Chapter 5. Instabilities in Geophysical Flows.

Part 1. Plane-parallel flows

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Plan

Arbitrary dynamical system

$$\dot{\mathcal{U}} = \mathcal{M}[\mathcal{U}], \tag{1}$$

$\mathcal{U}$ - dynamical variable(s) , $\mathcal{M}$ - operator defined by the structure of the model. Solutions : trajectoires in the space of $\mathcal{U}$:

$$\mathcal{U}(t_0) \longrightarrow \mathcal{U}(t) \tag{2}$$

In hydrodynamics $\mathcal{U} = (\mathbf{v}, \rho, p, ...)$.
$\mathcal{U}_0$: exact solution, for example the state of rest $\mathcal{M}[\mathcal{U}_0] = 0$, or other.
Linearisation : $\mathcal{U} = \mathcal{U}_0 + u$, $||u|| << 1 \ \Rightarrow$ linear equations :

$$\dot{u} = \hat{\mathcal{L}}[\mathcal{U}_0] \circ u, \tag{3}$$

$\hat{\mathcal{L}}$ - linear operator

Linear and non-linear (Lyapunov) stability

Linear stability

Linearised system $\to$ Fourier transformation :
$u(t) \to \hat{u}(\omega)e^{i\omega t} \to$ eigenproblem for eigenvalues $\omega \to$ spectrum of ω (dispersion relation).
Complex eigenvalues, in general : $\omega = \omega_r + i\omega_i$
Linear (in)stability : $\omega_i \geq 0 (\omega_i < 0) \leftrightarrow$ exponential growth (decay) of small perturbations of the solution.

Stability according to Lyapunov

$$\forall \epsilon \ \ \exists \delta : ||u||_{t=0} < \delta \ \Rightarrow \ ||u||_{\forall t>0} < \epsilon. \tag{4}$$

Plane-parallel barotropic incompressible flow

Flow the plane (x, y)($\equiv$ homogeneity in z- direction) :
$\mathbf{v} = U(y)\hat{\mathbf{x}}$ - exact solution of the 2D Euler equations (= QG equation on the f - plane).
Velocity via streamfunction ψ : $u = \psi_y$, $v = -\psi_x \Rightarrow$

$$\nabla^2 \psi_t + \mathcal{J}(\nabla^2 \psi, \psi) = 0. \quad (5)$$

Plane-parallel stationary flow solution :

$$\psi_0 = \int^y dy' \, U(y'). \quad (6)$$

Linearisation : $\psi = \psi_0 + \phi \Rightarrow \nabla^2 \psi = U'(y) + \nabla^2 \phi \rightarrow$

$$\nabla^2 \phi_t + U(y) \nabla^2 \phi_x - \phi_x U''(y) = 0. \quad (7)$$

Fourier transformation : $\phi(x, y, t) \rightarrow \hat{\phi}(y) e^{ik(x-ct)} \Rightarrow$

$$\hat{\phi}''(y) - \left[k^2 + \frac{U''(y)}{U(y) - c} \right] \hat{\phi}(y) = 0. \quad (8)$$

Boundary conditions :

Channel : $y_1 \leq y \leq y_2$ with free-slip conditions (no viscosity), or the entire plane ($y_{1,2} \to \infty$) :

$$v|_{y=y_{1,2}} = \phi_x|_{y=y_{1,2}} = 0, \Rightarrow \hat{\phi}\Big|_{y=y_{1,2}} = 0$$

Multiplication by the conjugate solution (*) and integration by y

$$\int_{y_1}^{y_2} dy \left[\hat{\phi}^*(y)\left(\hat{\phi}''(y) - \left[k^2 + \frac{U''(y)}{U(y)-c}\right]\hat{\phi}(y)\right)\right] = 0 \quad (9)$$

Integration by parts + boundary conditions :

$$\int_{y_1}^{y_2} dy \left(\hat{\phi}^{*\prime}(y)\hat{\phi}'(y) + \left[k^2 + \frac{U''(y)}{U(y)-c}\right]\hat{\phi}^*(y)\hat{\phi}(y)\right) = 0 \quad (10)$$

Imaginary part :

$$c_i \int_{y_1}^{y_2} dy \, \frac{U''(y)}{|U(y) - c|^2} \hat{\phi}^*(y)\hat{\phi}(y) = 0 \quad \Rightarrow$$

$$\int_{y_1}^{y_2} dy \, \frac{U''(y)}{|U(y) - c|^2} \hat{\phi}^*(y)\hat{\phi}(y) = 0 \quad \text{if} \qquad c_i \neq 0. \tag{11}$$

In the absence of critical layers ($U(y) - c \neq 0$), if the flow is unstable, then $U(y)$ has an inflexion point $\exists y_0 : U''(y_0) = 0$.

Plane-parallel stratified flow in the vertical plane (x, z)

$\mathbf{v} = U(z)\hat{\mathbf{x}}$ - exact solution of the non-hydrostatic primitive equations with pressure $P_0(z)$ and density $\rho_0(z)$ in hydrostatic equilibrium.

Linearisation about this solution :

$$\begin{aligned} \rho_0(z)\left(u_t + U(z)u_x + wU'(z)\right) &= -p_x, \\ \rho_0(z)\left(w_t + U(z)w_x\right) + g\rho &= -p_z, \\ u_x + w_z = 0,\ \rho_t + +U(z)\rho_x + w\rho_0'(z) &= 0. \end{aligned} \tag{12}$$

Streamfunction : $u = \psi_z,\ w = -\psi_x$.
Fourier transformation in x, t :

$$\psi \to \phi(z)e^{ik(x-ct)}, \quad \rho \to r(z)e^{ik(x-ct)} \Rightarrow \tag{13}$$

$$u \to \phi'(z)e^{ik(x-ct)}, \quad w \to -ik\phi(z)e^{ik(x-ct)} \tag{14}$$

Elimination of variables

Elimination of p by cross - differentiation :

$$\begin{aligned} (U-c)\rho_0(\phi'' - k^2\phi) - gr &+ \\ \rho_0' \left[(U-c)\phi' - U'\phi\right] - \rho_0 U''\phi &= 0 \\ (U-c)r - \rho_0'\phi &= 0 \end{aligned} \quad (15)$$

Elimination of r :

$$\begin{aligned} (U-c)^2\rho_0(\phi'' - k^2\phi) - g\rho_0'\phi &+ \\ \rho_0'(U-c)\left[(U-c)\phi' - U'\phi\right] - \rho_0 U''(U-c)\phi &= 0 \end{aligned}$$

Integral form

Change of variable :

$$\phi = (U-c)^{\frac{1}{2}}\Phi \;\Rightarrow \tag{16}$$

$$\left[\rho_0(U-c)\Phi'\right]' + \left[-\frac{1}{2}\left(\rho_0 U'\right)' \quad + \right.$$
$$\left. \rho_0 k^2(U-c) - \rho_0 \frac{\frac{U'^2}{4} + g\frac{\rho_0'}{\rho_0}}{U-c}\right]\Phi \quad = \quad 0. \tag{17}$$

Multiplication by Φ^* and integration (by parts) in z with b. c. of the channel type

$$\int_{z_1}^{z_2} dz \left[\rho_0(U-c)\left(|\Phi'|^2 + k^2|\Phi|^2\right) + \frac{(\rho_0 U')'}{2}|\Phi|^2 \quad + \right.$$
$$\left. \rho_0\left(\frac{U'^2}{4} + g\frac{\rho_0'}{\rho_0}\right)(U-c)^*\frac{|\Phi|^2}{|U-c|^2}\right] \quad = \quad 0.$$

Integral estimate and instability criterion

Imaginary part :

$$c_i \int_{z_1}^{z_2} dz \left[\rho_0 \left(|\Phi'|^2 + k^2 |\Phi|^2 \right) - \rho_0 \left| \frac{\Phi}{U-c} \right|^2 \left(\frac{U'^2}{4} + g \frac{\rho_0'}{\rho_0} \right) \right] = 0. \tag{18}$$

Miles-Howard criterion : $Ri < \frac{1}{4}$

$$c_i \neq 0 \Rightarrow \frac{U'^2}{4} + g \frac{\rho_0'}{\rho_0} > 0, \tag{19}$$

Brunt - Väisälä frequency : $N^2 = -g \frac{\rho_0'}{\rho_0} \Rightarrow \frac{N^2}{U'^2} < \frac{1}{4} \leftrightarrow$ $Ri < \frac{1}{4}$, where $Ri = \frac{N^2}{U'^2}$ - Richardson number.

