Chapter 4: Dynamics in the equatorial region

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Course GFD M2 MOCIS

Plan

Equatorial dynamics in the RSW model

- Equations of motion and scaling
- Kelvin waves
- Yanai waves
- Rossby and inertia-gravity waves

Two-layer RSW model on the equatorial β- plane

Equatorial waves in the primitive equations

Observations of equatorial waves

- Satellite observations
- Extracting equatorial waves from data
- Spectrum of equatorial waves from OLR

RSW model on the equatorial β- plane

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \beta y \hat{\mathbf{z}} \wedge \mathbf{v} + g \nabla h = 0 \,. \tag{1}$$

$$\partial_t h + \nabla \cdot (\mathbf{v} h) = 0 \,, \tag{2}$$

Boundary conditions: decay at $y \to \pm\infty$

Equatorial scaling

Characteristic scales:

- Spatial scale - equatorial deformation radius: $L \sim \left(\frac{\sqrt{gH}}{\beta}\right)^{\frac{1}{2}}$
- Time-scale - $T \sim (\beta L)^{-1}$
- Velocity scale - $U \sim \frac{g \Delta H}{\beta L^2}$, where ΔH - typical deviation of h with respect to H

Equatorial Rossby number:

$$Ro = \epsilon = \frac{U}{\beta L^2} = \frac{\Delta H}{H}. \tag{3}$$

Non-dimensional equations

$$\partial_t \mathbf{v} + \epsilon \mathbf{v} \cdot \nabla \mathbf{v} + y \hat{\mathbf{z}} \wedge \mathbf{v} + \nabla \eta = 0\,, \tag{4}$$

$$\partial_t \eta + \nabla \cdot \mathbf{v} + \epsilon \nabla \cdot (\mathbf{v} \eta) = 0\,, \tag{5}$$

Linearisation ($\equiv \epsilon \to 0$ limit

$$\begin{aligned} u_t - yv + h_x &= 0, & (6)\\ v_t + yu + h_y &= 0, & (7)\\ h_t + u_x + v_y &= 0. & (8) \end{aligned}$$

Change of variables

$$f = \frac{1}{2}(u+h); \;\; g = \frac{1}{2}(u-h). \qquad (9)$$

Equations (6) - (8) are simplified:

$$\begin{aligned} f_t + f_x + \frac{1}{2}(v_y - yv) &= 0, & (10)\\ g_t - g_x - \frac{1}{2}(v_y + yv) &= 0, & (11)\\ v_t + y(f+g) + (f-g)_y &= 0. & (12) \end{aligned}$$

Parabolic cylindre (Gauss - Hermite) functions

Functional basis $\phi_n(y)$ at the infinite interval $-\infty \leq y \leq +\infty$ with decay b.c.:

$$\phi_n''(y) + (2n + 1 - y^2)\phi_n(y) = 0, \tag{13}$$

$$\phi_n(y) = \frac{H_n(y) e^{-\frac{y^2}{2}}}{\sqrt{2^n n! \sqrt{\pi}}}, \tag{14}$$

H_n Hermite polynomials:

$$H_0 = 1, \;\; H_1 = 2y, \;\; H_2 = 4y^2 - 2, \;\; \ldots . \tag{15}$$

$$\phi_n' + y\phi_n = \sqrt{2n}\phi_{n-1}, \;\; \phi_n' - y\phi_n = -\sqrt{(2n+1)}\phi_{n+1} \tag{16}$$

Kelvin waves

Particular solution with $v \equiv 0 \Rightarrow$:

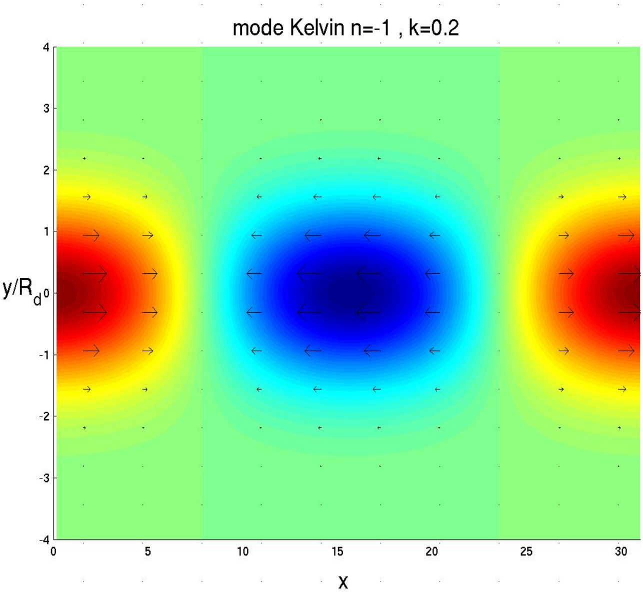
$$f_t + f_x = 0,\ g_t - g_x = 0,\ \Rightarrow f = F(x-t, y),\ g = G(x+t, y). \tag{17}$$

$$y(f+g) + (f-g)_y = 0,\ \Rightarrow F \propto e^{-\frac{y^2}{2}},\ G \propto e^{+\frac{y^2}{2}} \tag{18}$$

