Chapter 5. Instabilities in Geophysical Flows.

Part 2. Circular flows

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Plan

Vortex solutions in 1-layer RSW

RSW in polar coordinates (r, θ) in terms of radial and azimuthal components of velocity $\boldsymbol{v} = u\,\hat{r} + v\,\hat{\theta}$

$$\begin{cases} \frac{du}{dt} - \frac{v^2}{r} - fv = -g\partial_r h, \\ \frac{dv}{dt} + \frac{uv}{r} + fu = -g\frac{1}{r}\partial_\theta h, \\ \partial_t h + \frac{1}{r}\partial_r(hru) + \frac{1}{r}\partial_\theta(hv) = 0. \end{cases} \quad (1)$$

where $\frac{d}{dt} = \partial_t + u\partial_r + \frac{v}{r}\partial_\theta$.
Any axisymmetric flow with velocity $u = 0,\ v = V(r)$ and thickness $h = H(r)$ in cyclo-geostrophic equilibrium

$$\left(\frac{V}{r} + f\right) V = g\partial_r H \quad (2)$$

is an exact solution. Isolated vortex : zero circulation at infinity.

Scaling and vortex profile

Scaling :

$\sqrt{gH_0}$ for velocity, $R_d = \frac{\sqrt{gH_0}}{f}$ for r, and $1/f$ for time, where H_0 is the non-perturbed thickness of the layer. Non-dimensional variables denoted by *.

Example of vortex profile

Alpha-Gaussian isolated vortices :

$$V^*(r^*) = \pm\epsilon r^{*\frac{\alpha}{2}} e^{\frac{(-r^{*\alpha}+1)}{2}}, \quad \alpha \geq 1. \tag{3}$$

Positive sign - cyclones, negative sign - anticyclones. The corresponding profile of $H(r^*)$ is given by the primitive of the l.h.s. of (2) calculated with (3). Parameters α and ϵ control the steepness of the azimuthal velocity profile and the amplitude of the velocity, respectively.

Linearised system

Small perturbation of the axisymmetric background flow :

$$\begin{cases} u(r,\theta,t) = u'(r,\theta,t), \\ v(r,\theta,t) = V(r) + v'(r,\theta,t), \\ h(r,\theta,t) = H(r) + h'(r,\theta,t). \end{cases} \tag{4}$$

Linearised non-dimensional equations :

$$\begin{cases} (\partial_{t^*} + \frac{V^*}{r^*}\partial_{\theta^*})u^* - (1 + \frac{2V^*}{r^*})v^* = -\partial_{r^*}\eta^*, \\ (\partial_{r^*}V^* + 1 + \frac{V^*}{r^*})u^* + (\partial_{t^*} + \frac{V^*}{r^*}\partial_{\theta^*})v^* = -\frac{1}{r^*}\partial_{\theta^*}\eta^*, \\ (\partial_{t^*} + \frac{V^*}{r^*}\partial_{\theta^*})\eta^* + \left[H^*\partial_{r^*} + (\frac{1}{r^*}\partial_{r^*}r^*H^*)\right]u^* + \frac{1}{r^*}H^*\partial_{\theta^*}v^* = 0. \end{cases} \tag{5}$$

Linear stability problem

Normal-mode solutions : harmonic dependence on time and polar angle

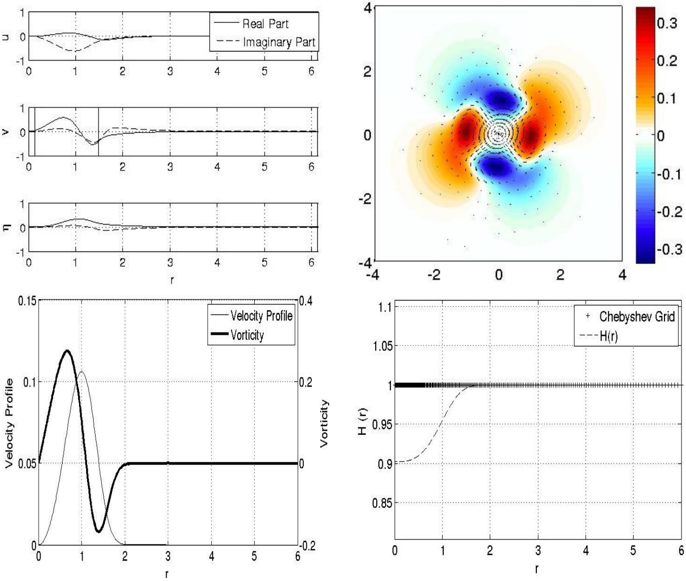
$$(u^*, v^*, \eta^*)(r^*, \theta^*, t^*) = Re[(i\tilde{u}, \tilde{v}, \tilde{\eta})(r^*)e^{i(l\theta^* - \omega t^*)}], \quad (6)$$

