(c - U_1)(\mathbf{k}^2 + F_1) + 1 + F_1(U_1 - U_2)\right] - A_2(c - U_1)F_1 &= 0, \\ -A_1(c - U_2)F_2 + A_2\left[(c - U_2)(\mathbf{k}^2 + F_2) + 1 - F_2(U_1 - U_2)\right] &= 0. \end{aligned}$$

Dispersion relation

$$
\begin{aligned}
c &= U_2 \\
&+ \frac{1}{2\mathbf{k}^2(\mathbf{k}^2+F_1+F_2)}\Big[\big(\Delta U \mathbf{k}^2(\mathbf{k}^2+2F_2) \\
&\quad -\mathbf{k}^2(2\mathbf{k}^2+F_1+F_2)\big) \\
&\pm \Big[(F_1+F_2)^2 + 2\Delta U \mathbf{k}^4(F_1-F_2) \\
&- \mathbf{k}^4\,(\Delta U)^2\,(4F_1F_2-\mathbf{k}^4)\Big]^{\frac{1}{2}}\Big]
\end{aligned}
$$

Sufficiently strong shear ΔU, sufficiently small $|\mathbf{k}|$ $\rightarrow$ frequency (or c) has non-zero imaginary part $\rightarrow$ growth of the amplitude $\rightarrow$ instability.

Remark :

Instability exists on the f- plane as well : interface slope $\Rightarrow$ gradient of PV due to the shear sufficient to make Rossby waves propagate :

$$c = \frac{1}{2(\mathbf{k}^2 + F_1 + F_2)} \left[U_1(\mathbf{k}^2 + 2F_2) + U_2(\mathbf{k}^2 + 2F_1) \right. \\ \left. \pm \left[(\Delta U)^2 \left(\mathbf{k}^4 - 4F_1F_2 \right) \right]^{\frac{1}{2}} \right] \quad (43)$$

Exercise

- Find the threshold for the instability in the case $F_1 = F_2$,
- Find the wavelength corresponding to the maximal growth rate.

PE, oceanic case

Characteristic scales

- Typical horizontal velocity : U
- Typical horizontal scale : L
- Time-scale : $T \sim L/U$ -turn-over time
- Typical vertical scale : H
- Typical vertical velocity :W $\frac{W}{H} \sim \lambda \frac{U}{L}$ - to confirm *aposteriori*
- Presssure scale : $\rho_0 gH$

Parameters

- Rossby number : $Ro = \frac{U}{f_0 L} \equiv \epsilon$
- Typical dimensionless deviation of the isopycnal surfaces : λ
- Dimensionless gradient of the Coriolis parameter : $\tilde{\beta}$
- Stratification parameter : $N = \frac{\text{variable part of density}}{\text{constant part of density}}$
- Burger number : $Bu = \frac{R_d^2}{L^2}$, where R_d baroclinic deformation radius with reduced gravity $g' = Ng$.

Pressure and density related via hydrostatics :

$$\begin{aligned} \rho &= \rho_0 \left[1 + N \left(\rho_s(z) + \lambda \sigma(x, y, z; t)\right)\right], \Rightarrow \\ P &= \rho_0 g H \left[(1 - z) + N \left(p_s(z) + \lambda \pi(x, y, z; t)\right)\right] \end{aligned} \quad (44)$$

Non-dimensional equations

$$\epsilon \frac{d}{dt}\mathbf{v}_h + (1+\tilde{\beta}y)\hat{z}\wedge \mathbf{v}_h = -\vec{\nabla}_h \pi. \tag{45}$$

$$\frac{d}{dt}\sigma + \rho_s' w = 0, \quad \partial_z \pi + \sigma = 0. \tag{46}$$

$$\vec{\nabla}_h \cdot \mathbf{v}_h + \lambda \partial_z w = 0; \tag{47}$$

where

$$\frac{d}{dt} = \partial_t + \mathbf{v}_h \cdot \nabla_h + \lambda w \partial_z \tag{48}$$

Boundary conditions - rigid lid/flat bottom, for simplicity :

$$w|_{z=0,1} = 0. \tag{49}$$

If bathymetry $b(x,y)$ then $w|_{z=b} = \frac{db}{dt}$.

