Chapter6: Simplified models of the dynamics of fronts and frontogenesis.

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Course GFD M2 MOCIS

Plan

Fronts in the RSW model

RSW model in dimension 1.5

Lagrangian approach to 1.5d RSW

Geostrophic Adjustment

Mechanism of breaking.

Hyperbolic systems. Method of characteristics.

Fronts/density currents in the RSW model

Double density fronts

Fronts and frontogenesis in the PE model

PE model in 2.5 dimensions

Lagrangian approach in EP 2.5d

Geostrophic adjustment and frontogenesis

General properties of 1.5d RSW

Equations of the model

"Dimension 1.5": no dependence on y:

$$\begin{aligned}
\partial_t u + u\partial_x u - fv + g\partial_x h &= 0 \,, \\
\partial_t v + u\partial_x v + fu &= 0 \,, \\
\partial_t h + u\partial_x h + h\partial_x u &= 0 \,.
\end{aligned} \tag{1}$$

Frontal configurations: localised distributions of $v(x)$, $h(x)$ with common compact support in x of v, $\partial_x h$.

Lagrangian invariants

- Potential Vorticity:

$$Q = (\partial_x v + f)/h, \tag{2}$$

.

- Geostrophic momentum:

$$M = v + fx \tag{3}$$

$$(\partial_t + u\partial_x)M = 0, \;\; (\partial_t + u\partial_x)Q = 0. \tag{4}$$

Inertia - gravity waves

Linearisation with respect to the rest state $H = const$: zero mode (slow motions) and inertia- gravity waves (fast motions) with standard dispersion relation:

$$\omega = \pm(c_0^2 k^2 + f^2)^{\frac{1}{2}}, \;\; c_0 = \sqrt{gH}. \tag{5}$$

Geostrophic equilibrium

exact solution of the equations of motion:

$$fv = g\partial_x h, \quad u = 0 \;, \tag{6}$$

Slow motions. Vorticity is entierely determined by the perturbation of h and vice verse:

$$Q^{(g)} = \left(\frac{f + \frac{g}{f}\partial^2_{xx} h}{h} \right) . \tag{7}$$

Geostrophic adjustment

Adjustment $\rightarrow$ Relaxation towards equilibrium state. Equilibrium $\leftrightarrow$ minimum of energy $\Rightarrow$ necessity to evacuate energy. The only energy sink in the absence of dissipation: émission of inertia - gravity waves.

Equations of motion in Lagrangian coordinates

Lagrangian coordinates

Trajectories of "fluid parcels" $x \to X(x,t)$, where x is a position of the parcel at $t = 0$. $\dot{X} = u(X,t)$, notation: $X' = \frac{\partial X}{\partial x}$.

Momentum equations

$$\begin{aligned} \ddot{X} - fv + g\partial_X h &= 0\,, \\ \dot{v} + f\dot{X} &= 0\,, \end{aligned} \tag{8}$$

where v is considered as a function of x and t.

Conservation of mass:

$$h(X,t)\,dX = h_I(x)\,dx, \quad \Rightarrow \quad h(X,t) = h_I(x)\partial_X x. \tag{9}$$

Reduction to a single equation

Integration of the equation for v:

$$v(x,t) + fX(x,t) = M(x)\ . \qquad (10)$$

Determination of M from b.c.:

$$M(x) = fx + v_I(x). \qquad (11)$$

Chain differentiation:

$$\partial_X h = \partial_X \left(h_I(x)\partial_x X\right) = h_I' \left(X'\right)^{-2} - h_I(x)X'' \left(X'\right)^{-3}, \qquad (12)$$

Closed equation for X:

$$\ddot{X} + f^2 X + gh_I' \left(X'\right)^{-2} + \frac{gh_I}{2}\left[\left(X'\right)^{-2}\right]' = fM\,. \qquad (13)$$

Re-writing in terms of deviations of parcels from their initial positions: $X(x,t) = x + \phi(x,t)$:

$$\ddot{\phi} + f^2\phi + gh_I'\left(\frac{1}{(1+\phi')^2}\right) + \frac{gh_I}{2}\left(\frac{1}{(1+\phi')^{-2}}\right)' = fv_I\ . \quad (14)$$

To be solved with b.c.:

$$\phi(x,0) = 0, \quad \dot{\phi}(x,0) = u_I(x),$$

where u_I is the initial velocity in x direction.