B.C. at $y \pm \infty \ \Rightarrow G \equiv 0 \Rightarrow$

$$u = F_0(x-t)e^{-\frac{y^2}{2}};\ \ h = F_0(x-t)e^{-\frac{y^2}{2}};\ \ v = 0. \tag{19}$$

Velocity and pressure distribution in a Kelvin wave



Yanai waves

Particular solution with $g = 0$, $f \neq 0, v \neq 0$. From (6) - (8) we get:

$$f_t + f_x + \frac{1}{2}(v_y - yv) = 0, \quad (20)$$

$$v_y + yv = 0, \quad (21)$$

$$v_t + yf + f_y = 0, \quad (22)$$

Solution by separation of variables:

$$v = v_0(x,t)\phi_0(y), \quad f = F_1(x,t)\phi_1(y). \quad (23)$$

Equations for v_0 and F_1:

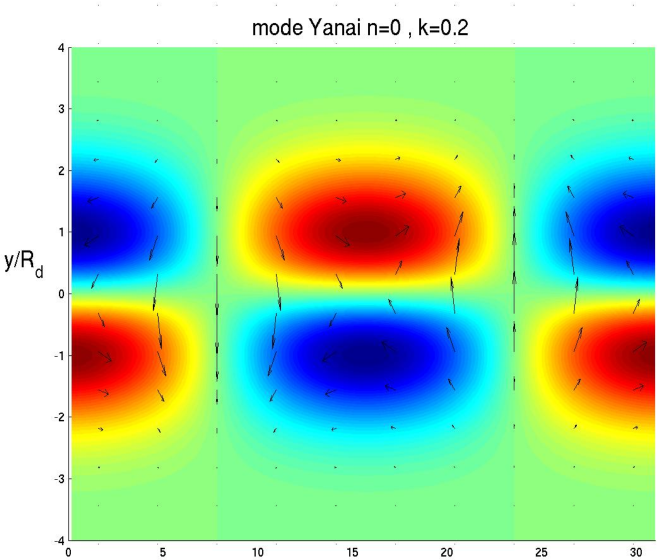
$$F_{1_t} + F_{1_x} - \frac{1}{\sqrt{2}} v_0 = 0, \quad v_{0_t} + \sqrt{2} F_1 = 0. \tag{24}$$

Dispersion relation:

Fourier-transformation $\propto e^{i(\omega t - kx)}$ → algebraic system for amplitudes. Solvability condition →

$$\omega = \frac{k}{2} \pm \sqrt{\frac{k^2}{4} + 1}, \tag{25}$$

Velocity and pressure distribution in a Yanai wave; eastward propagation



Rossby and inertia-gravity waves

General case: elimination of u and h (or f and g) in favour of v:

$$\partial_t \left(\nabla^2 v - y^2 v - \partial_{tt} v\right) + \partial_x v = 0. \tag{26}$$

Expansion of v in a series in ϕ_n, $v = \sum_n v_n(x,t)\phi_n(y)_; \Rightarrow$

$$\partial_t \left[\partial^2_{xx} v_n - (2n+1)v_n - \partial^2_{tt} v_n\right] + \partial_x v_n = 0. \tag{27}$$

Fourier - transformation $\tilde{v}_n(k,t) = \int dx e^{-ikx} v_n(x,t)$

$$\partial^3_{ttt}\tilde{v}_n + (k^2 + 2n + 1)\partial_t \tilde{v}_n - ik\tilde{v}_n = 0. \tag{28}$$