Kelvin -Helmholtz (KH) instability

The model :

Non-hydrostatic Euler equations for two layers of incompressible fluid with $\rho_i = \text{const}$, $i = 1, 2$ without rotation ($Ro \to \infty$) in the vertical plane x, z.

Equations of motion :

$$\begin{aligned} u_t^{(i)} + u^{(i)} u_x^{(i)} + w^{(i)} u_z^{(i)} &= -\frac{1}{\rho_i} P_x^{(i)}, \\ w_t^{(i)} + u^{(i)} w_x^{(i)} + w^{(i)} w_z^{(i)} + g &= -\frac{1}{\rho_i} P_z^{(i)}, \\ u_x^{(i)} + w_z^{(i)} &= 0. \end{aligned} \tag{20}$$

Boundary conditions :

- Dynamic b.c. :

$$P^{(1)}\Big|_{z=\eta} = P^{(2)}\Big|_{z=\eta}, \qquad (21)$$

- Kinematic b.c. :

$$\eta + u^{(i)}\eta_x = w^{(i)}\Big|_{z=\eta}, \quad i = 1, 2. \qquad (22)$$

where $\eta(x, t)$ - position of the interface between the layers 1 (superior) and 2 (inferior).

Stationary solution :

$$w^{(i)} = 0;\ u^{(i)} = U_i = \text{const};\ \eta = 0;\ P_z^{(i)} = -g\rho_i,\ i = 1, 2. \qquad (23)$$

Linearisation about this solution :

- Equations for perturbations :

$$
\begin{aligned}
u_t^{(i)} + U_i u_x^{(i)} &= -\frac{1}{\rho_i} p_x^{(i)}, \\
w_t^{(i)} + U_i w_x^{(i)} &= -\frac{1}{\rho_i} p_z^{(i)}, \\
u_x^{(i)} + w_z^{(i)} = 0 \; \Rightarrow \nabla^2 p^{(i)} &= 0.
\end{aligned} \tag{24}
$$

- Boundary conditions :

$$
p^1\Big|_{z=0} - p^2\Big|_{z=0} = g(\rho_1 - \rho_2)\eta. \tag{25}
$$

Solution of the Laplace equation :

$$p^{(1)} = \bar{p}_1 e^{-kz} e^{i(kx-\omega t)}, \;\; p^{(2)} = \bar{p}_2 e^{+kz} e^{i(kx-\omega t)} \tag{26}$$

Separation of variables in $w^{(i)}$:

$$w^{(i)} = \bar{w}_i(z) e^{i(kx-\omega t)} \Rightarrow \tag{27}$$

$$\bar{w}_1 = -i\frac{k\bar{p}_1 e^{-kz}}{\rho_1(kU_1-\omega)}, \;\; \bar{w}_2 = i\frac{k\bar{p}_2 e^{kz}}{\rho_2(kU_2-\omega)}. \tag{28}$$

Kinematic b.c. :

$$\eta = \bar{\eta} e^{i(kx-\omega t)} \Rightarrow -i(\omega - kU_i)\bar{\eta} = \bar{w}_i\big|_{z=0}, \Rightarrow \tag{29}$$

$$\bar{p}_1 = -\frac{\bar{\eta}}{k}\rho_1(\omega - kU_1)^2, \;\; \bar{p}_2 = +\frac{\bar{\eta}}{k}\rho_2(\omega - kU_2)^2 \tag{30}$$

Dynamic b.c. :

$$\rho_2(\omega - kU_2)^2 + \rho_1(\omega - kU_1)^2 = kg(\rho_2 - \rho_1) \equiv kg\Delta\rho, \quad \Delta\rho > 0. \Rightarrow \tag{31}$$

Dispersion relation :

$$(\rho_1+\rho_2)\omega^2 - 2k(U_1\rho_1 + U_2\rho_2)\omega + \left[k^2(\rho_1 U_1^2 + \rho_2 U_2^2) - kg\Delta\rho\right] = 0. \tag{32}$$

Solution in the moving frame $U_2 = 0, U_1 = U$:

$$c = \frac{\omega}{k} = \frac{U\rho_1 \pm \sqrt{(\rho_1+\rho_2)\frac{g\Delta\rho}{k} - \rho_1\rho_2 U^2}}{\rho_1+\rho_2} \tag{33}$$

Instability of short waves :

$$k > \frac{g\Delta\rho}{U^2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right). \tag{34}$$

Shear instability :

Particular case $g = 0$:

$$c = \frac{\omega}{k} = U\frac{\rho_1 \pm i\sqrt{\rho_1\rho_2}}{\rho_1 + \rho_2} \Rightarrow \tag{35}$$

always unstable

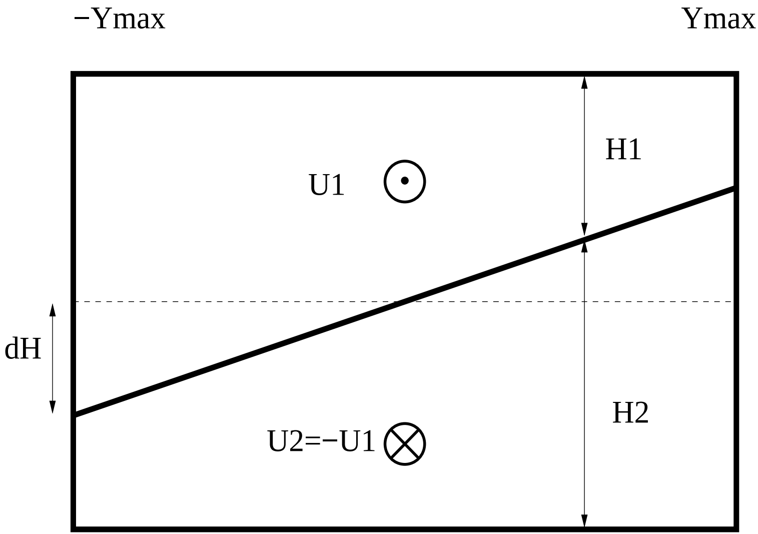
Example of KH instability

Exercise :

Analyse the KH instability in the two-layer RSW model without rotation, with layers of non-perturbed depths $H_{1,2}$, with flat bottom and the rigid lid. Demonstrate that the instability threshold corresponds to the critical shear :

$$U_c = \frac{1}{2}\sqrt{g\Delta\rho\left(\frac{H_2}{\rho_2} + \frac{H_1}{\rho_1}\right)} \quad (36)$$

Phillips model



2-layer RSW with rigid lid on the f-plane

$$\begin{aligned} \partial_t \mathbf{v}_j + \mathbf{v}_j \cdot \nabla \mathbf{v}_j + f\hat{\mathbf{z}} \wedge \mathbf{v}_j &= -\frac{1}{\rho_j} \nabla \pi_j \\ \partial_t h_j + \nabla \cdot (\mathbf{v}_j h_j) &= 0, \quad h_1 + h_2 = H \\ \pi_2 - \pi_1 &= g(\rho_2 - \rho_1)\eta, \end{aligned} \tag{37}$$

where $\mathbf{v}_j = (u_j,\ v_j)$, $j = 1, 2$, there is no summation over repeating index, and $\eta(x, y, t)$ is the deviation of the interface.