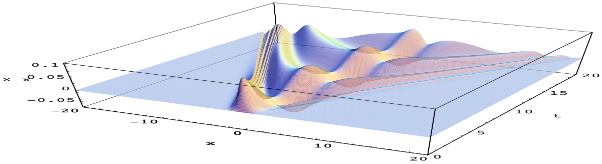
Exercises:

- Demonstrate the equivalence of (9) with the standard continuity equation in RSW,
- Linearise (14) and find solutions - inertia-gravity waves,
- Demonstrate that geostrophic equilibria are exact solutions of (14).

Example of direct simulation with MATHEMATICA of the 1.5d adjustment

Initial configuration :

$$h_I(x) = 1 + e^{-x^2},\ v_I(x) = -2(x + 0.2\sin(x))\, e^{-x^2},\ u_I(x) = 0.1\, e^{-x^2}$$



Archetype model of breaking: "simple" wave

Non-dispersive one-dimensional wave with advective non-linearity:

$$u_t + \epsilon u u_x + u_x = 0. \tag{15}$$

Constant phase speed$= 1$, solution of the linearised system: $u(x,t) = U(x-t)$.

Changing the reference frame, introducing slow time:

$$U_T + U U_x = 0 \tag{16}$$

Lagrangian description:

$$U = \dot{X}, \Rightarrow \ddot{X} = 0, \Rightarrow \dot{X} = U_I(x), \Rightarrow X(x, T) = x + U_I(x)t. \quad (17)$$

where U_I - initial distribution of U.

Breaking:

$$\forall x_1, x_2 : x_2 > x_1, \; U_I(x_2) < U_I(x_1), \quad (18)$$

intersection of Lagrangian trajectories $\equiv$ breaking.

$1d$ quasi-linear systems

Definition:

$$\partial_t V_i(x,t) + \sum_{j=1}^{N} A_{ij}\left(\vec{V}\right)\partial_x V_j(x,t) = B_i\left(\vec{V}\right),\ i = 1, 2, ..., N. \quad (19)$$

Eigenvectors and eigenvalues:

Let $\vec{l}^{(a)}$ - left eigenvectors and $\xi^{(a)}$ - corresponding eigenvalues, $a = 1, 2, ...$:

$$\vec{l}^{(a)} \cdot A = \xi^{(a)}\vec{l}^{(a)}, \Rightarrow \quad (20)$$

$$\vec{l}^{(a)} \cdot \left(\partial_t \vec{V} + A \circ \partial_x \vec{V}\right) = \vec{l}^{(a)} \cdot \left(\partial_t \vec{V} + \xi^{(a)}\partial_x \vec{V}\right). \quad (21)$$

Characteristics:

$$\frac{dx}{dt} = \xi^{(a)} \tag{22}$$

Advection along a characteristic:

$$\dot{\vec{V}} \equiv \frac{d\vec{V}}{dt} = \left(\partial_t + \xi^{(a)}\partial_x\right)\vec{V}. \tag{23}$$

$$\vec{l}^{(a)} \cdot \dot{\vec{V}} = \vec{l}^{(a)} \cdot \vec{B} \tag{24}$$

- ordinary differential equations (easy to integrate).

Hyperbolic systems:

N real and different eigenvalues $\xi^{(a)}$.

Riemann invariants :

If $\vec{l}^{(a)} = const$ (or integrating multiplier exists) - Riemann variables (invariants if $\vec{B} = 0$):

$$r^{(a)} = \vec{l}^{(a)} \cdot \vec{V} :, \quad \frac{dr^{(a)}}{dt} = \vec{l}^{(a)} \cdot \vec{B} \tag{25}$$

Shocks:

Intersection of characteristics $\leftrightarrow$ derivatives of Riemann invariants become infinite in finite time.