QG regime. Asymptotic expansion order by order

$$\lambda \sim \tilde{\beta} \sim \epsilon \ll 1, \quad \Rightarrow \quad L \sim R_d, \; \frac{W}{H} = \lambda \frac{U}{L}. \tag{50}$$

Order ϵ^0

$$u^{(0)} = -\partial_y \pi, \quad v^{(0)} = \partial_x \pi, \Rightarrow \partial_x u^{(0)} + \partial_y v^{(0)} = 0, \Rightarrow \partial_z w = 0. \tag{51}$$

Consistent with the choice of the scale W.

Thermal wind

Geostrophic + hydrostatic equilibria :

$$u = -\partial_y \pi, \; v = \partial_x \pi, \; \sigma = -\partial_z \pi \Rightarrow \partial_z v = -\partial_x \sigma, \; \partial_z u = +\partial_y \sigma \tag{52}$$

Horizontal density gradient $\leftrightarrow$ vertical shear of the horizontal wind. Atmosphere : $\sigma \to -\theta$.

Order ϵ^1

$$u^{(1)} = -\frac{d^{(0)}}{dt} v^{(0)} - y u^{(0)}, \quad v^{(1)} = \frac{d^{(0)}}{dt} u^{(0)} - y v^{(0)}, \Rightarrow \tag{53}$$

$$\partial_x u^{(1)} + \partial_y v^{(1)} = -\frac{d^{(0)}}{dt} \nabla_h^2 \pi - \partial_x \pi \equiv -\frac{d^{(0)}}{dt} \left(\nabla_h^2 \pi + y \right), \tag{54}$$

where $\frac{d^{(0)}}{dt} \cdots = \partial_t \cdots + \mathcal{J}(\pi, \dots)$ - horizontal advection by geostrophic wind.

Elimination of w

$$w^{(0)} = -\frac{1}{\rho_s'(z)} \frac{d^{(0)}}{dt} \sigma = \frac{1}{\rho_s'(z)} \frac{d^{(0)}}{dt} \partial_z \pi \tag{55}$$

Continuity equation :

$$
\begin{aligned}
-\frac{d^{(0)}}{dt}\left(\nabla_h^2\pi + y\right) + \partial_z\left(\frac{1}{\rho_s'(z)}\frac{d^{(0)}}{dt}\partial_z\pi\right) &= 0 \Rightarrow \\
\frac{d^{(0)}}{dt}\left(-\nabla_h^2\pi - y + \partial_z\left(\frac{1}{\rho_s'(z)}\partial_z\pi\right)\right) &= 0, \qquad (56)
\end{aligned}
$$

Meaning : advection of quasi-geostrophic potential vorticity by geostrophic wind.

Boundary conditions

Evolution equations (dynamics !)

$$
w|_{z=0,1} = 0 \Rightarrow \left.\frac{d^{(0)}}{dt}\partial_z\pi\right|_{z=0,1} = 0. \qquad (57)
$$

Surface quasi-geostrophy

Remark : On the f- plane and at constant stable stratification $\rho_s'(z) = \text{const} < 0$ the geostrophic PV becomes a three-dimensional Laplacian of π after rescaling of the vertical coordinate $\Rightarrow$ any solution of the Laplace equation gives a solution of the full problem provided the b.c. are verified $\Rightarrow$ dynamics is defined by evolution of density on the boundary $\Leftrightarrow$ surface quasi geostrophy (SQG).

Example : Solution of the 3D Laplace equation in the upper half-plane decaying at $z \to \infty$:

$$\pi(\mathbf{x}, z, t) = \int d\mathbf{k}\, \hat{\pi}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}} e^{-|\mathbf{k}|z},$$

where $\mathbf{x} = (x,\ y)$, $\mathbf{k} = (k\ l)$ are horizontal radius- and wave-vectors. Therefore

$$\sigma(\mathbf{x}, z, t) = \int d\mathbf{k}\, |\mathbf{k}|\, \hat{\pi}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}} e^{-|\mathbf{k}|z}$$

Setting $z = 0$ and substituting to (57) produces the SQG dynamics for π on the boundary.

Exercise

- Deduce (56) directly from the conservation of PV using the QG scaling,
- Demonstrate that

$$\partial_z \left(\frac{1}{\rho_s'(z)} \frac{d^{(0)}}{dt} \partial_z \pi \right) = \frac{d^{(0)}}{dt} \left(\partial_z \left(\frac{1}{\rho_s'(z)} \partial_z \pi \right) \right) \quad (58)$$

- Obtain the SQG evolution equation for $\hat{\pi}(\mathbf{k}, t)$ in Fourier space (k, l).