Linearisation about the mean shear flow

Mean flow : geostrophic equilibrium in y

$$h_j = H_j(y), \quad u_j = U_j(y) = -\frac{1}{\rho_j f}\partial_y \Pi_j, \quad v_j \equiv 0. \qquad (38)$$

Linearisation about this flow

$h_j = H_j(y) + (-1)^j \eta(x,y,t),\ \pi_j \to \Pi_j(y) + \pi_j(x,y,t).$

$$\begin{aligned}
\partial_t u_j + U_j \partial_x u_j + v_j \partial_y U_j - f v_j &= -\frac{1}{\rho_j}\partial_x \pi_j \\
\partial_t v_j + U_j \partial_x v_j + f u_j &= -\frac{1}{\rho_j}\partial_y \pi_j \\
\partial_t \eta + U_j \partial_x \eta &= (-1)^{j+1}\left(H_j \partial_x u_j + \partial_y (H_j v_j)\right) \\
\pi_2 - \pi_1 &= g(\rho_2 - \rho_1)\eta. \qquad (39)
\end{aligned}$$

Phillips model : $U_j = \text{const}$, $H_j(y)$ is linear function of y.

Non-dimensionalised linearised system

Mean flow : $U_1 = -U_2 \equiv U_0$
Scaling : time-scale f^{-1}, vertical scale $H_0 = H_2(0)$, horizontal scale $R_d = \frac{(g' H_0)^{\frac{1}{2}}}{f}$ - baroclinic deformation radius, where $g' = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}$ - reduced gravity, velocity scale U_0, pressure scales $\rho_i f U_0 R_d$. Width of the channel L, Burger number $Bu = R_d^2 / L^2$. Weak stratification limit $\rho_2 \to \rho_1$.

$$
\begin{aligned}
\partial_t u_j + F(-1)^{j+1} \partial_x u_j - v_j &= -\partial_x \pi_j \\
\partial_t v_j + F(-1)^{j+1} \partial_x v_j + u_j &= -\partial_y \pi_j \\
\partial_t \eta + F(-1)^{j+1} \partial_x \eta &= (-1)^{j+1} \left(H_j \partial_x u_j + \partial_y (H_j v_j) \right) \\
\pi_2 - \pi_1 &= \frac{2}{F} \eta, \qquad (40)
\end{aligned}
$$

where $F = \frac{U_0}{f R_d}$ - Froude ($\equiv$ Rossby) number.

Processing the linearised system

Difficulty : The linearised system has coefficients depending on y (even for constant U_j, H_j contain y).

Method :

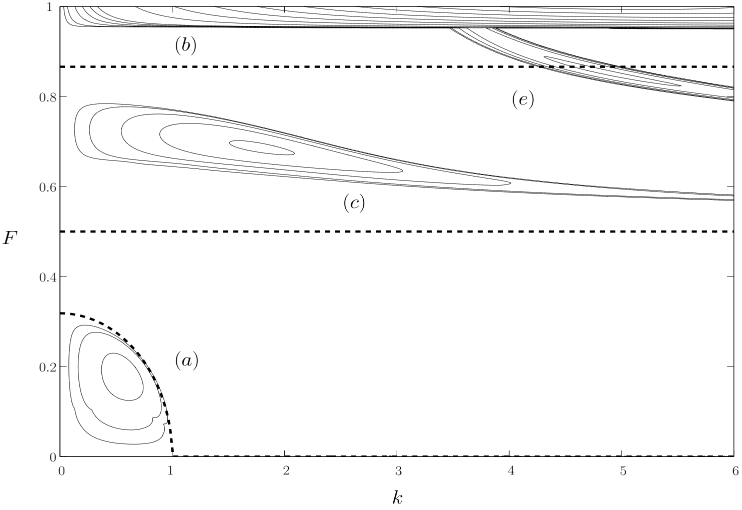
1. Fourier-transform in "good" variables :

$$(u_j,\ v_j,\ \pi_j)(x,y,t) = (\hat{u}_j,\ \hat{v}_j,\ \hat{\pi}_j)(y)e^{i(kx-\omega t)} \quad \rightarrow$$

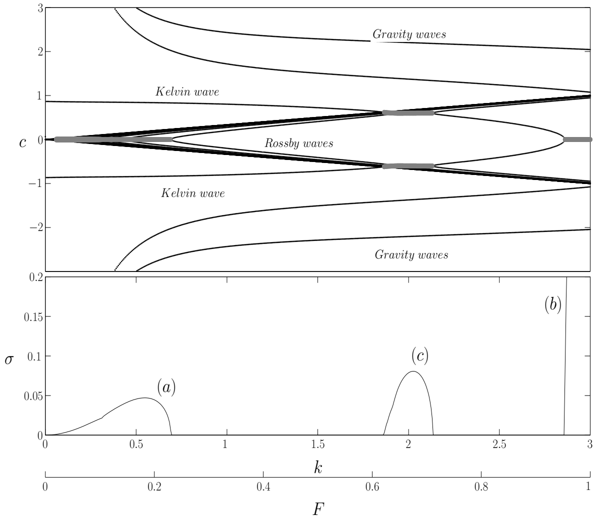
   a system of linear first-order ordinary differential equations for $(\hat{u}_j,\ \hat{v}_j,\ \hat{\pi}_j)(y)$,
2. Discretisation of this system on a regular or, better, Chebyshev, grid (collocation).
3. Numerical solution of the resulting algebraic system for eigenvalues ω for each fixed k.

Result : eigenvalues ω and corresponding eigenfunctions $(\hat{u}_j,\ \hat{v}_j,\ \hat{\pi}_j)(y)$ as functions of $k \Rightarrow$ stability diagram for $\Re\omega(k)$ (dispersion relation) and $\Im\omega(k)$ (growth rate)

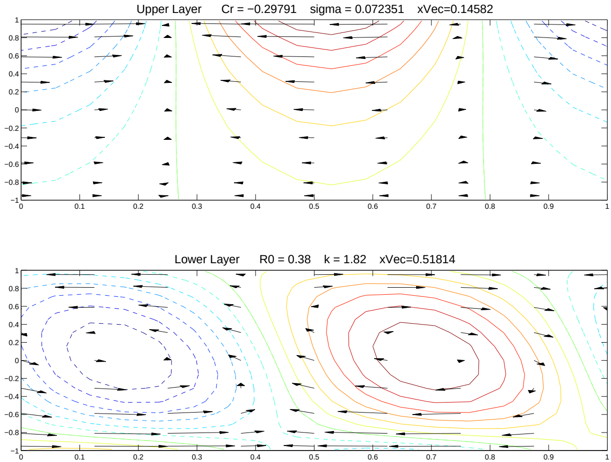
Stability diagram in the $F - k$ plane



Phase velocity (top) and growth rate bottom) of eigenmodes



Structure of the unstable Rossby-Kelvin mode



Résumé

Co-existence of geostrophic and ageostrophic instabilities of the balanced flow :

- Classical geostrophic baroclinic instability : $Ro \to 0$ et $k \to 0$
- Strongly ageostrophic KH-type instability : $Ro \to \infty$, all k
- New ageostrophic hybrid (Rossby-Kelvin) instability : $Ro \sim 1$

Heuristic analysis of instabilities in the 2-layer model

Dispersion relations layerwise :

$$D_{1,2}(\omega, k) = 0 - \text{no coupling} \tag{41}$$

Two curves close in the vicinity of a point k^* :

$$D_1(\omega^*, k^*) = 0, \quad D_2(\omega^* + \delta, k^*) = 0, \; |\delta| << \omega^*. \tag{42}$$

Weak coupling :

$$D_1(\omega, k) D_2(\omega, k) = \epsilon \tag{43}$$

At point k^* the eigenfrequencies become $\omega^* + \Delta$ and $\omega^* + \delta + \Delta$, $|\Delta| << \omega^*$.