Example: SW

Quasi - linear form of the SW equations:

$$\partial_t \begin{pmatrix} u \\ h \end{pmatrix} + \begin{pmatrix} u & 1 \\ h & u \end{pmatrix} \partial_x \begin{pmatrix} u \\ h \end{pmatrix} = 0, \qquad (26)$$

Eigenvectors and eigenvalues:

$$\vec{l}^{\pm} = (\pm\sqrt{h}, 1), \quad \xi^{\pm} = u \pm \sqrt{h}. \qquad (27)$$

Riemann invariants:

$$r^{\pm} = u \pm 2\sqrt{h}, \quad \frac{dr^{\pm}}{dt^{\pm}} = 0, \ \frac{d}{dt^{\pm}} \equiv \partial_t + \xi^{\pm}\partial_x. \qquad (28)$$

Wave-breaking in SW

Equation for derivatives of Riemann invariants:

$D^{\pm} \equiv \partial_x r^{\pm}$

$$\frac{dD^{\pm}}{dt^{\pm}} + \partial_x \xi^{\pm} D^{\pm} = 0, \; \xi^{\pm} = \frac{3}{4} r^{\pm} + \frac{1}{4} r^{\mp}, \Rightarrow \tag{29}$$

$$\frac{dD^{\pm}}{dt^{\pm}} + \frac{3}{4} \left(D^{\pm}\right)^2 + \frac{1}{4} D^{\pm} D^{\mp} = 0. \tag{30}$$

Suppose one of the invariants is identically zero $\Rightarrow$ Riccatti equation along the characteristic for remaining D:

$$\frac{dD}{dt} + \frac{3}{4} (D)^2 = 0, \; \rightarrow D = (D_I^{-1} + \frac{3}{4} t)^{-1} \tag{31}$$

$\Rightarrow$ singularity in finite time, if initial D is negative.

Exercises:

- Demonstrate that 1.5d RSW model in Lagrangian variables is a hyperbolic quasi-linear system.
- Determine the characteristics and Riemann variables for this system.
- Analyse the evolution of the derivatives of the Riemann variables and conditions of breaking.

Example: adjustment of a "wind blow" (Rossby, 1936)

I.c.: jet out of equilibrium: $h_I = H = \text{const}, \; v_I \neq 0$.
Notation $J = \partial X/\partial x = H/h(X,t)$.

$$g\,\partial_X h = \partial_x P, \quad P = gH/(2J^2) - \text{Lagrangian pressure}$$

Lagrangian equations:

$$\dot{u} - fv + \partial_x P = 0, \tag{32}$$

$$\dot{v} + fu = 0, \tag{33}$$

$$\dot{J} - \partial_x u = 0. \tag{34}$$

Equivalent to a single equation for J:

$$\ddot{J} + f^2 J + \partial^2_{xx} P = fHQ\,. \tag{35}$$

where

$$Q(x) = \frac{1}{H}\left(\partial_x v(x,t) + fJ(x,t)\right) = \frac{1}{H}\left(\partial_x v_I(x) + fJ_I(x)\right).$$

Adjusted stationary solution:

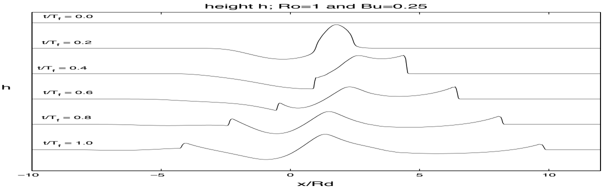
$$f^2 J + \partial^2_{xx} P = fHQ \tag{36}$$

- completely determined by Q

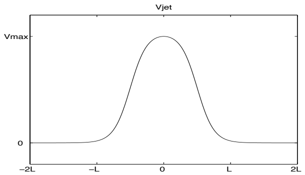
High-resolution simulations of the Rossby adjustment

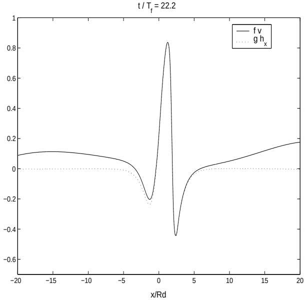
Finite-volume code (calculating fluxes in each cell of the grid).