Hence

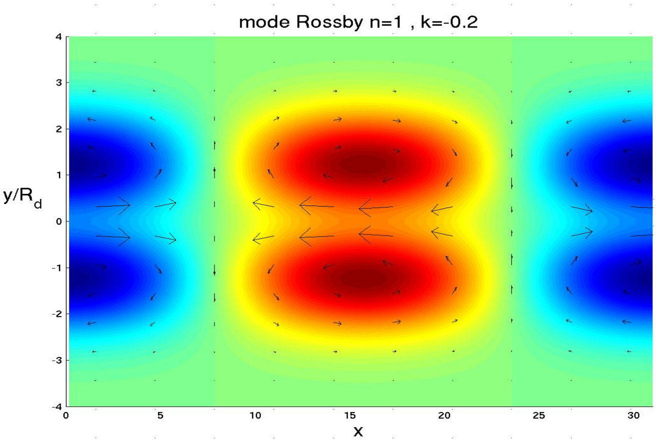
$$\tilde{v}_n = v_{n_1}(k)e^{i\omega_{n_1}t} + v_{n_2}(k)e^{i\omega_{n_2}t} + v_{n_3}(k)e^{i\omega_{n_3}t} \tag{29}$$

where $\omega_{n_\alpha}, \ \alpha = 1, 2, 3$ are 3 roots of the dispersion relation:

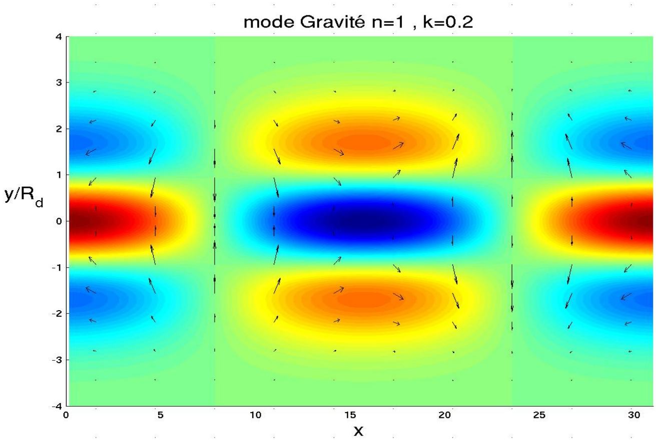
$$\omega_{n_\alpha}^3 - (k^2 + 2n + 1)\omega_{n_\alpha} - k = 0. \tag{30}$$

Lowest eigenvalue of ω - Rossby wave. Two others - inertia-gravity waves.

Velocity and pressure distribution in a Rossby wave; propagation uniquely westward



Velocity and pressure distribution in an inertie-gravity wave, eastward propagation



Remarks

- Formal application of the dispersion relation at $n = -1$:

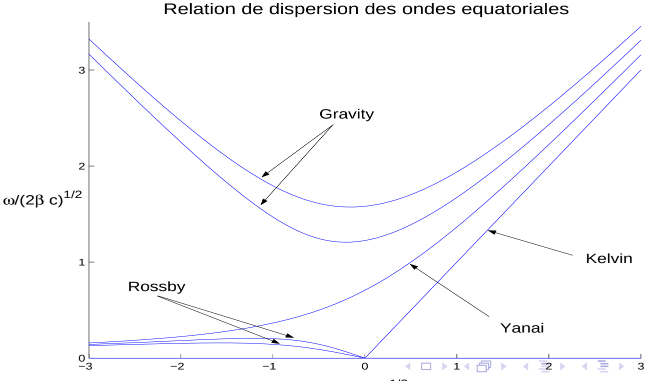
  $$\omega^3 - (k^2 - 1)\omega - k = 0, \Rightarrow (\omega - k)(\omega^2 + \omega k + 1) = 0. \quad (31)$$

  Positive root $\omega = k$: Kelvin wave,
- Formal application of the dispersion relation at $n = 0$:

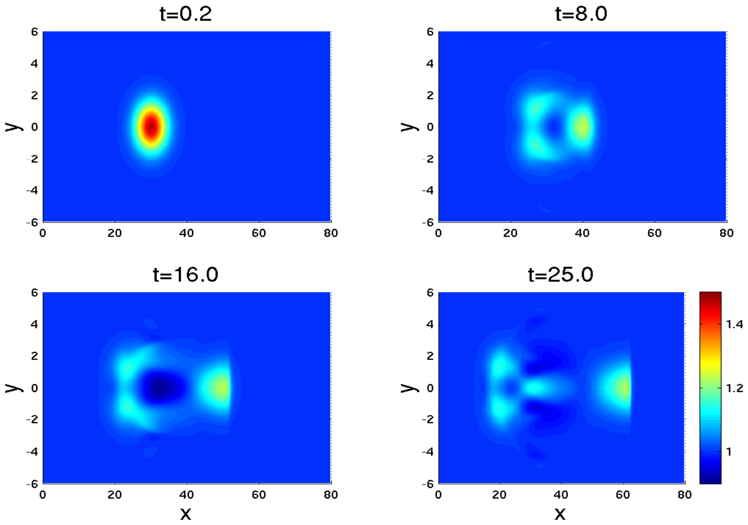
  $$\omega^3 - (k^2 + 1)\omega - k = 0, \Rightarrow (\omega + k)(\omega^2 - \omega k - 1) = 0. \quad (32)$$