Taylor series at point k^* :

$$\left[D_1(\omega^*, k^*) + \frac{\partial D_1}{\partial \omega}\Delta + ...\right]\left[D_2(\omega^*, k^*) + \frac{\partial D_2}{\partial \omega}(\delta + \Delta) + ...\right] = \epsilon \quad (44)$$

Quadratic equation for Δ :

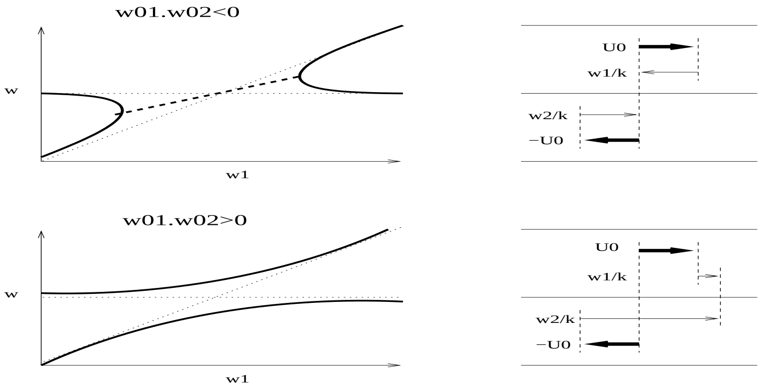
$$\Delta^2 - \delta\Delta - \epsilon\left(\frac{\partial D_1}{\partial \omega}\right)^{-1}\left(\frac{\partial D_2}{\partial \omega}\right)^{-1} = 0. \quad (45)$$

Instability : $Im(\Delta) \neq 0$:

- δ small enough and/or ϵ strong enough,
- $\epsilon\left(\frac{\partial D_1}{\partial \omega}\right)^{-1}\left(\frac{\partial D_2}{\partial \omega}\right)^{-1} < 0$

2-layer systems : the waves should propagate in the opposite directions, with close absolute frequencies (resonance)

Conditions of resonance

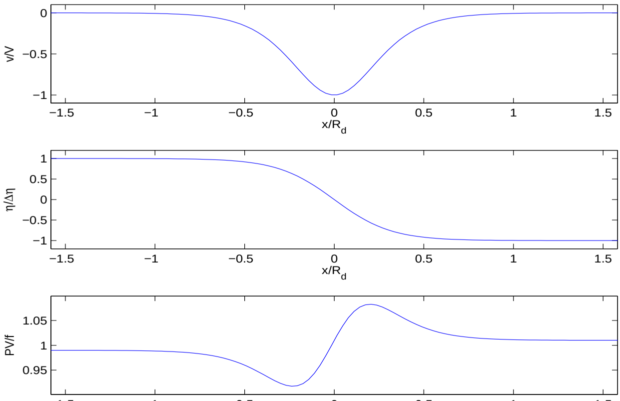


Barotropic jet on the f- plane in RSW

Exact solution of RSW equations : balanced jet :

$$\bar{u} = 0, \quad \bar{v} = -V_0 \operatorname{sech}^2\left(\frac{x}{L}\right), \quad h = H_0 + \bar{\eta} = H_0 - \Delta\eta \tanh\left(\frac{x}{L}\right),$$

$V_0 = \frac{g\Delta\eta}{fL}$ - peak velocity, L -jet width, H_0 - mean height.



Non-dimensional linearised equations

Small perturbations :

$$u \to u,\ v \to \bar{v} + v,\ \eta \to \bar{\eta} + \eta$$

$$\begin{cases} Ro\,(\partial_t u + \bar{v}\partial_y u) - v + \partial_x \eta = 0, \\ Ro\,(\partial_t v + u\partial_x \bar{v} + \bar{v}\partial_y v) + u + \partial_y \eta = 0, \\ Ro\,(\partial_t \eta + \partial_x (u\bar{\eta}) + \bar{v}\partial_y \eta + \bar{\eta}\partial_y v) + Bu\,(\partial_x u + \partial_y v) = 0. \end{cases} \tag{46}$$

Here $Ro = \frac{V_0}{fL}$, $Bu = \frac{R_d^2}{L^2} = \frac{gH_0}{f^2L^2}$ and standard geostrophic scaling is used.

Reduction to a system of ODEs

Fourier-transform in stream-wise direction

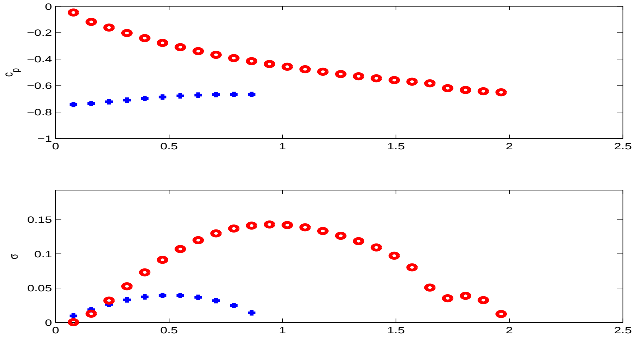
$$(u, v, \eta) = (ik\hat{u}, \hat{v}, \hat{\eta}) \exp\{i(ky - \omega t)\} + \text{c.c.} \;\rightarrow$$

eigenvalue problem : $\mathcal{M}\boldsymbol{a} = c\,\boldsymbol{a}$, with $\boldsymbol{a} = (\hat{u}, \hat{v}, \hat{\eta})$ and

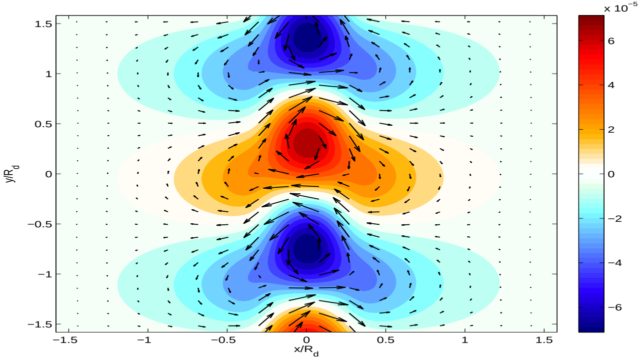
$$\mathcal{M} = \begin{pmatrix} \bar{v} & \frac{1}{Ro\,k^2} & -\frac{1}{Ro\,k^2}\partial_x \\ \frac{1}{Ro} + \partial_x \bar{v} & \bar{v} & 1 \\ (\partial_x \bar{\eta} + \bar{\eta}\partial_x) + \frac{Bu}{Ro}\partial_x & \bar{\eta} + \frac{Bu}{Ro} & \bar{v} \end{pmatrix}. \quad (47)$$

Solution by discretisation using Chebyshev collocation method.

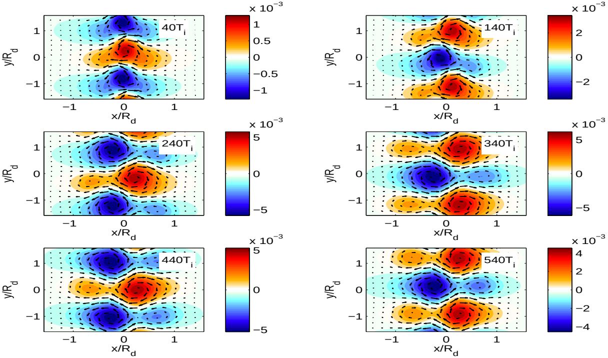
Stability diagram for a geostrophic jet with $Ro = 01$, $Bu = 10$