where l and ω are azimuthal wavenumber and frequency. Resulting eigen-problem :

$$\begin{bmatrix} \frac{lV^*}{r^*} & (1 + \frac{2V^*}{r^*}) & -D_{r^*} \\ (1 + \frac{V^*}{r^*} + D_{r^*}V^*) & \frac{lV^*}{r^*} & \frac{l}{r^*} \\ H^*D_{r^*} + \frac{1}{r^*}D_{r^*}(r^*H^*) & \frac{lH^*}{r^*} & \frac{lV^*}{r^*} \end{bmatrix} \times \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\eta} \end{bmatrix} = \omega \begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{\eta} \end{bmatrix}. \quad (7)$$

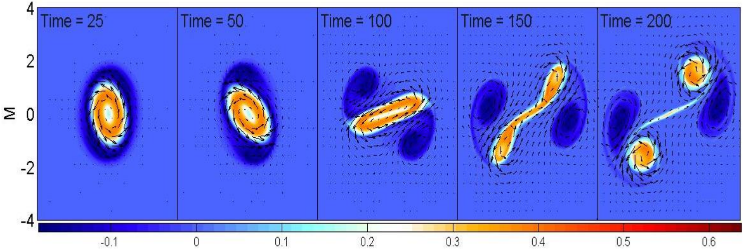
Solved numerically by pseudo-spectral collocation method, D_{r^*} Chebyshev differentiation. Complex eigenvalues $\omega = \omega_r + i\omega_I$ with $(\omega_I > 0) \Leftrightarrow$ instabilities with linear growth rate $\sigma = \omega_I$.

Most unstable mode of a localised cyclone



Upper left - radial profile $(u, v, \eta)(r)$, of the unstable mode; *Upper right* - pressure and velocity fields in the

Nonlinear evolution of the geostrophic instability with $l = 2$



Evolution of PV anomaly of the cyclonic vortex with $\alpha = 4$, $\epsilon = 0.1061$ with superimposed unstable mode with azimuthal wavenumber $l = 2$.

2-layer RSW in polar coordinates and vortex solutions

$$\begin{cases} \frac{\partial \boldsymbol{v}_i}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}_i + \left(f + \frac{v_i}{r}\right) \hat{\boldsymbol{z}} \wedge \boldsymbol{v}_i + g\nabla(s^{i-1}h_1 + h_2) = 0, \\ \frac{\partial h_i}{\partial t} + \nabla \cdot (h_i \boldsymbol{v}_i) = 0, \quad i = 1, 2. \end{cases} \tag{8}$$

$\boldsymbol{v}_i = (u_i, v_i)$ is velocity in layer i counted from the top, h_i is thickness of the layer i, $s = \rho_1/\rho_2 < 1$, $d = H_1/H_2$, and H_i - thickness of the layer i at rest. Scaling is the same as in the one-layer case, with $H_0 = H_1 + H_2$.

Stationary solutions (non-dimensional, same scaling as before) : cyclo-geostrophic equilibria layer-wise :

$$V_i \left(\frac{V_i}{r} + 1\right) = -\partial_r(s^{i-1}H_1 + H_2)\,, \quad i = 1, 2. \tag{9}$$

