# Chapter 3: Dynamical rôle of coasts and topography.

V. Zeitlin

Course GFD M2 MOCIS

# Plan

Introducing lateral boundaries and topography
- Dynamical role of lateral boundaries : an idealized coast
- Coast with a shelf

Topography without lateral boundary
- Topographic waves
- Long-wave approximation in RSW model

Mountain waves
- Mountain waves in QG model
- Mountain waves in RSW model

# Linearised RSW with a lateral boundary

Simplest configuration : non-dissipative 1-layer RSW equations in a half-plane with a rectilinear meridional boundary at $x = 0$.

Linearised non-dimensional RSW equations :

$$\begin{aligned} u_t - v + \eta_x &= 0, \\ v_t + u + \eta_y &= 0, \\ \eta_t + u_x + v_y &= 0 \end{aligned} \quad (1)$$

Rectlinear meridional west coast : b.c. : $u|_{x=0} = 0$.

Inhomogeneity in $x$, but Fourier-transform in $y, t$ possible :

$$(u, v, \eta) = (\bar{u}_0(x), \bar{v}_0(x), \bar{h}_0(x))e^{i(ly - \omega t)} \Rightarrow$$

$$\begin{aligned} -i\omega\bar{u}_0 - \bar{v}_0 + \bar{h}_0' &= 0, \\ -i\omega\bar{v}_0 + \bar{u}_0 + il\bar{h}_0 &= 0, \\ -i\omega\bar{h}_0 + il\bar{v}_0 + \bar{u}_0' &= 0, \end{aligned} \quad (2)$$

## Reduction to a single equation ($\omega \neq 1$)

$$\bar{h}_0'' + (\omega^2 - 1 - l^2)\bar{h}_0 = 0, \tag{3}$$

while

$$\bar{u}_0 = i\frac{l\bar{h}_0 - \omega\bar{h}_0'}{\omega^2 - 1}, \tag{4}$$

and hence the b. c. is :

$$l\bar{h}_0 - \omega\bar{h}_0'\big|_{x=0} = 0. \tag{5}$$

## Solutions of two different types :

- Free inertia-gravity waves :

$$\omega^2 - 1 - l^2 \equiv k^2 > 0, \tag{6}$$

$$\bar{h}_0 \propto e^{\pm ikx}, \quad \omega^2 = 1 + k^2 + l^2. \tag{7}$$

- Trapped at the boundary waves :

$$\omega^2 - 1 - l^2 \equiv -\kappa^2 < 0, \tag{8}$$

$$\bar{h}_0 \propto e^{-\kappa x}. \tag{9}$$

The second type of solution is exponentially growing for $x < 0$, this is why it was discarded on the whole plane.

# Trapped solutions - Kelvin waves

Kelvin waves are dispersionless. Boundary condition →

$$l\bar{h}_0 - \omega \bar{h}_0'\big|_{x=0} = 0 \;\Rightarrow \kappa = -\frac{l}{\omega},$$
$$\Rightarrow \omega^2 - 1 - l^2 + \frac{l^2}{\omega^2} = 0, \;\Rightarrow \omega^2 = l^2 \; (\omega \neq 1), \quad (10)$$

and

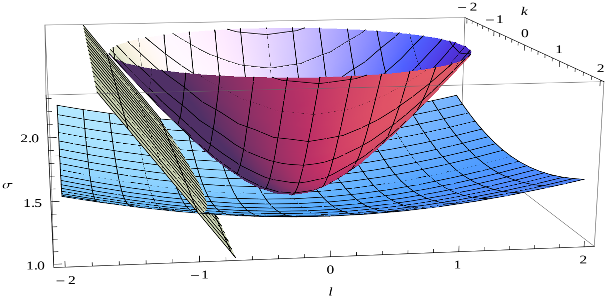
$$\kappa > 0 \Rightarrow \omega = -l, \quad \eta \propto e^{-x}. \quad (11)$$

Any packet of Kelvin waves :

$$(u, v, \eta) = (0, K(y+t), -K(y+t))e^{-x}, \quad (12)$$

where $K$ - an arbitrary function, is a solution of linearised RSW equations. Kelvin waves are traveling along the boundary leaving it on their right. Normal to the boundary component of the velocity is absent, and the along- boundary velocity and height anomaly are in quadrature.

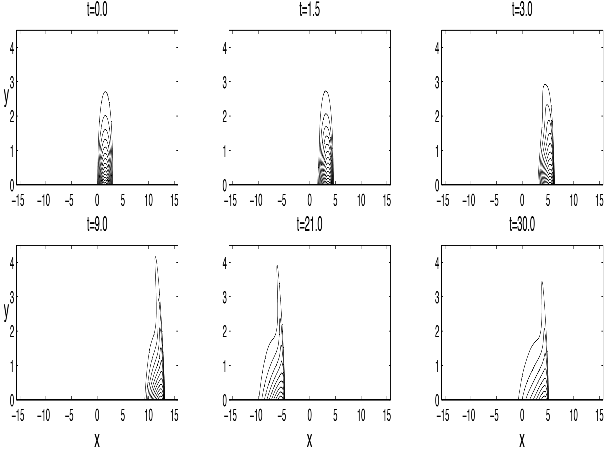
# Dispersion diagram of the 2-layer RSW with a lateral boundary




Dispersion relation for internal-gravity and Kelvin waves in the 2-layer RSW model. Baroclinic Kelvin waves are not shown. Upper surface : barotropic inertia-gravity waves, lower surface : baroclinic inertia-gravity waves, plane : barotropic Kelvin waves.