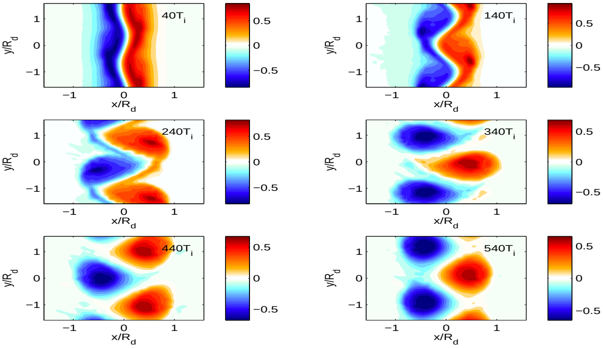
The most unstable mode



Evolution of the anomaly of H



Evolution of the relative vorticity : formation of secondary vortices



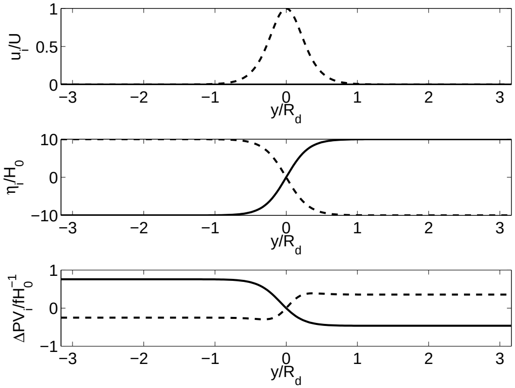
Baroclinic Bickley jet

Upper-layer jet in geostrophic equilibrium on the f- plane - exact solution of the 2-layer RSW equations with a free surface : Profiles of velocity and geopotential :

$$\bar{u}_1 = 0, \quad \bar{\eta}_1 = \frac{1}{\alpha - 1} \tanh(y),$$
$$\bar{u}_2 = \mathrm{sech}^2(y), \quad \bar{\eta}_2 = \frac{-1}{\alpha - 1} \tanh(y).$$

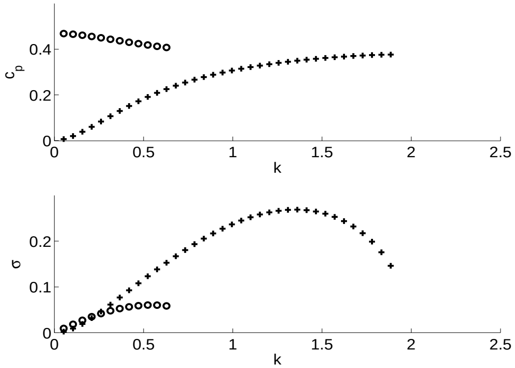
No deviation of the free surface : $\bar{\eta}_1 + \bar{\eta}_2 = 0$.
Parametres : $Ro = 0.1$, $Bu = 10$ - typical for the atmospheric jets.

Upper-layer Bickley jet



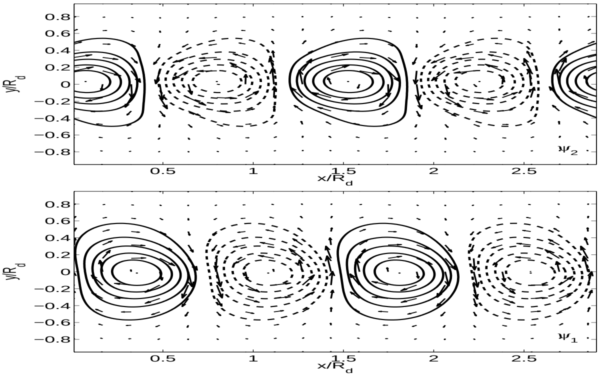
Zonal velocity $\bar{u}_i$, deviation of thickness $\bar{\eta}_i$, PV anomaly. Lower (upper) layer : continuous (dashed).

Linear stability diagram



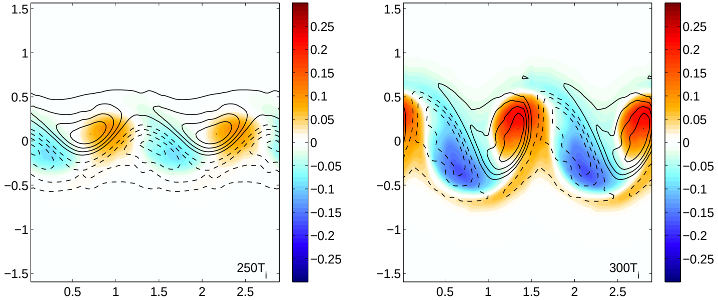
Phase velocity (top) and growth rate (bottom) of the unstable modes.

The most unstable mode :



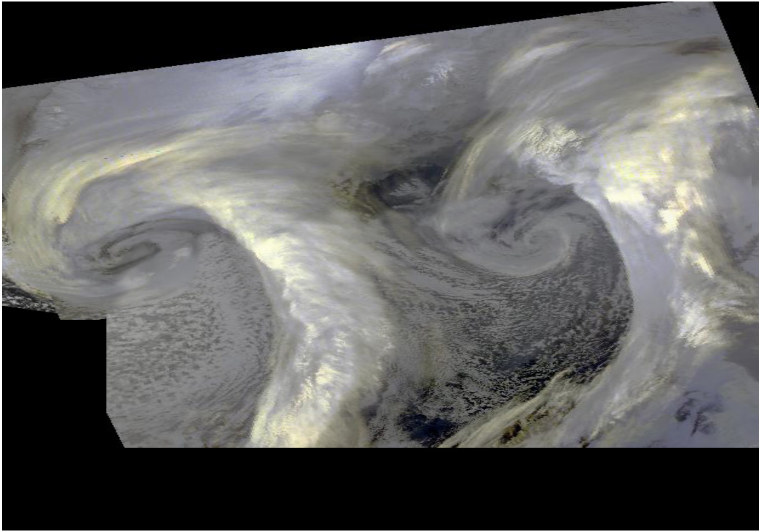
The most unstable mode of the upper-layer Bickley jet. Geostrophic streamfunctions and velocities in the upper and lower layers.

Non-linear saturation



Relative vorticity in the lower (colours) and upper (contours) layers.

Baroclinic instability in Nature



Barotropic Bickley jet in 2-layer system

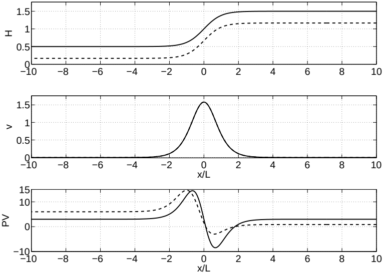
Jet in geostrophic equilibrium on the f- plane with the same velocity in both layers - another exact solution of the 2-layer RSW equations with the free surface.

$$\begin{cases} h_1 = H_1(x) &= H_{10} \\ h_2 = H_2(x) &= H_{20} + \delta \tanh\left(\frac{x}{L}\right) \end{cases}, \begin{cases} U_1(x) = U_2(x) = 0 \\ V_{1,2}(x) = V(x) = \\ \frac{g\delta}{fL}\left(1 - \tanh^2\left(\frac{x}{L}\right)\right) \end{cases}. \quad (48)$$

Parametres and scaling :

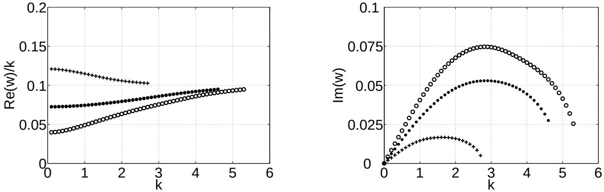
$H_{10}, H_{20} = const$, L and δ - width and intensity of the jet, $V_0 = \frac{g\delta}{fL}$ - max. velocity, $Bu = \frac{gH_0}{f^2L^2}$, $Ro = \frac{g\delta}{(fL)^2}$, $d = \frac{H_{20}}{H_{10}}$, r, $H_0 = H_{10} + H_{20}$. Scaling - standard geostrophic.

Barotropic Bickley jet



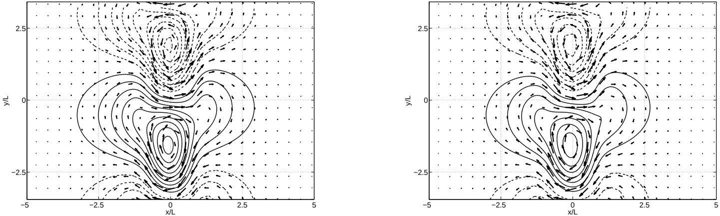
Profiles of thickness, velocity, and PV of the jet as functions of x/L for $\frac{\delta}{L} = \sqrt{\frac{5}{2}}$; continuous : layer 1 ; dashed : layer 2.