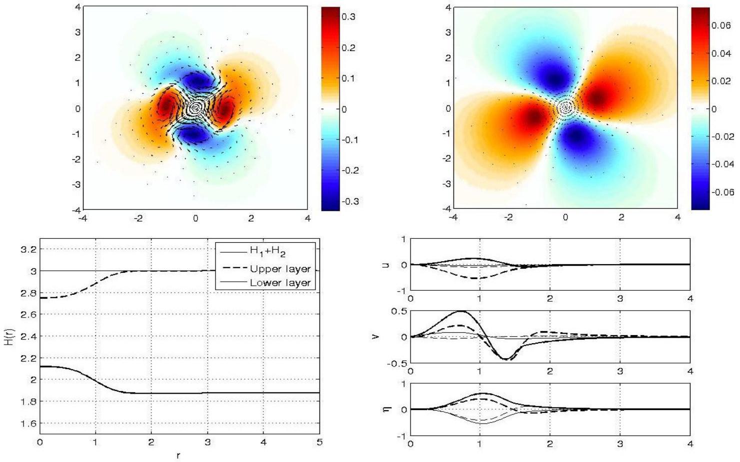
Below an upper layer vortex with quiescent lower layer will be considered.

Linearised system

$$\begin{cases} \left(\partial_t + \frac{V_i}{r}\partial_\theta\right) u_i - \left(1 + 2\frac{V_i}{r}\right) v_i + \partial_r(s^{i-1}\eta_1 + \eta_2) = 0, \\ \left(\partial_t + \frac{V_i}{r}\partial_\theta\right) v_i + \left(1 + \frac{V_i}{r} + \partial_r V_i\right) u_i + \frac{1}{r}\partial_\theta(s^{i-1}\eta_1 + \eta_2) = 0, \\ \left(\partial_t + \frac{V_i}{r}\partial_\theta\right) \eta_i + \left[H_i\partial_r + \frac{\partial_r(rH_i)}{r}\right] u_i + \frac{H_i}{r}\partial_\theta v_i = 0, i = 1, 2. \end{cases} \tag{10}$$

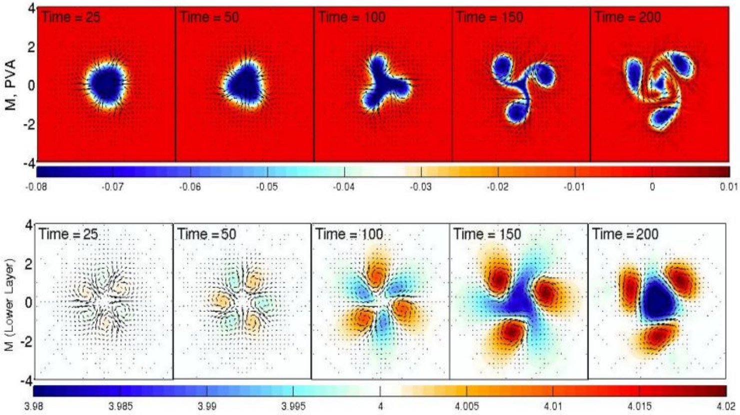
Solutions, as above, are sought in the harmonic form : $(u_i, v_i, \eta_i)(r, \theta, t) = (i\tilde{u}_i, \tilde{v}_i, \tilde{\eta}_i)(r)e^{i(l\theta - \omega t)}$, where l is the discrete azimuthal wavenumber.

Instability of an upper-layer cyclone



Most unstable mode of an upper-layer cyclone with $\alpha = 4$, , $H_2/H_1 = 0.6$, $l = 2, \epsilon = 0.08$, $s = 1.37$. Top : *left (right)* - pressure and velocity, upper (lower) layer. Bottom : *left* - $H_i(r)$, *right* - radial structure of the mode.

Nonlinear saturation of the $l = 3$ mode



Evolution of PV anomaly (top) and pressure and velocity (bottom) during the saturation of the baroclinic instability with most unstable mode $l = 3$.

Integral estimate for azimuthally symmetric trapped modes

Azimuthally symmetric system (10) is reduced, by elimination of variables, to

$$(-\omega^2 + \overline{\Phi}) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \partial_r \left(\frac{\partial_r}{r} \begin{pmatrix} rH_1 u_1 + rH_2 u_2 \\ rsH_1 u_1 + rH_2 u_2 \end{pmatrix} \right). \quad (11)$$

Here $\overline{\Phi} = 2\overline{L}_a \overline{\zeta}_a / r^2$, $\overline{L}_a = r^2/2 + rV$ and $\overline{\zeta}_a = 1 + \partial_r(rV)/r$ - non-dimensional absolute angular momentum density and vorticity of the vortex.