# Propagation of a packet of Kelvin waves

No dispersion → breaking and front formation :



# Reflexion of inertia-gravity waves

Any "free" wave is a sum of incident and reflected waves :

$$(u, v, \eta) = (u_i, v_i, \eta_i) + (u_r, v_r, \eta_r)$$

$$(u_i, v_i, \eta_i) = A_i \left( \frac{k\omega + il}{\omega^2 - 1}, \frac{l\omega - ik}{\omega^2 - 1}, 1 \right) e^{i(kx+ly-\omega t)} + \text{c.c.},$$
$$(u_r, v_r, \eta_r) = A_r \left( \frac{-k\omega + il}{\omega^2 - 1}, \frac{l\omega + ik}{\omega^2 - 1}, 1 \right) e^{i(-kx+ly-\omega t)} + \text{c.c.}.$$

Boundary condition :

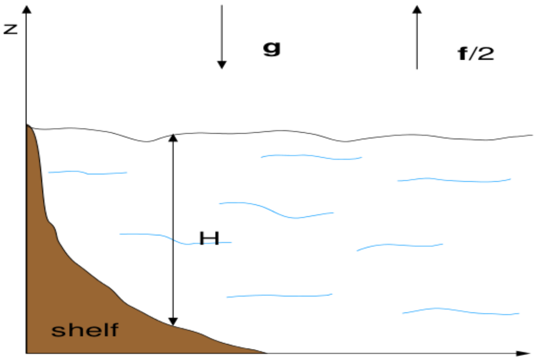
$$u_i + u_r|_{x=0} = 0, \Rightarrow A_r = A_i \frac{k\omega + il}{k\omega - il}, \; \omega^2 = 1+k^2+l^2. \quad (13)$$

Snell's law.

## Exercises

- Obtain (3),
- Consider Kelvin waves with the coast at the a) est, b) north ; determine their propagation direction,
- Demonstrate that Kelvin waves carry no PV anomaly.

# Shallow-water model with a shelf.



## Linearised non-dimensional RSW equations in the presence of bottom topography :

$$\begin{aligned} u_t - v + \eta_x &= 0, \\ v_t + u + \eta_y &= 0, \\ \eta_t + (Hu)_x + (Hv)_y &= 0. \end{aligned} \tag{14}$$

$H$ - unperturbed depth of the fluid.

- Abrupt shelf : typical scale $L << R_d \leftrightarrow \frac{L}{R_d} = \epsilon$.
- Shelf with gentle slope : typical scale $L \sim R_d$

# Abrupt shelf

Non-dimensional $H$ and boundary conditions are :

$$H = H\left(\frac{x}{\epsilon}\right), \quad H|_{x=0} = 0, \quad H|_{x=\infty} = 1.$$

Looking for wave solutions

$$(u, v, \eta) = (\bar{u}_0(x), \bar{v}_0(x), \bar{h}_0(x))e^{i(ly-\omega t)} + \text{c.c.}$$

we get

$$\begin{cases} -i\omega\bar{u}_0 - \bar{v}_0 + \bar{h}_0' = 0, \\ -i\omega\bar{v}_0 + \bar{u}_0 + il\bar{h}_0 = 0, \\ -i\omega\bar{h}_0 + ilH\bar{v}_0 + (H\bar{u}_0)' = 0, \end{cases} \tag{15}$$

which may be reduced to a single equation :

$$(H\bar{h}_0')' + (\omega^2 - 1 - l^2 H - \frac{l}{\omega}H')\bar{h}_0 = 0. \tag{16}$$

# Asymptotic analysis

- "Open-sea" domain :

$$\bar{h}_0'' + (\omega^2 - 1 - l^2)\bar{h}_0 = 0. \quad (17)$$

Solution - trapped wave : $\bar{h}_0^{(h)} = Ae^{-\kappa x}$, $\kappa > 0$

$$\kappa^2 = l^2 + 1 - \omega^2. \quad (18)$$

Suppose : $\kappa = \kappa_0 + \epsilon\kappa_1 + \dots$, $\omega = \omega_0 + \epsilon\omega_1 + \dots$.

- "Coastal" domain :

$$\frac{1}{\epsilon^2}\left(H(\xi)\bar{h}_0^{(c)}(\xi)'\right)' + \left(\omega^2 - 1 - l^2 H(\xi) - \frac{1}{\epsilon}\frac{l}{\omega}H'(\xi)\right)\bar{h}_0^{(c)} = 0. \quad (19)$$

$$\bar{h}_0^{(c)}(\xi) = \bar{\eta}^{(0)}(\xi) + \epsilon\bar{\eta}^{(1)}(\xi) + \dots, \quad \xi = \frac{x}{\epsilon} \quad (20)$$

## Hierarchy of equations for $\bar{\eta}^{(n)}$, $n = 0, 1, \dots$ :

$$
\begin{aligned}
\left(H(\xi)\bar{\eta}^{(0)}(\xi)'\right)' &= 0, \\
\left(H(\xi)\bar{\eta}^{(1)}(\xi)'\right)' - \frac{l}{\omega_0} H'(\xi))\bar{\eta}^{(0)}(\xi) &= 0, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned} \quad (21)
$$

## Order zero

$$
H(\xi)\bar{\eta}^{(0)}(\xi)' = C = \text{const.} \quad (22)
$$

$C \neq 0, \Rightarrow$ singularity at $x = 0$, $\Rightarrow \bar{\eta}^{(0)} = \text{const.}$
Matching with the domain $(h)$ à $x = \epsilon\xi$ :

$$
\bar{h}_0^{(h)} = A\left(1 - \kappa_0\epsilon\xi + \frac{1}{2}\kappa_0^2(\epsilon\xi)^2 - \epsilon^2\kappa_1\xi + \dots\right), \Rightarrow \quad (23)
$$