Stability diagram of the barotropic jet at small *Ro*



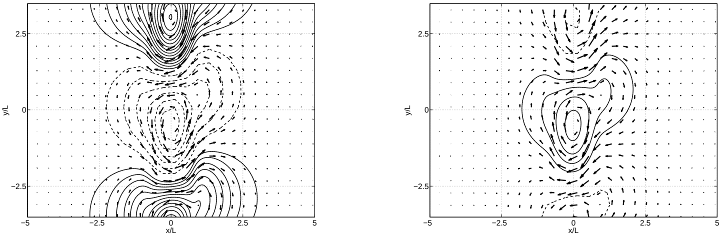
Left : phase velocity $Re(\omega)/k$ as a function of k ; *Right :* Growth rate $Im(\omega)$ as a function of k. Quasi-geostrophic jet : $H_0 = 1, Bu = 10, Ro = 0.5, d = 2, r = 0.5$.

2D structure of the most unstable mode



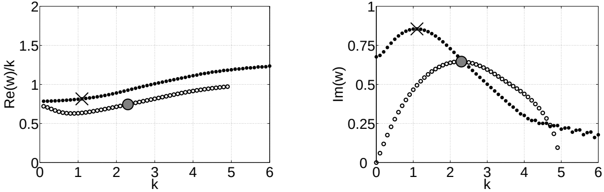
Left (Right) : upper (lower) layer. Layer-wise identical ⇒ barotropic instability.

2D structure of the most unstable mode on branch 2



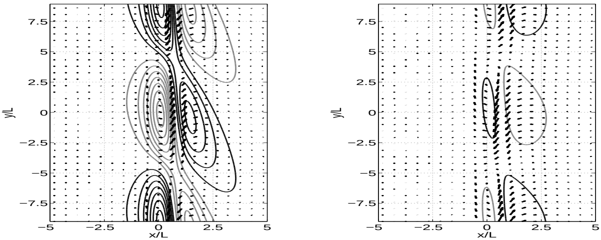
Motions in the layers are opposite → baroclinic instability

Stability diagram of the barotropic jet at large *Ro*



Strongly ageostrophic jet :
$H_0 = 1, Bu = 10, Ro = 5, d = 2, r = 0.5$. Non-zero limit of the growth rate at $k \to 0 \to$ symmetric instability (with respect to translations) $\equiv$ inertial instability.

2D structure of the most unstable mode



Baroclinic, concentrated in the anticyclonic part of the jet.

Exercise :

- Write down the 2-layer RSW equations with a free surface
- Demonstrate that baroclinic and barotropic jet configurations considered above are exact solutions
- Linearise the equations about these solutions

2-layer RSW model with rigid lid in "1.5" dimensions ("symmetric", no dependence on y)

$$\partial_t u_1 + u_1 \partial_x u_1 - f v_1 + \rho_1^{-1} \partial_x \pi = 0 \,, \tag{49a}$$
$$\partial_t v_1 + u_1 (f + \partial_x v_1) = 0 \,, \tag{49b}$$
$$\partial_t u_2 + u_2 \partial_x u_2 - f v_2 + \rho_2^{-1} \partial_x \pi + g' \partial_x \eta = 0 \,, \tag{49c}$$
$$\partial_t v_2 + u_2 (f + \partial_x v_2) = 0 \,, \tag{49d}$$
$$\partial_t (H_1 - \eta) + \partial_x ((H_1 - \eta) u_1) = 0 \,, \tag{49e}$$
$$\partial_t (H_2 + \eta) + \partial_x ((H_2 + \eta) u_2) = 0 \,, \tag{49f}$$

where $(u_1,\ v_1)$, $(u_2,\ v_2)$ are components of velocity in superior and inferior layers ; π -barotropic pressure ; η -displacement of the interface, H_1 and H_2 - layers' thicknesses at rest ; $H = H_1 + H_2 = const$, g' - reduced gravity : $g' = g(\rho_2 - \rho_1)/\rho_2$.

Exact solution : geostrophic equilibrium

$$V_{1g} = \frac{1}{f\,\rho_1}\,\partial_x \Pi_g \;, \tag{50a}$$

$$V_{2g} = \frac{1}{f\rho_2}\,\partial_x \Pi_g + \frac{g'}{f}\partial_x h_{2g} \;. \tag{50b}$$

Non-dimensionalising (bar notation for non-dimensional variables) :

$$\bar{V}_{1g} = \partial_x \bar{\Pi}_g \;, \tag{51a}$$

$$\bar{V}_{2g} = r\,\partial_x \bar{\Pi}_g + Bu\,\partial_x \bar{h}_{2g} \;. \tag{51b}$$

where $r = \frac{\rho_1}{\rho_2}$ and the Burger number : $Bu = \frac{g'\,H_2}{f^2 L^2}$.

Linearisation :

$$\partial_t u_1 - v_1 + \partial_x \pi = 0 \, , \tag{52a}$$

$$\partial_t v_1 + u_1 (1 + \partial_x \bar{V}_{1g}) = 0 \, , \tag{52b}$$

$$\partial_t u_2 - v_2 + r \partial_x \pi + Bu \partial_x \eta = 0 \, , \tag{52c}$$

$$\partial_t v_2 + u_2 (1 + \partial_x \bar{V}_{2g}) = 0 \, , \tag{52d}$$

$$\partial_t \eta - \partial_x (\bar{h}_{1g} u_1) = 0 \, , \tag{52e}$$

$$\partial_t \eta + \partial_x (\bar{h}_{2g} u_2) = 0 \, . \tag{52f}$$

π, η are non-dimensional perturbations of pressure and free surface with respect to the geostrophic balance (51a), (51b).

Reduction to a single equation (bars omitted) :

Constraint of rigid lid :

$$((H_1 - \eta)u_1) + ((H_2 + \eta)u_2) = HU_b(t). \qquad (53)$$

U_b - barotropic velocity in x-direction. Absence of global mass flux in $x \Rightarrow U_b = 0$. New variable $U = h_{2g}u_2 = -h_{1g}u_1 \Rightarrow$ single equation :

$$Bu\, \partial^2_{xx} U - \left[\frac{rh_{2g} + h_{1g}}{h_{1g}h_{2g}}(\partial^2_{tt} + 1) + \frac{r\, \partial^2_{xx}\Pi_g}{h_{1g}h_{2g}} + Bu\, \frac{\partial^2_{xx} h_{2g}}{h_{2g}} \right] U = 0\,. \qquad (54)$$

Trapped/unstable modes :

If the anti-cyclonic shear of the mean flow is sufficiently strong $\Rightarrow$ sub-inertial trapped modes and symmetric instability.

Demonstration :

Fourier transformation :
$U(x,t) = \int d\omega \tilde{U}(\omega, x) e^{-i\omega t} + c.c.$. Auxiliary functions :

$$F(x) = \frac{r h_{2g} + h_{1g}}{h_{1g} h_{2g}}, \tag{55}$$

$$G(x) = \frac{r\, \partial^2_{xx} \Pi_g}{h_{1g} h_{2g}} + Bu\, \frac{\partial_{xx} h_{2g}}{h_{2g}}. \tag{56}$$

Equation for $\tilde{U}(\omega, x)$:

$$Bu\, \partial^2_{xx} \tilde{U} - \left((1 - \omega^2)\, F(x) + G(x) \right) \tilde{U} = 0\,. \tag{57}$$

Demonstration, continued :

Multiplication by $\tilde{U}^*$ and integration in x supposing that the modes are localised :

$$\omega^2 = 1 + \frac{Bu \int |\partial_x \tilde{U}|^2 \, dx + \int \, G(x)|U|^2 \, dx}{\int \, F(x)|\tilde{U}|^2 \, dx} \, . \qquad (58)$$

F is by definition positive, but G may be negative, particularly in the anticyclonic regions where $\partial^2_{xx}\Pi_g < 0$ $\Rightarrow \exists \omega^2 < 1$, even $\omega^2 < 0 \Rightarrow$ instability.

Example : barotropic jet with $\eta = 0$.