Integral estimate for trapped modes :

$$\omega^2 = \frac{\int \overline{\Phi} \cdot H_{eq} |u_b|^2 dr}{\int H_{eq} |u_b|^2 dr} + (1-s) \left[\frac{\int \left[|\partial_r (H_{eq} u_b)|^2 + \frac{|H_{eq} u_b|^2}{4r^2} \right] dr}{\int H_{eq} |u_b|^2 dr} - \frac{\int H_{eq} u_b^* \partial_r \left(\frac{\partial_r (rH_1 u_B)}{r} \right) dr}{\int H_{eq} |u_b|^2 dr} \right]. \quad (12)$$

$u_B = \frac{H_1 u_1 + H_2 u_2}{H_1 + H_2}$, $u_b = u_1 - u_2$, $H_{eq} = \frac{H_1 H_2}{H_1 + H_2}$.

Qualitative analysis of the integral estimate

- The sign of $\overline{\Phi}$ defines the sign of the first term in the r.h.s. of this relation
- The second term is positive-definite.
- The third term is not sign-definite, but it is the only one containing the barotropic velocity u_B, and thus vanishes for purely baroclinic modes

Hence, we infer that for sufficiently large negative values of $\overline{\Phi}$ there exist trapped baroclinic modes with imaginary eigenfrequencies. $\Rightarrow$ classical Rayleigh criterion for the centrifugal instability :

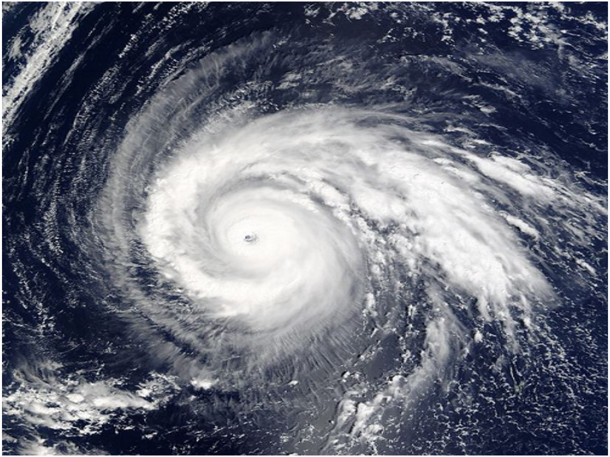
$$\overline{L}_a \overline{\zeta}_a < 0,$$

where the last product is called Rayleigh discriminant.

Exercise

- Obtain the expression for absolute angular momentum density $\overline{L}_a = r^2/2 + rV$ from the definition of angular momentum
- Obtain (11) from the axisymmetric version of primitive equations in polar coordinates
- Derive (12)

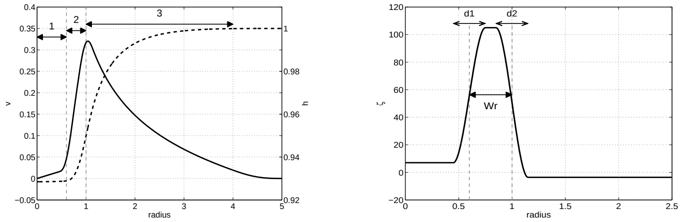
Tropical cyclone - essentially ageostrophic vortex



Hydrodynamic characteristics of TC

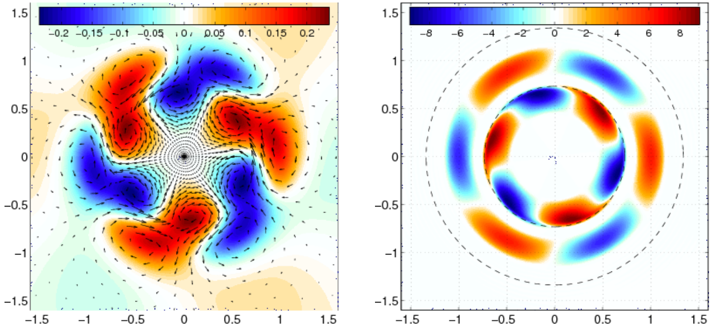
- Radius of max wind (RMW) : $20 - 50 km$
- Max velocity V_{max} : $40 - 60 m/s$
- Typical value of f (at $20^\circ N$) : $5 \cdot 10^{-5}$
- Relative vorticity ζ : up to $100f$, typical Rossby numbers : $10 - 40$
- Vertical wind distribution : $\approx$ barotropic
- Barotropic Froude number : $Fr = \frac{V_{max}}{\sqrt{gH_0}} = Ro/\sqrt{Bu} = \mathcal{O}(10^{-1})$
- Radial wind profile : U-shape in the core, decreasing as $1/r$ in the outer region $\rightarrow \approx$ constant vorticity core surrounded by higher vorticity ring, zero vorticity in the outer region.