Adjustment process

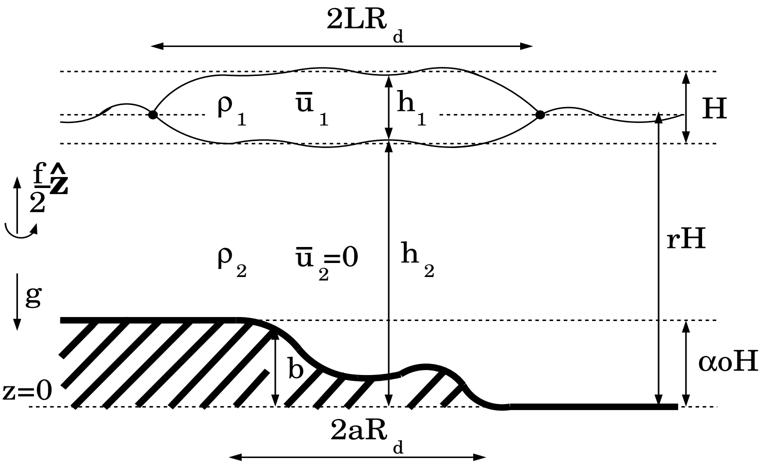


Initial and quasi- adjusted states





Double density fronts



RSW equations (no bathymetry)

Equations of motion :

$$\begin{aligned} u_t + uu_x + vu_y - fv + gh_x &= 0, \\ v_t + uv_x + vv_y + fu + gh_y &= 0, \\ h_t + (hu)_x + (hv)_y &= 0. \end{aligned} \qquad (37)$$

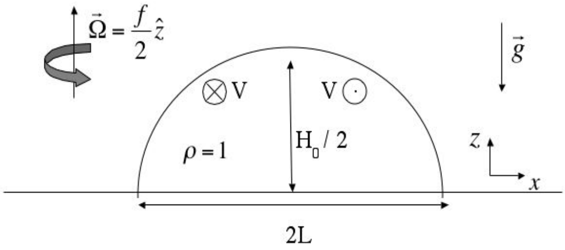
Boundary conditions:

$$H(y) + h(x, y, t) = 0, \quad D_t Y_0 = v \quad at \quad y = Y_0 \ , \qquad (38)$$

where $Y_0(x, t)$ - position of the "free" streamline, D_t Lagrangian derivative .

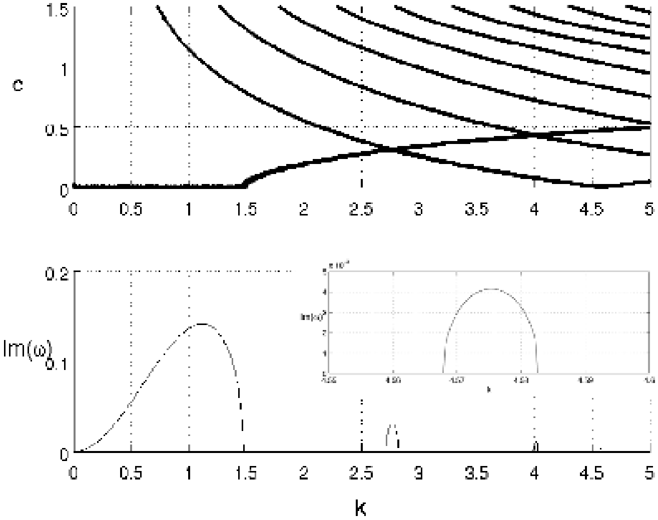
Classical example: balanced double front with parabolic profile of h.

$$\begin{cases} h = H & = \begin{cases} \frac{H_0}{2}\left(1 - \left(\frac{x}{L}\right)^2\right), & |x| \leq L \\ 0, & |x| > L \end{cases} \\ u = 0 & \\ v = V & = \begin{cases} \frac{g}{f} H_x = -\frac{gH_0}{fL^2}\, x, & |x| \leq L \\ 0, & |x| > L \end{cases} \end{cases} \tag{39}$$