Equation for U :

$$Bu\, \partial^2_{xx} U - \left[(\partial^2_{tt} + 1)\, H_e^{-1} + r\partial^2_{xx}\Pi_g\, (H_1 H_2)^{-1} \right] U = 0\ . \tag{59}$$

Solutions in the form $\tilde{U}\, e^{i\omega t} + c.c.$:

$$\partial^2_{xx}\tilde{U} + \frac{1}{Bu}\left[\omega^2 H_e^{-1} - (H_e^{-1} + (H_1 H_2)^{-1} r\partial^2_{xx}\Pi_g) \right] \tilde{U} = 0\ . \tag{60}$$

where $H_e = \frac{H_1 H_2}{H_1 + H_2}$.

This is the Shrödinger equation of quantum mechanics :

$$\partial^2_{xx}\psi + (E - V(x))\psi = 0 \qquad (61)$$

for a particle with the energy

$$E = \omega^2 (H_e\, Bu)^{-1}$$

moving in the potential

$$V(x) = Bu^{-1}\,(H_e^{-1} + (H_1 H_2)^{-1} r \partial^2_{xx}\Pi_g).$$

Potential well sufficiently deep (anti-cyclonic shear sufficiently strong) $\Rightarrow$ trapped modes. Well even deeper $\Rightarrow$ eigenvalues $< -1 \Rightarrow \omega^2 < 0 \Rightarrow$ symmetric (inertial) instability.

Ageostrophic Eady model

Zonally symmetric non-hydrostatic primitive equations on the f- plane

$$
\begin{aligned}
(\partial_t + v\partial_y + w\partial_z)\, u - fv &= 0 \\
(\partial_t + v\partial_y + w\partial_z)\, v + fu + \partial_y \phi &= 0 \\
(\partial_t + v\partial_y + w\partial_z)\, b &= 0 \\
(\partial_t + v\partial_y + w\partial_z)\, w - b + \partial_z \phi &= 0 \\
\partial_y v + \partial_z w &= 0. \qquad (62)
\end{aligned}
$$

$b = -g\frac{\rho}{\rho_0}$ - buoyancy.

Exact solution - zonal thermal wind with linear vertical shear :

$$
\bar{v} = \bar{w} = 0, \quad \bar{u} = -\frac{M^2}{f} z, \quad \bar{b} = M^2 y + N^2 z \qquad (63)
$$

Brunt - Väisälä frequency N^2 is constant, as well as M^2

Scaling and inearisation

Scaling

Vertical scale H, horizontal scale L, time-scale $T \sim f^{-1}$, horizontal and vertical velocity scales, U and W, such that $\frac{H}{L} \sim \frac{W}{U}$. The natural horizontal velocity scale in the Eady model is $U \sim \frac{M^2 H}{f}$, the natural geopotential scale is $\Phi \sim N^2 H^2$, and the natural buoyancy scale is $B \sim N^2 H$.

Non-dimensional parameters

- Aspect ratio $\delta = \frac{H}{L}$,
- Rossby number $Ro = \frac{M^2}{f^2}\delta$,
- Richardson number $Ri = \frac{f^2 N^2}{M^4}$,

Linearisation

Linearisation about (63) :

$$
\begin{aligned}
\partial_t u - Ro\, w - v &= 0\\
\partial_t v + u + Ri\, Ro\, \partial_y \phi &= 0\\
Ri\, \partial_t b + v + Ri\, Ro\, w &= 0\\
\delta^2 \partial_t w - Ri\, Ro\, b + Ri\, Ro\, \partial_z \phi &= 0\\
\partial_y v + \partial_z w &= 0.
\end{aligned}
$$

Streamfunction :

$$
v = \partial_z \psi, \quad w = -\partial_y \psi, \tag{64}
$$

Reduction to a single equation, Fourier-transform and dispersion relation

Elimination of b and u :

$$\partial_t^2\left(\delta^2\partial_{yy}^2\psi+\partial_{zz}^2\psi\right)+\partial_{zz}^2\psi-2Ro\,\partial_{yz}^2\psi+RiRo^2\partial_{yy}^2\psi=0. \qquad (65)$$

Normal-mode solutions : $\psi \propto e^{i(ly+mz)+\sigma t}$.
Real and positive σ correspond to unstable modes.
Dispersion relation :

$$\sigma=\pm\sqrt{\frac{2Ro\,\alpha-RiRo^2\alpha^2-1}{1+\delta^2\alpha^2}}, \qquad (66)$$

where $\alpha=\frac{l}{m}$ is the slope of the wave-vector of the eigenmodes in the $y-z$ plane.

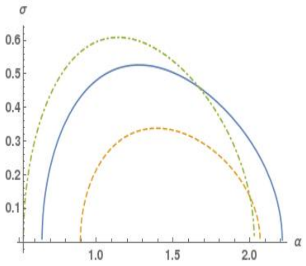
Analysis of the dispersion relation

Ri is positive-definite $\rightarrow$ numerator of the square root in (66) represents a downward oriented quadratic parabola, in terms of $Ro\,\alpha$. The parabola extends to the upper half-plane, and hence corresponds to instability, only in the limited range of $Ro\,\alpha$:

$$\frac{1-\sqrt{1-Ri}}{Ri} < Ro\,\alpha < \frac{1+\sqrt{1-Ri}}{Ri}. \qquad (67)$$

Instability exists at $Ri < 1$ for any *Ro*, with a well-defined maximum of the growth rate and the most unstable mode corresponding to $Ro\,\alpha = 1/Ri$. Orientation of the unstable wavenumbers is correlated with the sign of *Ro* (i.e. with the sign of horizontal relative vorticity of the background flow : anticyclonic for positive M^2, and cyclonic for negative M^2). Non-hydrostatic effects, when $\delta \neq 0$, diminish the hydrostatic growth rate.

Stability diagram

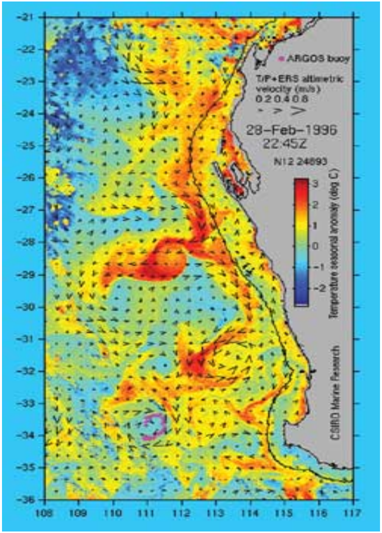


Non-dissipative non-hydrostatic growth rates as functions of *Ro* α (blue solid) $\frac{\delta^2}{Ro} = 0.3$, $Ri = 0.7$.

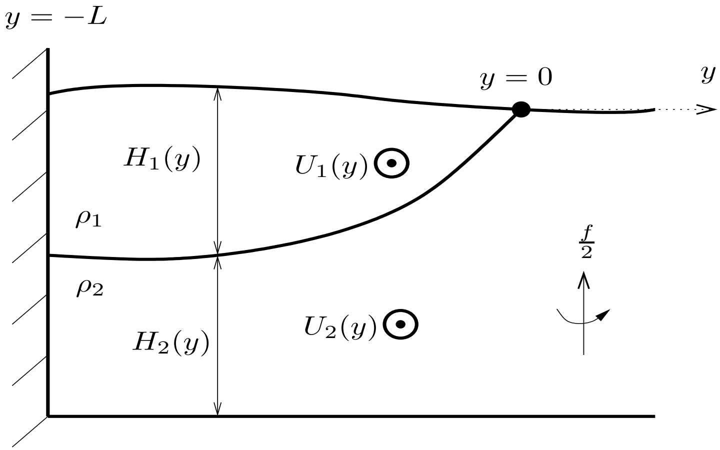
Exercise

- Derive equation (65)
- Introduce viscous terms in the equations (62) and analyse how they affect the instability.