Typical category 3 cyclone's profile : RSW modeling



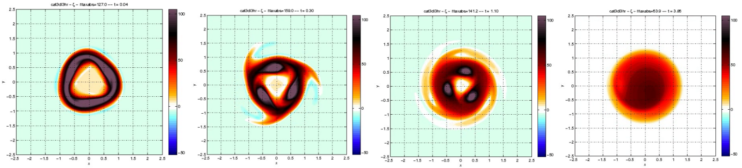
Left panel : azimuthal velocity and thickness (dashed). *Right panel :* relative vorticity. Parameters : ratio of relative vorticities ring/core $\frac{\zeta_r}{\zeta_c} = 105/7$, $Ro_{loc} = \frac{\zeta_r}{f} = 105$, $Ro = V_{max}/fL = 32$, $Fr = 0.3$.

The most unstable mode

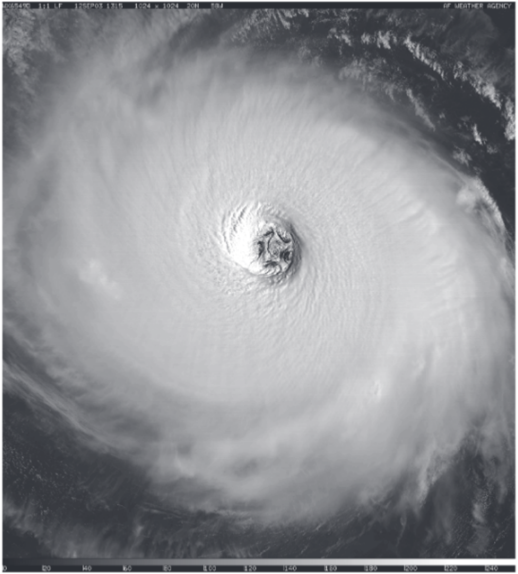


Left panel : velocity and pressure anomalies. *Right panel :* relative vorticity anomaly ⇒ Ageostrophic barotropic instability

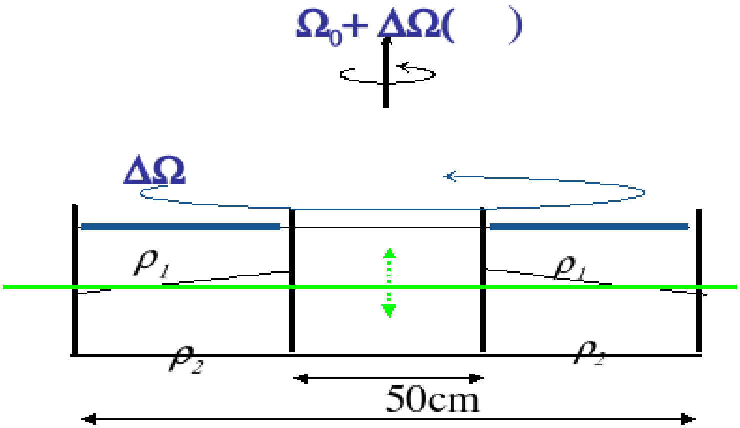
Evolution of the unstable mode and formation of secondary mesovortices



Mesovortices as observed in the hurricane Isabel



Sketch of classical experiments on baroclinic instability



Preliminaries : 1-layer RSW in the rotating annulus

RSW in cylindrical geometry, annulus $r_1 \leq r \leq r_2$:

$$\begin{aligned} Du - (f + \tfrac{v}{r})v &= -g\partial_r h \,, \\ Dv + (f + \tfrac{v}{r})u &= -\tfrac{g}{r}\partial_\theta h \,, \\ Dh + h(\partial_r u + \partial_\theta v/r + u/r) &= 0 \,. \end{aligned} \qquad (13)$$