  Positive root $\omega^2 - \omega k - 1 = 0$: Yanai waves
- $\omega_n = -\frac{k}{2n+1+k^2}$ - excellent approximate formula (precision 2%) for the Rossby branch.

Dispersion and spectral separation of equatorial waves


Relation de dispersion des ondes equatoriales

Adjustement of geopotential anomaly in the equatorial region



Exercises

- Obtain formula (26),
- Demonstrate that for equatorial Rossby waves

$$(u, v, \eta) = (U(y), V(y), H(y))\, e^{i(kx-\omega t)}, \tag{33}$$

$$\begin{aligned} V(y) &= \phi_n(y),\; U(y) = -i\frac{k\phi_n'(y) - \omega_n y\phi_n(y)}{\omega_n^2 - k^2}, \\ H(y) &= i\frac{\omega_n\phi_n'(y) - ky\phi_n(y)}{\omega_n^2 - k^2}. \end{aligned} \tag{34}$$

Equations of the 2-layer RSW model with a rigid lid onthe equatorial β- plane:

$$\partial_t \mathbf{v}_i + \mathbf{v}_i \cdot \nabla \mathbf{v}_i + \beta y \hat{\mathbf{z}} \wedge \mathbf{v}_i + \frac{1}{\rho_i} \nabla \pi_i = 0\,, i = 1, 2; \tag{35}$$

$$\partial_t h_i + \nabla \cdot (h_i \mathbf{v}_i) = 0 \tag{36}$$

$$(\rho_2 - \rho_1) g \eta = \pi_2 - \pi_1, \;\; h_1 + h_2 = H. \tag{37}$$

Simplifying hypotheses:

- $\rho_1 \to \rho_2, \; \pi_2 = \pi_1 + \rho_1 g' h_1, \; g' = g \frac{\rho_2 - \rho_1}{\rho_1}$
- $H_1 = H_2$

Scaling

- Spatial scale - baroclinic equatorial deformation radius : $L \sim \left(\frac{\sqrt{g'H}}{\beta}\right)^{\frac{1}{2}}$
- Time-scale - $T \sim (\beta L)^{-1}$
- Velocity scale - $U \sim \frac{g'\Delta H}{\beta L^2}$
- Pressure scale - $P_i \sim \rho_i U \beta L^2$

Barotropic and baroclinic velocities:

$$\mathbf{v}_{bt} = \frac{h_1 \mathbf{v}_1 + h_2 \mathbf{v}_2}{H}, \quad \mathbf{v}_{bc} = \mathbf{v}_1 - \mathbf{v}_2 \qquad (38)$$

Barotropic streamfunction:

$$h_1 + h_2 = \text{const} \Rightarrow \nabla\cdot(h_1\mathbf{v}_1 + h_2\mathbf{v}_2) = H\nabla\cdot\mathbf{v}_{bt} = 0 \Rightarrow \mathbf{v}_{bt} = \hat{\mathbf{z}}\wedge\nabla\psi$$

Non-dimensional linearised equations for ψ, $\mathbf{v}_{bc} \equiv \mathbf{v} = (u, v)$, $\eta = h_1 - H_1$

$$\nabla^2 \psi_t + \psi_x = 0 \tag{39}$$

$$\mathbf{v}_t + \nabla \eta + y \hat{\mathbf{z}} \times \mathbf{v} = 0 \tag{40}$$

$$\eta_t + \nabla \cdot \mathbf{v} = 0, \tag{41}$$

Exercise: Derive the equations (39 - 41)