Instability of a coastal current



Idealised configuration of the coastal current



RSW equations with a coast and outcropping (layer 2 passive $H_2 \to \infty$)

Equations of motion :

$$
\begin{aligned}
u_t + uu_x + vu_y - fv + gH_x &= 0, \\
v_t + uv_x + vv_y + fu + gH_y &= 0, \\
H_t + (Hu)_x + (Hv)_y &= 0.
\end{aligned} \tag{68}
$$

Boundary conditions :

$$
H(x, y, t) = 0, \quad D_t Y_0 = v \qquad y = Y_0 \,, \tag{69}
$$

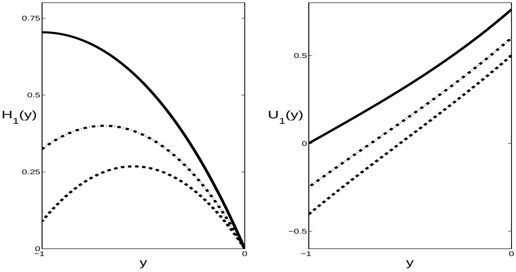
where $Y_0(x, t)$ is the position of the free streamline, D_t - Lagrangian derivative.

Flows in geostrophic equilibrium :

$u = U(y)$, $v = 0$, and $H = H(y)$,

$$U(y) = -\frac{g}{f} H_y(y) \tag{70}$$

- stationary exact solution .



Examples of profiles of depth (left) and velocity (right) for currents with constant PV, $U_0 = -sinh(-1)/cosh(-1)$ (bold), $U_0 = 1/2$ (dashed).

Linearised non-dimensional system :

$$\begin{aligned} u_t + Uu_x + vU_y - v &= -h_x, \\ v_t + Uv_x + u &= -h_y, \\ h_t + Uh_x &= -(Hu_x + (Hv)_y). \end{aligned} \tag{71}$$

linearised b.c. :

- $$Y_0 = -\frac{h}{H_y}\Big|_{y=0}, \tag{72}$$
- continuity equation evaluated at $y = 0$.

The only constraint is regularity of solutions at $y = 0$.

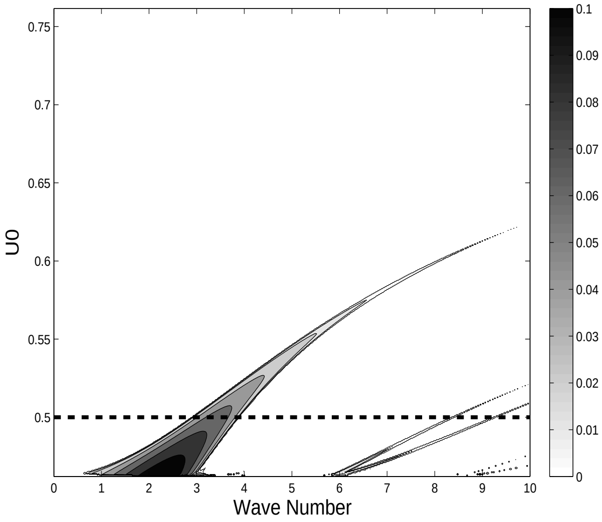
PV of the mean flow :

$$Q(y) = \frac{1 - U_y}{H(y)}, \tag{73}$$

Geostrophic equilibrium $\Rightarrow$

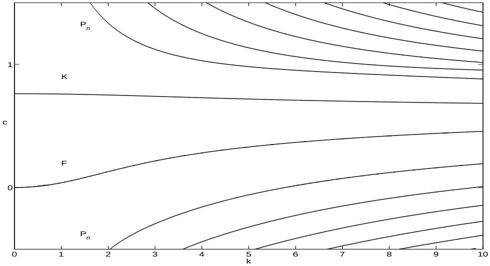
$$H_{yy}(y) - Q(y)H(y) + 1 = 0, \ with \begin{cases} H(0) = 0 \\ H_y(0) = -U_0, \end{cases} \tag{74}$$

$U(0) = U_0$ is the current velocity at the front.



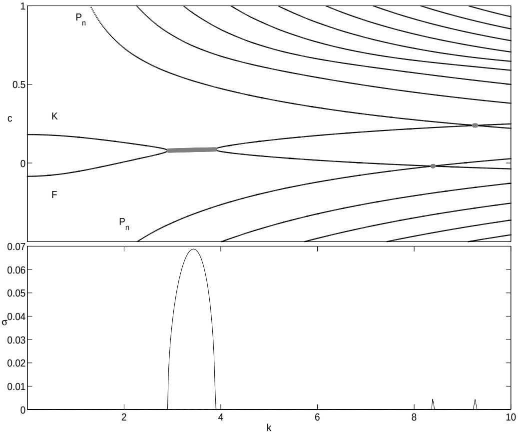
Stability diagram in the plane $(\frac{U_0}{fL}, k)$ for a current with constant PV. Values of the growth rate - right bar.

Dispersion diagramd : stable current



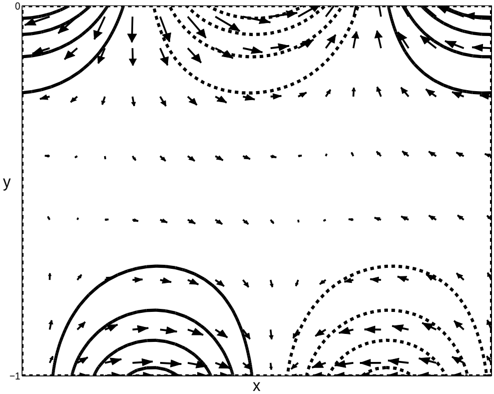
Dispersion diagram for $U_0 = -sinh(-1)/cosh(-1)$ et $Q_0 = 1$.

Dispersion diagram : unstable current



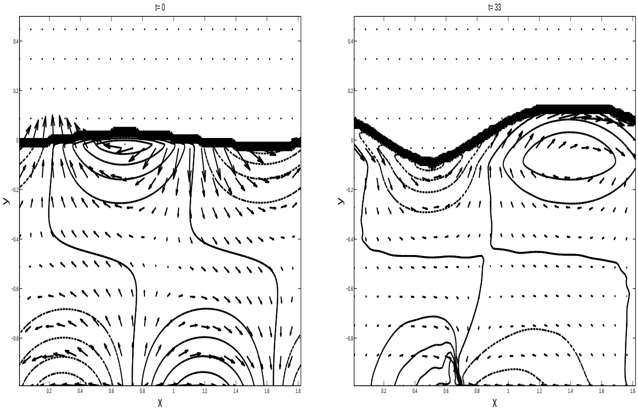
Dispersion diagram for $U_0 = 0.5$ and $Q_0 = 1$. Crossings of the dispersion curves on top correspond to instability zones at the bottom.

The most unstable mode : resonance Kelvin wave-Frontal wave



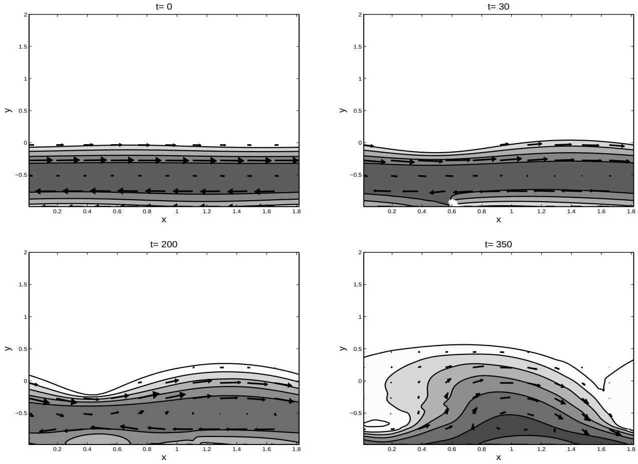
Anomalies of the thickness and velocity for the unstable mode $k = 3.5$.

Saturation of the instability : initial stage



Depth and velocity of the perturbation at $t = 0$ (left) and $t = 30$ (right). Kelvin front is visible on the right.

Saturation of the instability



Evolution of PV : $t = 0$, $t = 30$, $t = 200$, $t = 350$.

Exercise

- Obtain the equations (71) et (72)
- Starting from (74) obtain the profiles of the costal currents with constant PV