$\bar{\eta}^{(0)} = A.$

## Order 1

$$\left(H(\xi)\bar{\eta}^{(1)}(\xi)'\right)' - \frac{l}{\omega_0}H'(\xi))A = C_1 = \text{const.} \quad (24)$$

Solution regular for $\bar{u}_0, \bar{v}_0$ $\quad C_1 = 0 \Rightarrow$

$$\bar{\eta}^{(1)} = \frac{l}{\omega_0}A\xi + \text{const.} \quad (25)$$

Matching of $\bar{\eta}^{(0)} + \epsilon\bar{\eta}^{(1)}$ with $\bar{h}_0^{(h)}$ à $x = \epsilon\xi$
$\Rightarrow \frac{l}{\omega_0} = -\kappa_0$, const $= 0$.
Since $\kappa^2 = l^2 + 1 - \omega^2$, $\omega^2 \neq 1 \quad \Rightarrow \kappa_0 = 1$.
Kelvin wave. Further corrections $\rightarrow$ corrections to the dispersion relation.

# Shelf with gentle slope.

Reduction to a single non-dimensional wave equation (wave with $\omega$, $l$ ) for the free-surface perturbation $\bar{h}_0(x)$ :

$$(H\bar{h}_0')' + (\omega^2 - 1 - l^2 H - \frac{l}{\omega} H')\bar{h}_0 = 0. \qquad (26)$$

Ball's model :

$$H(x) = (1 - e^{-ax}). \qquad (27)$$

Change of variables(trapped solutions) $x \to s = e^{-ax}$, $\bar{h}_0 \to s^p \tilde{h}_0$, where $p$ is defined by

$$\omega^2 - 1 - l^2 = -p^2 < 0, \Rightarrow \qquad (28)$$

Hypergeometric equation :

$$s(1-s)\tilde{h}_0''(s) + [\gamma - (\alpha+\beta+1)]\,\tilde{h}_0'(s) - \alpha\beta\tilde{h}_0(s) = 0, \qquad (29)$$

solutions $F(\alpha, \beta, \gamma, s)$ - hypergeometric functions,

$$\gamma = 2p+1, \;\; \alpha = p + \frac{1}{2} - \sqrt{l^2 - \frac{l}{\omega} + \frac{1}{4}}, \;\; \beta = p + \frac{1}{2} + \sqrt{l^2 - \frac{l}{\omega} + \frac{1}{4}}. \qquad (30)$$

## Trapped wave solutions

A regular at $x = 0$ and decaying at $x \to \infty$ solution $\Rightarrow$ $\alpha = -n, \; n = 0, 1, \ldots$.. In this case

$$\bar{h}_0 = s^p F(-n, \beta, \gamma, s), \quad n = 0, 1, \ldots, \tag{31}$$

where

$$F(-n, \beta, \gamma, s) = \sum_{m=0}^{n} \frac{(-n)_m (\beta)_m}{(\gamma)_m m!} s^m, \quad (a)_m := a(a+1)\ldots(a+m-1) \tag{32}$$

$\alpha = -n \to$ dispersion relation :

$$p + \frac{1}{2} + n = \sqrt{l^2 - \frac{l}{\omega} + \frac{1}{4}}, n = 0, 1, \ldots. \tag{33}$$

## Free wave solutions

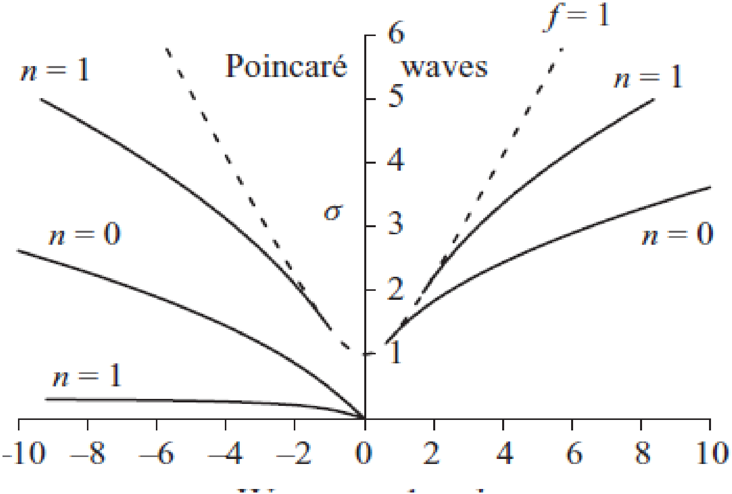
Solution for propagating (incident and reflected) Poincaré waves : $p \to ik$ in the above-displayed formulas. Solution is then given in terms of hypergeomeric functions :

$$\bar{h}_0 = A\left[e^{-ikx}F(\alpha^*, \beta^*, \gamma^*, s) - re^{ikx}F(\alpha, \beta, \gamma, s)\right], \quad (34)$$

$A$ is the amplitude of the wave, * means complex conjugation, $r$ is reflection coefficient :

$$r = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha^* + \beta^*)}{\Gamma(\alpha^*)\Gamma(\beta^*)\Gamma(\alpha + \beta)}, \ \Gamma - \text{gamma-function.} \quad (35)$$

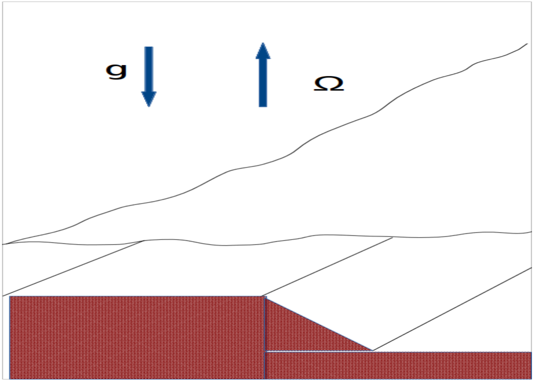
# Dispersion relation for the coastal waves ($n$ - number of nodes in the $x$ direction