Baroclinic Rossby waves : continuous stratification

Formal linearisation

$$\partial_t \left[\nabla_h^2 \pi - \partial_z \left(\frac{1}{\rho_s'(z)} \partial_z \pi \right) \right] + \partial_x \pi = 0, \quad \partial_{tz}^2 \pi \big|_{z=0,1} = 0. \tag{59}$$

Separation of variables

$$\pi(x, y, z; t) = p(x, y; t) S(z) \Rightarrow \tag{60}$$

$$\partial_t \nabla_h^2 p(x, y; t) S(z) - \partial_t p(x, y; t) \left[\frac{1}{\rho_s'(z)} S'(z) \right]' + \partial_x p(x, y; t) S(z) = 0 \Rightarrow$$

Equations in z and in x, y, t :

- $$\frac{1}{S(z)}\left[\frac{1}{\rho_s'(z)}S'(z)\right]' = \kappa^2 \qquad (61)$$
- $$\partial_t \nabla_h^2 p(x,y;t) - \kappa^2 \partial_t p(x,y;t) + \partial_x p(x,y;t) = 0, \qquad (62)$$

κ - separation constant

Vertical modes

Sturm - Liouville problem :

$$\left[\frac{1}{\rho_s'(z)}S'(z)\right]' - \kappa^2 S(z) = 0, \quad S'(z)\big|_{z=0,1} = 0 \qquad (63)$$

Eigenfunctions $S_n(z)$ and eigenvalues κ_n, $n = 0, 1, 2, ...$

Example : linear stratification $\rho_s = -N^2 z$

$$S''(z) + (N\kappa)^2 S(z) = 0, \quad S_n \propto \cos(\pi n z), \ \kappa_n = \frac{\pi n}{N}. \quad (64)$$

Horizontal motion

Wave solutions : $p(x, y; t) \propto e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \rightarrow$

$$\omega = -\frac{k_x}{\mathbf{k}^2 + \kappa_n^2} - \text{Rossby waves.} \qquad (65)$$

$n \nearrow$ (stronger vertical shear) $\Rightarrow c_{phase} \searrow$.

Eady model

QG with constant stratification $N = \text{const}$ on the f-plane

$$\frac{d^{(0)}}{dt}\left(\partial_x^2\pi + \partial_y^2\pi + \frac{1}{N^2}\partial_z^2\pi\right) = 0, \quad \frac{d^{(0)}}{dt}\partial_z\pi\Big|_{z=0,1} = 0 \quad (66)$$

Thermal wind

Exact solution : $\vec{v} = U_0(z)\hat{\mathbf{x}}$ for any $U_0(z)$. We take $U_0 = z$.
Linearisation : $\pi = -U_0(z)y + \phi(x, y, z; t)$, $||\phi|| \ll 1$:

$$\begin{aligned}(\partial_t + U_0(z)\partial_x)\left(\partial_x^2\phi + \partial_y^2\phi + \frac{1}{N^2}\partial_z^2\phi\right) &= 0\\ \left[(\partial_t + U_0(z)\partial_x)(-y\partial_z U_0(z) + \partial_z\phi) - \partial_x\phi\partial_z U_0(z)\right]\Big|_{z=0,1} &= 0\end{aligned}$$

Solution by separation of variables

Fourier modes

B.C. in y-direction : zonal channel $-1 \leq y \leq 1 \Rightarrow$
$\partial_x \phi|_{y=\pm 1} = 0$.

$$\phi = A(z) \cos l_n y \, e^{ik(x-ct)}, \quad l_n = (n + \frac{1}{2})\pi, n = 0, 1, 2, \dots$$

Equations and b.c. :

$$(z - c)\left(A''(z) - \mu^2 A(z)\right) = 0, \quad \mu^2 = (k^2 + l_n^2)N^2, \tag{67}$$

$$cA'(0) + A(0) = 0, \quad (c - 1)A'(1) + A(1) = 0. \tag{68}$$

Solution

Non-singular solutions $\Leftrightarrow$ absence of critical layers z_c :
$c = U_0(z_c) \Rightarrow A''(z) - \mu^2 A(z) = 0.$
General solution : $A(z) = a\cosh\mu z + b\sinh\mu z$:

$$\begin{aligned} a + c\mu b &= 0, \\ a\left[(c-1)\mu\sinh\mu + \cosh\mu\right] &+ \\ b\left[(c-1)\mu\cosh\mu + \sinh\mu\right] &= 0. \end{aligned} \tag{69}$$