B.c. : free-slip : $u = 0$ at $r = r_1, r_2$; $D = \partial_t + u\partial_r + \frac{v}{r}\partial_\theta$.
Exact solution : cyclo-geostrophic equilibrium with profiles of the thickness and velocity $H(r)$, $V(r)$:

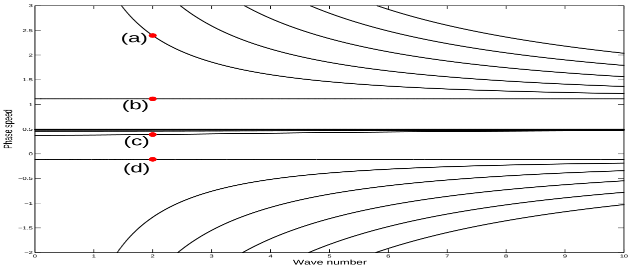
$$fV + \frac{V^2}{r} = g\partial_r H. \qquad (14)$$

Linearisation

$$\begin{aligned}
\partial_t u + \frac{V}{r}\partial_\theta u - v - 2\frac{Vv}{r} &= -Bu\partial_r h \,, \\
\partial_t v + u\partial_r V + \frac{V}{r}\partial_\theta v + u + \frac{Vu}{r} &= -Bu\frac{\partial_\theta h}{r} \,, \\
\partial_t h + \frac{1}{r}(rHu)_r + \frac{1}{r}H\partial_\theta v + \frac{V}{r}\partial_\theta h &= 0 \,,
\end{aligned} \qquad (15)$$

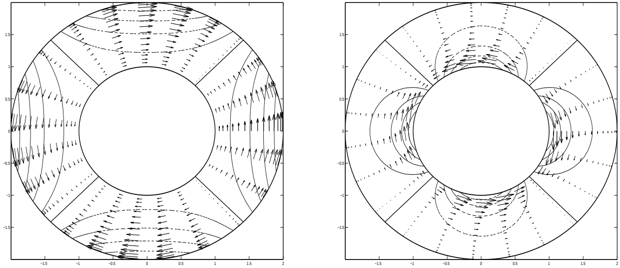
where $Bu = (R_d/r_0)^2$ is the Burger number,
$R_d = (gH_0)^{\frac{1}{2}}/(\Omega r_0)$ is the deformation radius, $r_0 = r_2 - r_1$.
Solution $\rightarrow$ eigenmodes and eigenvalues (dispersion diagram).

Dispersion diagram



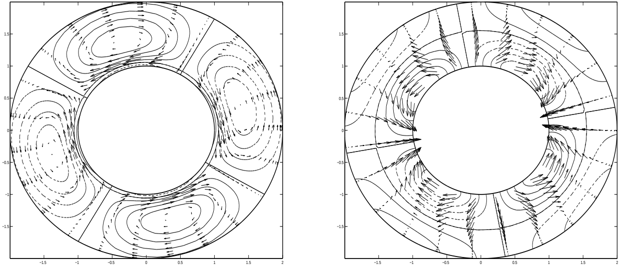
Dispersion diagram $c = c(k)$. (a) Poincaré modes, (b) and (d) Kelvin modes, (c) Rossby modes. Fast Poincaré and Kelvin modes are separated from the slow Rossby modes ; only the values $k \in \mathbf{N}$ correspond to realisable solutions.

Kelvin modes



Pressure and velocity for the Kelvin modes propagating along the exterior (left) and interior (right) walls with $k = 2$. These modes correspond to (b) and (d), respectively.

Rossby and Poincaré modes



Pressure and velocity of the Rossby (left) and Poincaré (right) modes with $n = 1$ and $k = 2$. These modes correspond to (c) and (a), respectively.