# General properties of the coastal waves :

- Unique Kelvin wave,
- Discrete spectrum of sub-inertial trapped waves with $\omega < f$ (shelf waves) with unique sense of propagation (coast at their right)
- Discrete spectrum of supra-inertial trapped waves with $\omega > f$ (edge waves) with double sense of propagation
- Continuous spectrum of incident/reflected supra-inertial inertia-gravity (Poincaré) waves

# Escarpment topography



# Wave spectrum over escarpment

Same non-dimensional wave equation as before :

$$(H\bar{h}_0')' + (\omega^2 - 1 - l^2 H - \frac{l}{\omega}H')\bar{h}_0 = 0. \qquad (36)$$

At $x \to \pm\infty$ depth is constant, albeit different : $H = H_\pm =$ const. Asymptotics of $\bar{h}_{0\pm}$ :

$$H_\pm \bar{h}_0{}''_\pm + (\omega^2 - 1 - l^2 H_\pm)\bar{h}_{0\pm} = 0. \qquad (37)$$

Two kinds of solutions, depending on the signs of $p_\pm^2 = \omega^2 - 1 - l^2 H_\pm$.

- $p_\pm^2 > 0 \to$ a wave propagating to or out of escarpment,
- $p_\pm^2 < 0 \to$ trapped at the escarpment wave.

# Linear escarpment

Wave equation at the escarpment :

$$\left((H_m + x)\bar{h}_0'\right)' + (\omega^2 - 1 - l^2(H_m + x) - \frac{l}{\omega})\bar{h}_0 = 0, \quad (38)$$

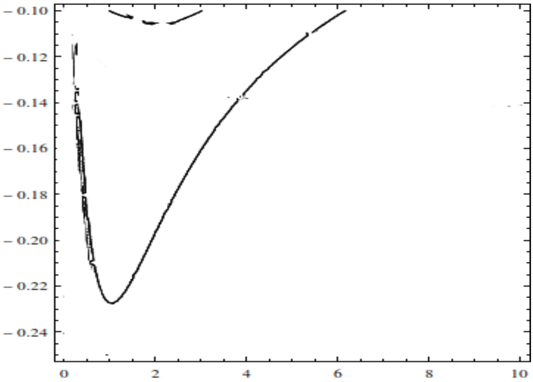
where $H_m$ - mean depth. May be explicitly solved in terms of confluent hypergeometric functions $M$ and $U$ :

$$\begin{aligned} \bar{h}_0(x) &= C_1 U\left(-\frac{-l - \omega - l\omega + \omega^3}{2l\omega}, 1, 4l + 2l\,x\right) \\ &+ C_2 M\left(\frac{-l - 1\omega - l\omega + \omega^3}{2l\omega}, 1, 4l + 2l\,x\right), (39) \end{aligned}$$

where $C_{1,2} = \text{const}$.

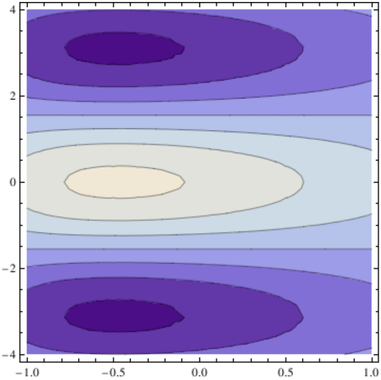
To be matched to the asymptotics $\bar{h}_0(x) = C_{\pm} e^{\mp\sqrt{-p_{\pm}^2}x}$ for trapped waves. Continuity of $\bar{h}_0$ and $\bar{h}_0'$ at $x = \pm 1$ - four homogeneous linear algebraic equations for the constants $C_{\pm}$, $C_{1,2}$, solvability condition $\rightarrow$ dispersion relation $\omega = \omega(l)$.

# Dispersion diagram for topographic waves trapped by the linear escarpment




Only two lowest modes with, respectively, zero and one node across the escarpment are shown ⇒ topographic Rossby waves, PV gradient is produced by topography.

# Phase portrait of the $n = 0$ mode




Isolines of $h$ for the gravest topographic wave of maximal frequency over the escarpment region ($x \in (-1, 1)$. Trapped waves can propagate only in the negative direction along the escarpment, i.e. leaving the shallower region on their right.

# "Long-wave" approximation (rigid lid)

Rigid lid $\leftrightarrow$ regime with $\lambda \to 0$. Equation for $h$ $\to$

$$(Hu)_x + (Hv)_y = 0 \quad \Rightarrow \text{stream-function } \psi : \tag{40}$$

$$Hu = -\psi_y, \quad Hv = +\psi_x \tag{41}$$

## One-dimensional topography $H = H(x)$

$$H = H(x) \Rightarrow u_x + v_y = -\frac{H'(x)}{H(x)} u = D - \text{divergence.} \tag{42}$$

$$v_x - u_y = \left(\frac{\psi_x}{H(x)}\right)_x + \frac{\psi_{yy}}{H(x)} = \zeta - \text{vorticity} \tag{43}$$

Vorticity equation $\frac{d}{dt}(\zeta + f) + (\zeta + f)D = 0 \to$ evolution equation for $\psi$.