Dispersion relation :

$$c^2 - c + \frac{\coth\mu}{\mu} - \frac{1}{\mu^2} = 0 \Rightarrow \tag{70}$$

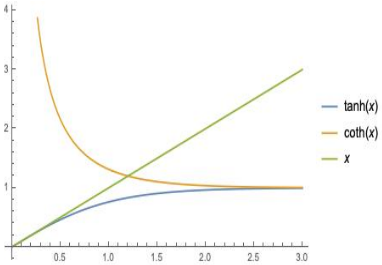
$$c = \frac{1}{2} \pm \left(\frac{1}{4} + \frac{1}{\mu^2} - \frac{\coth\mu}{\mu}\right)^{\frac{1}{2}} \tag{71}$$

Analysis of dispersion relation

Identity : $\coth\mu = \frac{1}{2}(\tanh\frac{\mu}{2} + \coth\frac{\mu}{2})$:

$$c = \frac{1}{2} \pm \frac{1}{\mu}\left[\left(\frac{\mu}{2} - \coth\frac{\mu}{2}\right)\left(\frac{\mu}{2} - \tanh\frac{\mu}{2}\right)\right]^{\frac{1}{2}}. \qquad (72)$$

$\forall x\ \tanh x \leq x \Rightarrow$ instability at
$\coth\frac{\mu}{2} > \frac{\mu}{2} \Rightarrow \mu < \mu_c \approx 2.4 \Rightarrow$ instability of long waves.



RSW quations at small Rossby number with two time-scales

Hypotheses :

- f- plane, infinite domain,
- Unique spatial scale L,
- Small Rossby number ϵ, regime QG : $\lambda \sim \epsilon$,
- Fast $t \sim f_0^{-1}$ and slow $t_1 \sim (\epsilon f_0)^{-1}$ time-scales

Non-dimensional equations :

$$(\partial_t + \epsilon \partial_{t_1})\mathbf{v} + \epsilon(\mathbf{v} \cdot \nabla \mathbf{v}) + \hat{\mathbf{z}} \wedge \mathbf{v} + \nabla h = 0 \,, \tag{73}$$

$$(\partial_t + \epsilon \partial_{t_1})h + (1 + \epsilon h)\nabla \cdot \mathbf{v} + \epsilon \mathbf{v} \cdot \nabla h = 0 \,, \tag{74}$$

$$\partial_t Q + \epsilon \mathbf{v} \cdot \nabla Q = 0 \,, \quad Q = \epsilon \frac{\zeta - h}{1 + \epsilon h} \text{ – PV anomaly.} \tag{75}$$

Geostrophic adjustment

Cauchy problem with localised initial conditions

$$u\big|_{t=0} = u_I\ ,\ v\big|_{t=0} = v_I\ ,\ h\big|_{t=0} = h_I\ . \tag{76}$$

Multi-scale asymptotic expansions

$$\begin{aligned} \mathbf{v} &= \mathbf{v}_0(x, y; t, t_1, ...) + \epsilon \mathbf{v}_1(x, y; t, t_1, ...) + ... \\ h &= h_0(x, y; t, t_1, ...) + \epsilon h_1(x, y; t, t_1, ...) + ..., \end{aligned} \tag{77}$$

Decomposition slow - fast order by order in ϵ :

$$h_i = \bar{h}_i(x, y; t_1, ...) + \tilde{h}_i(x, y; t, t_1, ...),\ i = 0, 1, 2, ... \tag{78}$$

$$\bar{h}_i(x, y; t_1, ...) = \lim_{T\to\infty} \frac{1}{T}\int_0^T h_i(x, y, t, t_1, ...)\, dt, \tag{79}$$

Approximation ϵ^0

$$\partial_t \mathbf{v}_0 + \hat{\mathbf{z}} \wedge \mathbf{v}_0 = -\nabla h_0 \,, \tag{80}$$

$$\partial_t (\zeta_0 - h_0) = 0 \,, \tag{81}$$

where $\zeta_0 = \hat{\mathbf{z}} \cdot \nabla \wedge \mathbf{v}_0$ - relative vorticity, and equation for PV is used. I.C.. :

$$u_0|_{t=0} = u_I \;,\; v_0|_{t=0} = v_I \;,\; h_0|_{t=0} = h_I \,. \tag{82}$$

Re-writing (80) in terms of relative vorticity ζ and divergence $D = \nabla \cdot \mathbf{v}_0$:

$$\partial_t \zeta_0 + D_0 = 0 \,, \tag{83}$$

$$\partial_t D_0 - \zeta_0 = -\nabla^2 h_0 \,. \tag{84}$$

Immediate integration of (81) in fast time t :

$$\zeta_0 - h_0 = \Pi_0 \,, \tag{85}$$

where Π_0 is yet unknown function of x, y, t_1 (integration "constant").