Exercise

- Consider a parabolic profile of $H(r)$ in (47), find the corresponding profile of $V(r)$,
- Reducee the equations (47) to a single differential equation for the propagating wave solutions $\propto e^{i(\omega t - n\theta)}$, solve it and find the structure in r of different types of waves

2-layer RSW in the annulus with a rigid lid

$$
\begin{aligned}
D_j u_j - (f + \frac{v_j}{r}) v_j &= -\partial_r \Pi_j \ , \\
D_j v_j + (f + \frac{v_j}{r}) u_i &= -\frac{1}{r} \partial_\theta \Pi_i \ , \\
D_j h_j + h_j(\partial_r u_j + \partial_\theta v_j / r + u_j / r) &= 0 \ ,
\end{aligned}
\tag{16}
$$

Π_j - geopotentials, D_j - Lagrangian derivative per layer.

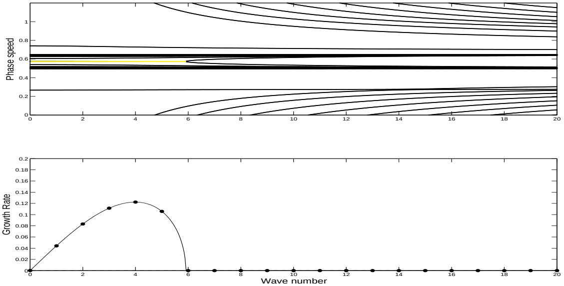
Linearisation about a stationary state with constant azimuthal velocities $V_1 \neq V_2$

$$
\begin{aligned}
\partial_t u_j + \frac{V_j}{r}\partial_\theta u_j - v_j - 2\frac{V_j v_j}{r} &= -Bu\,\partial_r \pi_j \,, \\
\partial_t v_j + u_j \partial_r V_j + \frac{V_j}{r}\partial_\theta v_j + u_j + \frac{V_j u_j}{r} &= -Bu\frac{\partial_\theta \pi_j}{r} \,, \\
\partial_t h_j + \frac{1}{r}(rH_j u_j)_r + \frac{1}{r}H_j \partial_\theta v_j + \frac{V_j}{r}\partial_\theta h_j &= 0 \,,
\end{aligned}
\tag{17}
$$

Pressure perturbations π_j are coupled via perturbation of the interface :

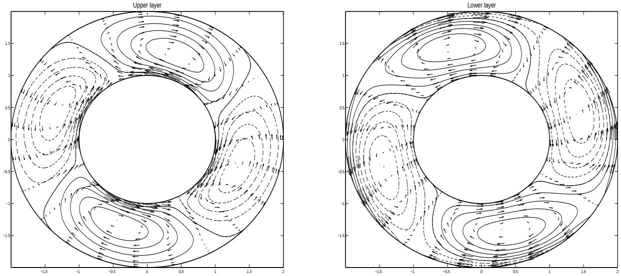
$$
\pi_2 - \pi_1 + s(\pi_2 + \pi_1) = Bu\,\eta \,, \tag{18}
$$

Baroclinic instability : RR resonance



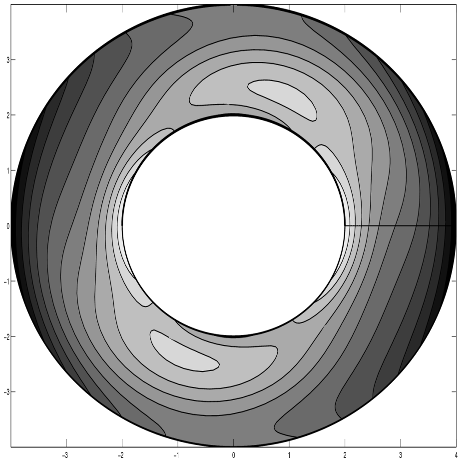
Dispersion iagram (top) and growth rates (bottom) for $Ro = 0.15$ and $F = 2.75$. Yellow line marks the RR resonance and instability.

Baroclinic instability : unstable mode

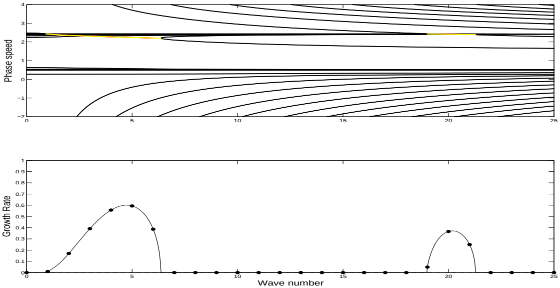


Structure of the unstable mode with $k = 2$, pressure and velocity in the superior (left) and inferior (right). Solid - positive values , dashed - negative values . Note the balanced character of the mode.