## Linearised vorticity equation on the $f$- plane :

$$\zeta_t + fD = 0 \leftrightarrow \left[\left(\frac{\psi_x}{H(x)}\right)_x + \frac{\psi_{yy}}{H(x)}\right]_t + f\frac{H'(x)}{H^2(x)}\psi_y = 0. \tag{44}$$

## Wave solutions :

$\psi = \phi(x)e^{i(\omega t - ly)} + \text{c.c.}$ :

$$\left(\frac{\phi'}{H}\right)' - \frac{l^2}{H}\phi - \frac{f}{c}\frac{H'}{H^2}\phi = 0, \quad c = \frac{\omega}{l}. \tag{45}$$

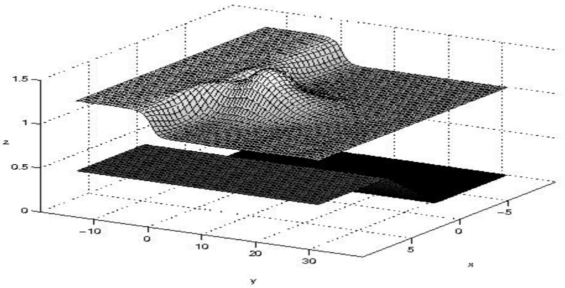
## Wave spectrum

$$\left(\frac{\phi'}{H}\right)' - \left(\frac{l^2}{H} + \frac{f}{c}\frac{H'}{H^2}\right)\phi = 0, \tag{46}$$

Sturm - Liouville problem for eigenfunctions $\phi_n$ and eigenvalues $c_n(l)$ (où $l_n(c)$) :

- No continuous spectrum.
- Discrete spectrum - trapped waves with number of nodes $n = 0, 1, 2, ...$.

# Example of the influence of topography : geostrophic adjustment of a pressure front over escarpment



## Exercises

Consider topography profile

$$H(x) = H_0 e^{2\Lambda x}, \quad 0 \leq x < \infty, \quad H_0 = \text{const.}$$

Show that topographic waves in the long-wave approximation obey the dispersion relation

$$c = -2f \frac{\Lambda}{k^2 + l^2 + \Lambda^2}.$$

Compare these waves with Rossby waves on the beta-plane.

# Potential vorticity in the presence of topography





PV conservation :

$$\frac{d}{dt}\left(\frac{\zeta + f}{h - b}\right) = 0. \tag{47}$$

Topography of weak amplitude $|b| \sim Ro$ the QG equation on the $\beta$-plane :

$$\nabla^2 \eta_t - \eta_t + \eta_x + \mathcal{J}(\eta, \nabla^2 \eta + b) = 0. \tag{48}$$

# Stationary solutions

Stationarity :

$$\mathcal{J}(\eta, \nabla^2\eta + b + y) = 0. \quad (49)$$

General solution :

$$\nabla^2\eta + b + y = \mathcal{F}(\eta), \quad (50)$$

$\mathcal{F}$ - arbitrary function. Zonal flow $U$ plus any perturbation : $\eta = -Uy + \psi$,

$$\nabla^2\psi + b + y = \mathcal{F}(\psi - Uy). \quad (51)$$

Looking for waves generated by localised topgraphy$\Rightarrow$ far upstream, at $x \to +\infty$ for $U < 0$, and $x \to -\infty$ for $U > 0$, the perturbation $\psi$ vanishes and (51) becomes

$$y = \mathcal{F}(-Uy), \quad (52)$$

$\Rightarrow \mathcal{F}(x) = -\frac{x}{U} \Rightarrow$ linear equation for $\psi$ :

$$U\nabla^2\psi + \psi = -Ub. \quad (53)$$

# One-dimensional topography

Ridge : $b = b(x) \Rightarrow \psi = \psi(x)$. Equation (53) becomes

$$\psi''(x) + \frac{1}{U}\psi(x) = -b(x). \tag{54}$$

Solution : inversion of the operator, Green's function :

$$\psi(x) = -U \int_{-\infty}^{+\infty} dx' \, G(x-x')b(x'), \tag{55}$$

$$G''(x-x') + \frac{1}{U}G(x-x') = \delta(x-x'). \tag{56}$$

Fourier transformation using $\delta(x) = \int_{-\infty}^{+\infty} dk \, e^{ikx} \rightarrow$

$$G(x-x') = \int_{-\infty}^{+\infty} dk \, \frac{e^{ik(x-x')}}{k^2 - \frac{1}{U}} \tag{57}$$

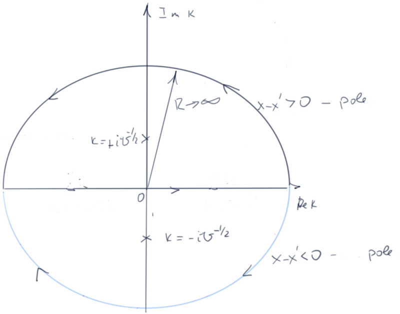
# Green's function for easterly flow $U < 0$

Calculation by the method of residues :

$$G(x-x') = \int_{-\infty}^{+\infty} dk \, \frac{e^{ik(x-x')}}{2iU^{-\frac{1}{2}}} \left( \frac{1}{k - iU^{-\frac{1}{2}}} - \frac{1}{k + iU^{-\frac{1}{2}}} \right)$$

$$= \pi \sqrt{U} e^{-\frac{|x-x'|}{\sqrt{U}}}, \qquad (58)$$

- decaying at both sides of the "ridge".

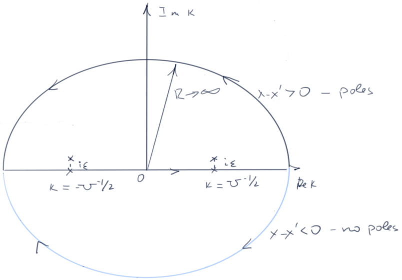


# Green's function for westerly flow $U > 0$

Integrand is singular Upstream decay $\rightarrow$ singularity shifted to the upper half-plane of complex $k$.

$$
\begin{aligned}
G(x-x') &= \int_{-\infty}^{+\infty} dk\, \frac{e^{ik(x-x')}}{2U^{-\frac{1}{2}}} \left( \frac{1}{k-U^{-\frac{1}{2}}} - \frac{1}{k+U^{-\frac{1}{2}}} \right) \\
&= \begin{cases} \pi\sqrt{U}\sin\frac{(x-x')}{\sqrt{U}}, & x-x'>0 \\ 0, & x-x'<0. \end{cases}
\end{aligned}
\qquad (59)
$$

- oscillating (waves) behind the "ridge".