Wave solutions

- "Free" barotropic Rossby waves

$$\psi_0 = A_\psi e^{i(\theta + ly)} + c.c.;\ \theta = kx - \omega t, \tag{42}$$

with dispersion

$$\omega = -k/(k^2 + l^2), \tag{43}$$

- "Trapped" baroclinic waves:

$$(u, v, \eta) = (iU_n, V_n, iH_n)\, Ae^{i\theta_n} + c.c.;\ \theta_n = kx - \omega_n t \tag{44}$$

with dispersion

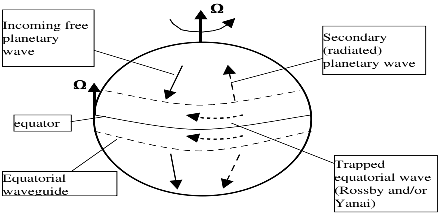
$$\omega_n^3 - (k^2 + 2n + 1)\omega_n - k = 0;\ n = -1, 0, 1, 2, \dots, \tag{45}$$

- Kelvin, Yanai, Rossby, Inertia-Gravity

Equator = waveguide transparent for barotropic waves

Equatorial waveguide and planetary waves

Interaction free planetary waves -trapped equatorial waves



Linearised primitive equations on the equatorial β-plane

$$\begin{aligned} u_t - \beta y v + \phi_x &= 0, \\ v_t + \beta y u + \phi_y &= 0, \qquad (46) \\ u_x + v_y - \left(N(z)^{-2}\phi_{zt}\right)_z &= 0, \qquad (47) \end{aligned}$$

Separation of variables 1

$(u, v, \phi) = (\bar{u}, \bar{v}, \bar{\phi})(y, z)e^{i(kx-\omega t)}$:

$$\begin{aligned} -i\omega\bar{u} - \beta y\bar{v} + ik\bar{\phi} &= 0, \\ -i\omega\bar{v} + \beta y\bar{u} + \bar{\phi}_y &= 0, \qquad (48) \\ ik\bar{u} + \bar{v}_y + i\omega\left(N(z)^{-2}\bar{\phi}_z\right)_z &= 0, \qquad (49) \end{aligned}$$

Elimination of $\bar{u}, \bar{\phi}$:

$$\left(\beta k\omega + \omega^2 k^2\right)\bar{v} + \omega^2\left[\left(\omega^2 - \beta^2 y^2\right)\left(\frac{\bar{v}_z}{N^2(z)}\right)_z - \bar{v}_{yy}\right] = 0. \tag{50}$$

Separation of variables 2

$\bar{v} = \chi(z)\nu(y) \Rightarrow$

- $$\frac{1}{\chi}\left(\frac{\chi'}{N^2(z)}\right)' = -\kappa^2 = \mathrm{const}, \tag{51}$$
- $$-\frac{\nu''}{\nu} + k^2 + \frac{\beta k}{\omega} + (\beta^2 y^2 - \omega^2)\kappa^2 = 0 \tag{52}$$

Particular case: $N = \text{const}$

$\kappa^2 N^2 = l^2 = \text{const}$, vertical modes -harmonic functions

Renormalisation: $y \to \left(\frac{N}{\beta l}\right)^{\frac{1}{2}} y$, $\left(\frac{N}{\beta l}\right)^{\frac{1}{2}}$ - equivalent height $\Rightarrow$

$$\nu''(y) + \left(\frac{\omega^2 l}{N\beta} - y^2 - \frac{k^2 N}{\beta l} - \frac{kN}{\omega l}\right)\nu(y) = 0 \qquad (53)$$

Expansion in $\phi_n(y)$ $\Rightarrow$

Dispersion relation :

$$\omega^3 - \left[(2n+1)\frac{N\beta}{l} + \frac{k^2 N^2}{l^2}\right]\omega - \frac{\beta k N^2}{l^2} = 0. \qquad (54)$$

Equatorial waves in Primitive Equations

- Kelvin waves, $n = -1$:

$$\left(\omega - \frac{kN}{l}\right)\left[\omega\left(\omega + \frac{Nk}{l}\right) + \frac{\beta N}{l}\right] = 0, \qquad (55)$$

Positive roots: $\omega = \frac{N}{l} k$.

- Yanai waves, $n = 0$:

$$\left(\omega + \frac{kN}{l}\right)\left[\omega\left(\omega - \frac{Nk}{l}\right) - \frac{\beta N}{l}\right] = 0, \qquad (56)$$

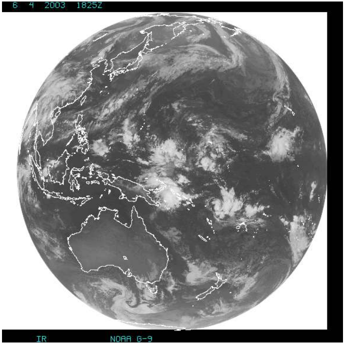
Positive roots: $\omega = \frac{N}{2l} k \pm \sqrt{\frac{N^2 k^2}{4l^2} + \frac{\beta N}{l}}$.