Baroclinic instability : deviation of the interface

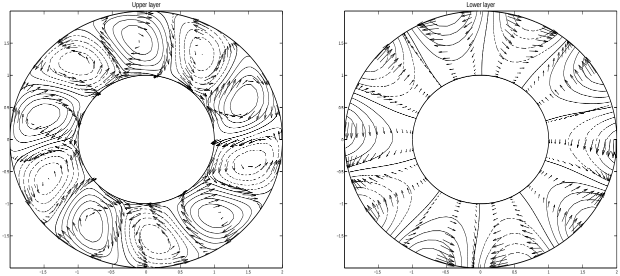


Rossby-Kelvin instability : RK resonance



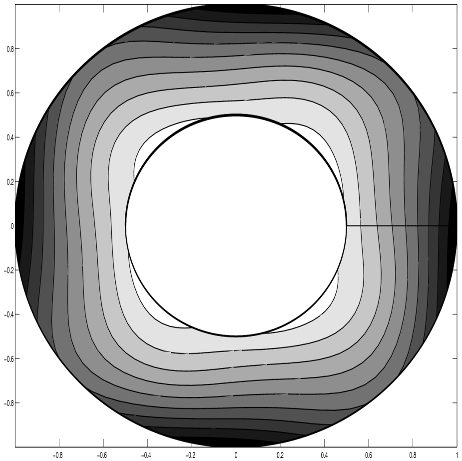
Dispersion diagram (top) and growth rates (bottom) at $Ro = 1.9$ and $F = 0.1$. Yellow lines mark the RK et RP resonances and respective instabilities .

Rossby-Kelvin instability : structure of the unstable mode

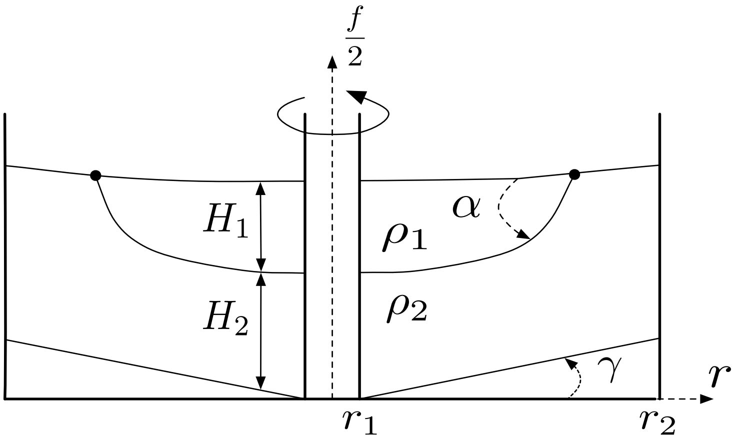


Structure of the unstable RK mode at $k = 5$, pressure and velocity in the superior (left) and inferieor (right). Solid - positive values , dashed - negative values.

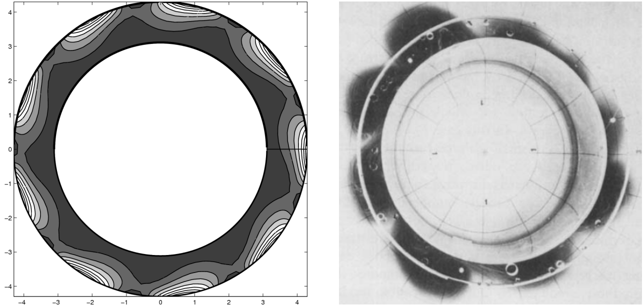
Rossby-Kelvin Instabiliti : deviation of the interface



Scheme of the classical experiments with gravity currents on the rotating turntable



Comparison theory/experiment



Deviation of the interface for the most unstable mode, as follows from the linear stability analysisin 2-layer RSW (left), and development of the instability in the experiment by Griffiths and Linden (1982)