# Qualitative analysis based on PV conservation of mountain Rossby waves in the westerly flow in Southern hemisphere

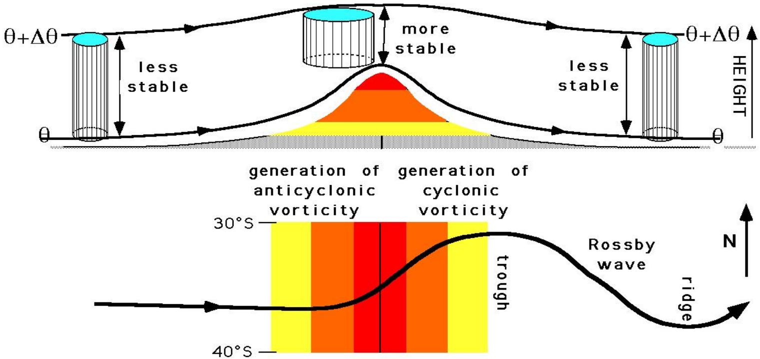



**Fig 12.K.4** The formation of a trough in the lee of a mountain range in the southern hemisphere. The top figure is a vertical cross section, and the bottom one is a plan view. The mountain ridge is at the same location in both figures.

# Mountain Rossby waves in the westerly flow over Andes Cordillera

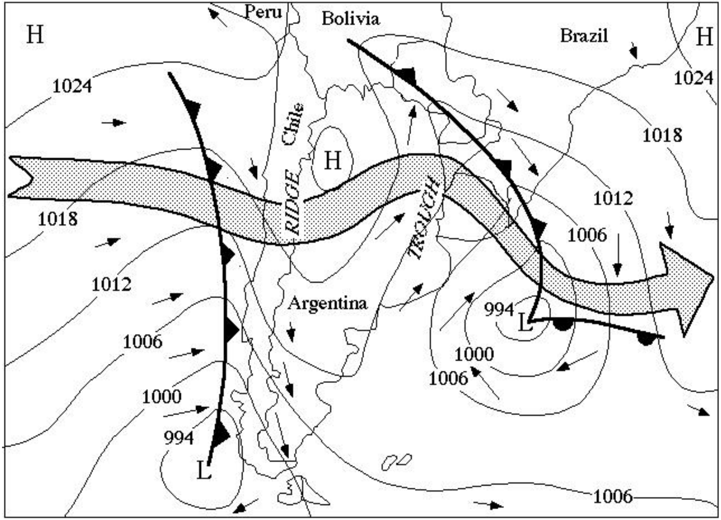



**Fig 12.K.5** The effect of the Andes on the upper westerly winds, in terms of the isobar pattern (in hPa) on 4 June 1995. The small arrows show the direction and speed of surface winds, and the bold wavy band is the jet stream. There is a ridge in the 300hPa flow near the mountains and a lee trough to the east, which promote a Rossby wave whose northward swing cradles a low on the right of the diagram.

# 2-dimensional topography

Green's function $G(x-x', y-y')$ :

$$U\nabla^2 G + G = \delta(x-x')\delta(y-y'); \qquad (60)$$

Bessel $Y_0$ (oscillating), or modified Bessel $K_0$ (decaying) function, depending on the sign of $U$. Fourier-transform : $G(x-x', y-y') \to G(k,l)$,

$$\left[-U(k^2+l^2)+1\right] G(k,l) = 1 \qquad (61)$$

$$\begin{aligned} G(x-x', y-y') &= \int_{-\infty}^{+\infty} dkdl \, \frac{e^{i(k(x-x')+l(y-y'))}}{-U(k^2+l^2)+1} \\ &= \int_0^{+\infty} |\mathbf{k}|d|\mathbf{k}| \int_0^{2\pi} d\theta \, \frac{e^{i|\mathbf{k}||\mathbf{x}-\mathbf{x}'|\cos\theta}}{-U\mathbf{k}^2+1} \\ &= \int_0^{+\infty} \frac{|\mathbf{k}|d|\mathbf{k}|}{-U\mathbf{k}^2+1} \int_0^{2\pi} d\theta \, \cos\left(|\mathbf{k}||\mathbf{x}-\mathbf{x}'|\cos\theta\right) \\ &= 2\pi \int_0^{+\infty} \frac{|\mathbf{k}|d|\mathbf{k}|}{-U\mathbf{k}^2+1} J_0\left(|\mathbf{k}||\mathbf{x}-\mathbf{x}'|\right) := 2\pi\mathcal{I} \end{aligned} \qquad (62)$$

$J_0$ - Bessel function

# Calculating $\mathcal{I}$

- $U > 0$

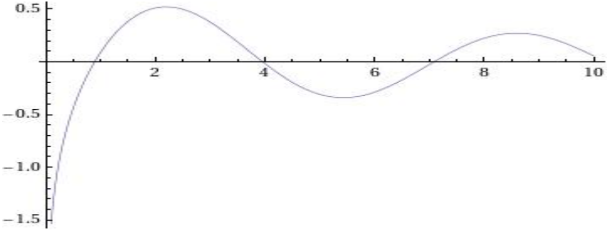
$$\mathcal{I} = -\frac{\pi}{2U} Y_0 \left( \frac{|\mathbf{x} - \mathbf{x}'|}{U} \right) \tag{63}$$

- $U < 0$

$$\mathcal{I} = \frac{1}{|U|} K_0 \left( \frac{|\mathbf{x} - \mathbf{x}'|}{|U|} \right) \tag{64}$$

# Bessel functions

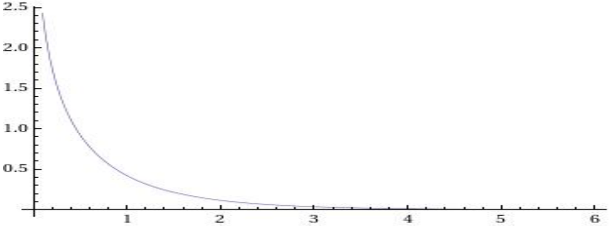
- Bessel function $Y_0$ : oscillating, and weakly decaying




$\rightarrow$

inconsistent with b.c. of strong decay upstream - ? ?.