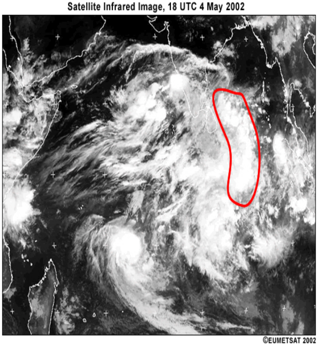
- Rossby waves, $n > 0$, lower branch :

$$\omega \approx -\frac{\beta k}{(2n+1)\frac{\beta l}{N} + k^2}. \qquad (57)$$

- Inertia-gravity waves: $n > 0$, upper branches.

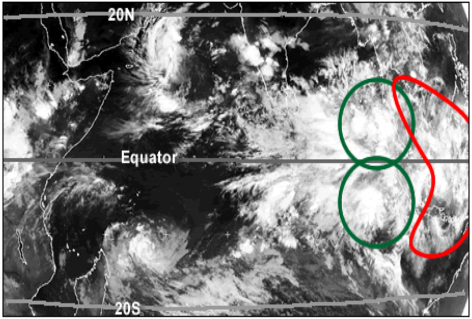
Observations of atmospheric equatorial waves





Left panel: Rossby wave. *Right panel:* Kelvin wave.

Rossby and Kelvin waves produced by heating due to the oceanic warmpool



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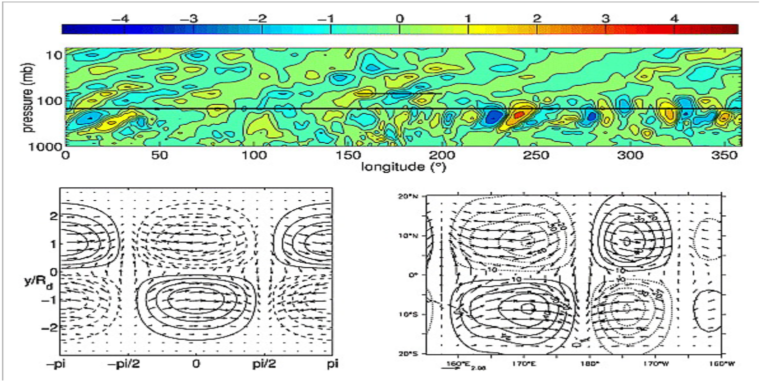
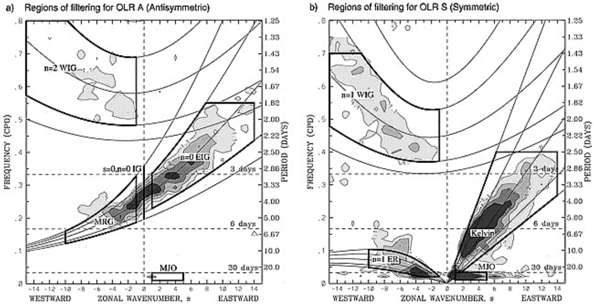
Yanai waves from ERA40 dataset



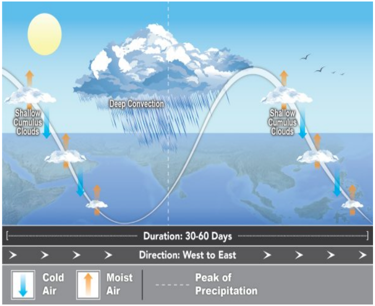
Top - vertical section along the equator of the eastward propagating meridional velocity anomaly, with average tropopause. Short black line - section of the lower right plot. *Bottom left* - horizontal section of wind and geopotential anomaly of YW, as predicted by shallow water theory; *Bottom right* - horizontal section at 70 hPa.

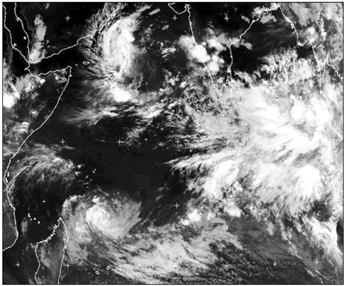
Outgoing long-wave radiation (OLR) vs dispersion diagram of equatorial waves (Wheeler & Kiladis, 1999)



Good agreement with the only discrepancy: presence of much slower than Kelvin waves signal, MJO.

Madden-Julian Oscillation (MJO)

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Periodic moving slowly eastward over Indo-Pacific warm-pool, dying out in the Pacific.