ϵ^0 - next

Elimination of ζ_0 and D_0- linear inhomogeneous equation for h_0 :

$$-\frac{\partial^2 h_0}{\partial t^2} - h_0 + \nabla^2 h_0 = \Pi_0(x, y; t_1, t_2, ...) \,. \qquad (86)$$

Solution : slow + fast :

$$h_0 = \tilde{h}_0(x, y; t, ...) + \bar{h}_0(x, y; t_1, ...) \qquad (87)$$

$$-\frac{\partial^2 \tilde{h}_0}{\partial t^2} - \tilde{h}_0 + \nabla^2 \tilde{h}_0 = 0 \,; \qquad (88)$$

$$-\bar{h}_0 + \nabla^2 \bar{h}_0 = \Pi_0 \qquad (89)$$

Klein - Gordon (KG) and Helmholtz equations.
Π_0 : geostrophic PV constructed from the slow component $\bar{h}_0$.

Initialisation problem : How to separate the i.c. in slow/fast ?

Response (unique at $\epsilon \to 0$)

- By definition :

$$\Pi_0(x, y; 0) = \partial_x v_I - \partial_y u_I - h_I \equiv \Pi_I(x, y) \quad (90)$$

- Determination of the initial value $\bar{h}_{0I}$ of $\bar{h}_0$ by inversion :

$$-\bar{h}_{0I} + \nabla^2 \bar{h}_{0I} = \Pi_I, \Rightarrow \bar{h}_{0I} = -(\nabla^2 - 1)^{-1}\Pi_I. \quad (91)$$

- Determination of the initial value $\tilde{h}_{0I}$ of $\tilde{h}_0$:

$$\tilde{h}_{0I} = h_I - \bar{h}_{0I}. \quad (92)$$

- Second i.c. for $\tilde{h}_0$ (PV and ζ - D) :

$$\partial_t \tilde{h}_0 \Big|_{t=0} = -D_I \equiv \partial_x u_I + \partial_y v_I\,. \quad (93)$$

Approximation ϵ^0 - continued

Analogous decomposition for $\mathbf{v}$:

$$\mathbf{v}_0 = \tilde{\mathbf{v}}_0(x, y; t, ...) + \bar{\mathbf{v}}_0(x, y; t_1, ...)\,, \qquad (94)$$

slow components verify the geostrophic relation :

$$\bar{\mathbf{v}}_0 = \hat{\mathbf{z}} \wedge \nabla \bar{h}_0 \qquad (95)$$

and the fast ones obey the equations

$$\partial_t \tilde{\mathbf{v}}_0 + \hat{\mathbf{z}} \wedge \tilde{\mathbf{v}}_0 = -\nabla \tilde{h}_0 \qquad (96)$$

with i.c. :

$$\tilde{u}_I^{(0)} = u_I - \bar{u}_{0I}\,; \;\; \tilde{v}_I^{(0)} = v_I - \bar{v}_{0I}\,, \qquad (97)$$

where $\bar{u}_{0I}, \bar{v}_{0I}, \bar{h}_{0I}$ verify (95). linearised PV $\tilde{\zeta}_0 - \tilde{h}_0$ of the fast component is identically zero.

Approximation ϵ^0 - continued

Fast component : solution for h :

Inertia-gravity waves propagating out of the initial perturbation ; created by its non-balanced par $\tilde{u}_I^{(0)}, \tilde{v}_I^{(0)}, \tilde{h}_{0I}$:

$$\tilde{h}_0(\mathbf{x}; t) = \sum_{\pm} \int d\mathbf{k}\, H_0^{(\pm)}(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x} \pm \Omega_{\mathbf{k}} t)}, \qquad (98)$$

where

$$H_0^{(\pm)}(\mathbf{k}) = \frac{1}{2}\left(\hat{\tilde{h}}_{0I}(\mathbf{k}) \pm i\,\frac{\hat{D}_I(\mathbf{k})}{\Omega_{\mathbf{k}}}\right), \qquad (99)$$

and the notation $\hat{\ldots}$ is used for the Fourier transformations of the corresponding quantities.