- Modified Bessel function $K_0$ : exponentially decaying



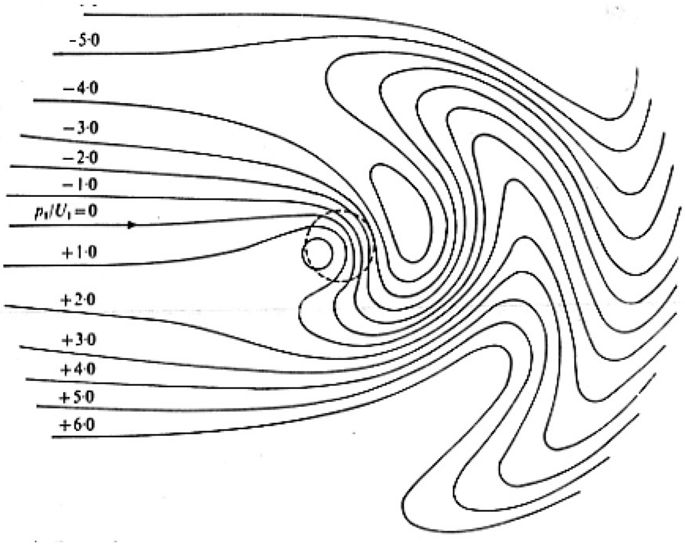

$\rightarrow$

localised topography (a mountain) produces exponentially decaying non-oscillating perturbation

# Recipe for correcting westerly flow result

Solution for westerly flow : to be corrected by a solution of the homogeneous problem "killing" the oscillations far upstream. The correction can not be found in closed form, expressed as a series of Bessel functions $\sum_{n=1}^{\infty} \frac{1}{2n-1} J_{2n-1}\left(\frac{|\mathbf{x}-\mathbf{x}'|}{\sqrt{U}}\right) \cos(2n-1)\phi$, where $\phi$ is the polar angle on the $x-y$ plane.

# Mountain Rossby waves in the westerly flux (calculated)



# Exercises

- Derive the equations (47) and (48),
- Analyse the passage of an easterly flow across a meridional mountain ridge,
- Analyse the passage of easterly and westerly flows across a meridional mountain ridge in the $f$- plane approximation.

# Non-dimensional RSW equations in the presence of a mean constant zonal flow $-U\hat{\mathbf{x}}$

$$
\begin{aligned}
u_t + (u - F)u_x + vu_y - \gamma^{-\frac{1}{2}} v + h_x &= 0, \\
v_t + (u - F)v_x + vv_y + \gamma^{-\frac{1}{2}} u + h_y &= 0, \\
h_t + ((u - F)(h - Mb))_x + (v(h - Mb))_y &= 0, \quad (65)
\end{aligned}
$$

$h$ - position of the free surface, $b(x, y)$ - topography.
Typical scales : horizontal - $L$, vertical - $H$ (non-perturbed thickness), velocity - $c = \sqrt{gH}$, time - $L/c$.
Parametres : Froude number $F = \frac{U}{c}$, non-dimensional height of the mountain : $M = \frac{b_{max}}{H}$, Burger number : $Bu = \gamma = \frac{c^2}{f^2 L^2}$.

# Linear limit (topography of weak amplitude)

Linearisation → single equation for $\eta$, the perturbation of $h$, with a stationary source due to topography :

$$\left((\partial_t - F\partial_x)^2 - \nabla^2 + \gamma^{-1}\right)\eta = M(F^2\partial_x^2 + \gamma^{-1})b \quad (66)$$

Stationary solutions in terms of Green's function

$$\eta(x,y) = M\int dx'dy'(F^2\partial_x^2 + \gamma^{-1})b(x-x', y-y')G(x', y') \quad (67)$$

Definition of $G$ :

$$\left((-F\partial_x)^2 - \nabla^2 + \gamma^{-1}\right) G(x-x', y-y') = \delta(x-x')\delta(y-y') \quad (68)$$

2 regimes : "supersonic" $F^2 > 1$ and "subsonic" $F^2 < 1$.

# Calculation of Green's function : Fourier method



$$G(x-x', y-y') = \int_{-\infty}^{+\infty} dkdl \, \frac{e^{i(k(x-x')+l(y-y'))}}{l^2 - (F^2-1)k^2 + \gamma^{-1}} \quad (69)$$

- $F^2 < 1$ : rescaling of $k$, polar Fourier coordinates $\Rightarrow$

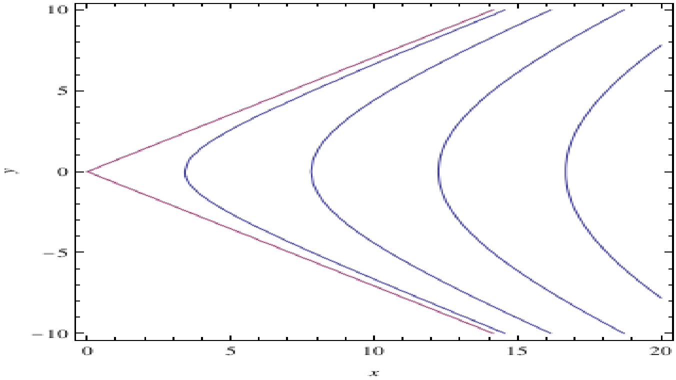
$$G \propto K_0 \left( \frac{\gamma^{-\frac{1}{2}} \sqrt{(x-x')^2 - (F^2-1)(y-y')^2}}{\sqrt{F^2-1}} \right) \quad (70)$$

- $F^2 > 1$ singular denominator $\Rightarrow$ method of residues $\Rightarrow$

$$G \propto J_0 \left( \frac{\gamma^{-\frac{1}{2}} \sqrt{(x-x')^2 - (F^2-1)(y-y')^2}}{\sqrt{F^2-1}} \right), \quad (71)$$

if $x < 0, x^2 < (F^2-1)y^2$, and 0 otherwise.

# Green's function $F^2 > 1$



# Stationary mountain waves in the rotating tank

20 J. G. Esler, O. J. Rump, and E. R. Johnson

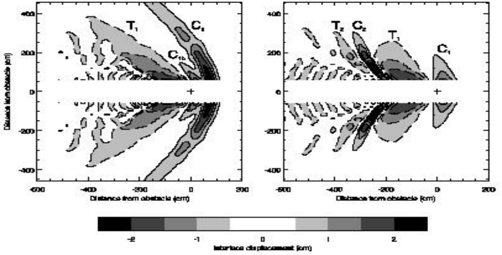
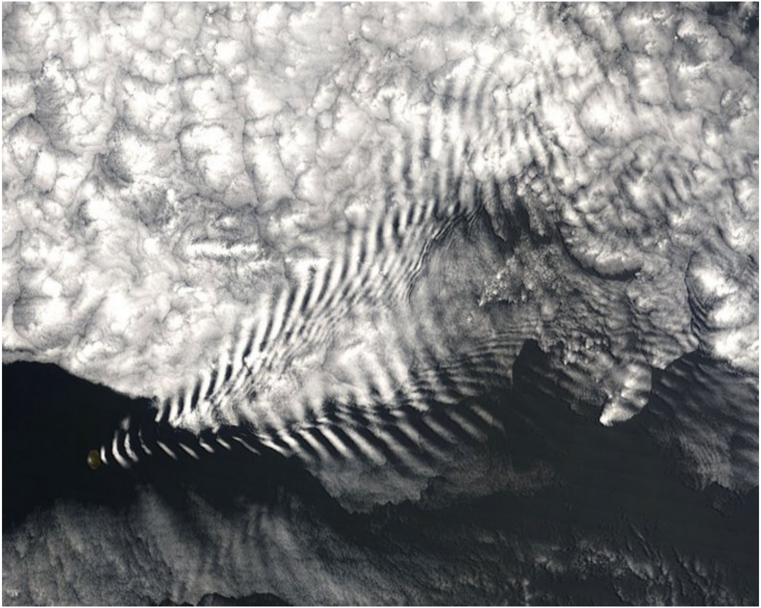
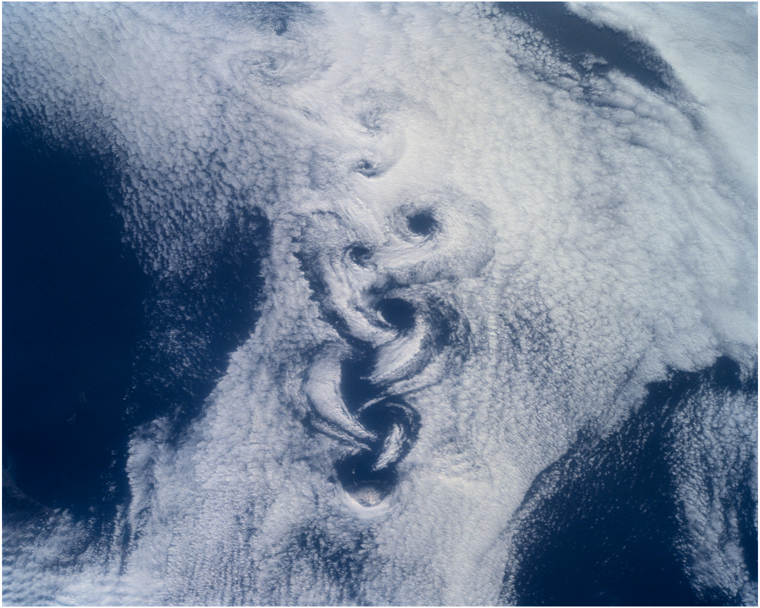



FIGURE 1. Observed experimental interface elevations for an oblong obstacle towed at speed $U = 10\text{cm s}^{-1}$ through the shallow layer of a two-layer fluid ($H_1 = 6$cm, $H_2 = 54$cm). Left panel: Non-rotating experiment. Right panel: Rotating experiment. The Froude number for both experiments (ratio of towing speed to interfacial gravity wave speed) is estimated to be in the range $F = 1.1 - 1.5$, and the nondimensional mountain height is $M = 0.5$. For the rotating experiment the period is $T = 120$s, corresponding to an inverse Burger number $B \approx 0.5$. In each panel the centre of the obstacle is marked by the '+' at the origin. Solid contours show regions where the interface rises (crests, marked C) and dashed contours depressed regions (troughs, marked T). Adapted from Johnson *et al.* (2006, see their Fig. 5).

# Observed stationary mountain waves : $b_{max} < H$



# "Penetrating" topography : $b_{max} > H$



# Exercise

- Derive (66) using the given scaling