Approximation ϵ^0 - end

Résumé of the first approximation

- Slow and fast components are defined unambiguously
- Fast and slow motions are separated dynamically (non-interacting)
- Fast part completely resolved : inertia-gravity waves propagating out of the initial perturbation
- Evolution of the slow part is still to determine

Approximation ϵ^1

Momentum equations :

$$\partial_t \mathbf{v}_1 + \hat{\mathbf{z}} \wedge \mathbf{v}_1 = -\nabla h_1 - (\partial_{t_1} + \mathbf{v}_0 \cdot \nabla)\, \mathbf{v}_0 \,. \tag{100}$$

Equation for PV in first order :

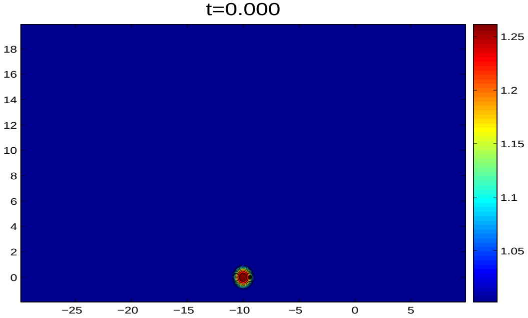
$$\partial_t (\zeta_1 - h_1) - \Pi_0\, \partial_t \tilde{h}_0 + \tilde{u}^{(0)} \partial_x \Pi_0 + \tilde{v}^{(0)} \partial_y \Pi_0 = -\partial_{t_1} \Pi_0 - J(\bar{h}_0, \Pi_0) \,. \tag{101}$$

Integrability condition $\leftrightarrow$ averaging over t :

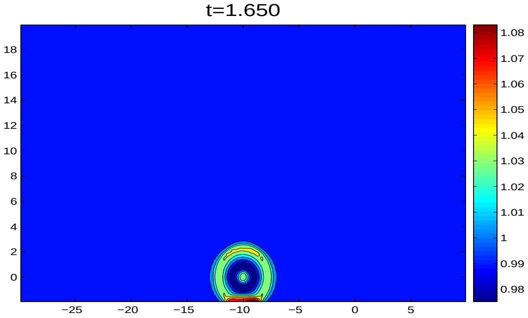
$$\partial_{t_1} \Pi_0 + J(\bar{h}_0, \Pi_0) \equiv \partial_{t_1} (\nabla^2 \bar{h}_0 - \bar{h}_0) + J(\bar{h}_0, \nabla^2 \bar{h}_0) = 0 \,. \tag{102}$$

$\Rightarrow$ QG equation. Arise from elimination of resonances in the equation for fast component at order 1.

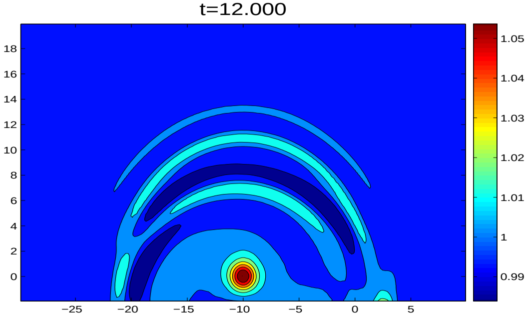
Numerical simulations of the geostrophic adjustement. Initial perturbation of h.



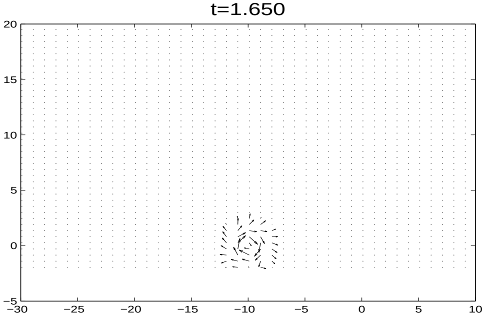
Initiale stage of adjustement, h.



Advanced stage of adjustement, *h*.



Initial stage of adjustement, velocity.



Advanced stage of adjustement, velocity.