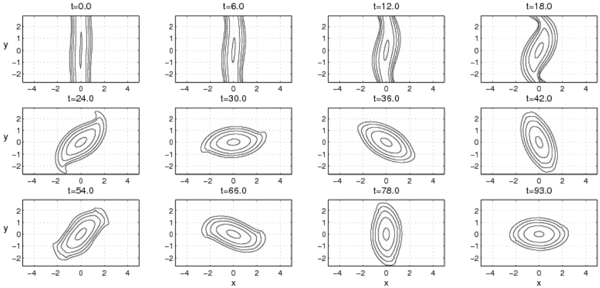


Instabilities (ageostrophic) of a double front

Real and imaginary (growth rate) parts of the perturbations of the parabolic front :



Non-linear evolution of the principal instability



Exercises:

- Demonstrate that (39) verifies equation (14),
- Consider solutions of (14) in a form (separation of variables): $\phi = f(x)\xi(t)$ and demonstrate that oscillations of finite amplitude ("pulsons") of the parabolic front exist.

General properties of 2.5d PE

Équations of the model

"Dimension 2.5": no dependence of y:

$$\frac{Du}{Dt} - fv + \phi_x = 0\,, \quad (40)$$

$$\frac{Dv}{Dt} + fu = 0\,, \quad (41)$$

$$\phi_z = g\frac{\theta}{\theta_r}\,, \quad (42)$$

$$u_x + w_z = 0\,, \quad (43)$$

$$\frac{D\theta}{Dt} = 0\,, \quad (44)$$

$$\frac{D}{Dt} = \partial_t + u\partial_x + w\partial_z \quad (45)$$

Lagrangian invariants

- Potential temperature θ,
- Potential vorticity:

$$Q = (\partial_x v + f)\theta_z - v_z \theta_x, \tag{46}$$

- Geostrophic momentum

$$M = v + fx \tag{47}$$

$$\frac{D}{Dt}(\theta,\ M,\ Q) = 0. \tag{48}$$

Expression of Q in terms of M:

$$Q = \frac{\partial(M, \theta)}{\partial(x, z)} . \tag{49}$$

Inertia - gravity waves

Linearisation about state of rest with $N = const$: zero mode (slow motions) and inertia- gravity waves (fast motions) with standard dispersion relation:

$$\omega = \pm(N^2 \frac{k^2}{m^2} + f^2)^{\frac{1}{2}} , \tag{50}$$

where wavenumber in (x, z) space is:

$$\mathbf{k} = k\hat{\mathbf{x}} + m\hat{\mathbf{z}}. \tag{51}$$

Thermal wind

Stationary states

$$u = w = 0, \quad fv = \phi_x, \quad g\frac{\theta}{\theta_r} = \phi_z. \tag{52}$$

Elimination of ϕ, use of M:

$$f\frac{\partial M}{\partial z} = \frac{g}{\theta_r}\frac{\partial \theta}{\partial x}\ , \tag{53}$$

$\Rightarrow$ a " potential" Φ may be introduced for equilibrium states:

$$M = f^{-1}\frac{\partial \Phi}{\partial x}\ , \tag{54a}$$

$$\theta = \frac{\theta_r}{g}\frac{\partial \Phi}{\partial z}\ . \tag{54b}$$

Equations of motion in Lagrangian coordinates

Lagrangian coordinates

Trajectories of fluid "parcels" $(x,\, z) \rightarrow (X(x,z,t),\, Z(x,z,t))$, where (x,z) is a position of a parcel at $t=0 \Rightarrow (\dot{X},\dot{Z}) = (u(X,Z,t), w(X,Z,t))$.

Incompressibility equation - conservation of volume:

$$\frac{\partial(X,Z)}{\partial(x,z)} = 1\;. \tag{55}$$

Hydrostatic equation

$$\partial_Z \phi \equiv \frac{\partial(X,\phi)}{\partial(x,z)} = g\frac{\theta_l}{\theta_r}. \tag{56}$$

Horizontal momentum equations

$$\begin{aligned} \ddot{X} - fv + \partial_X \phi &= 0 \,, \\ \dot{v} + f\dot{X} &= 0 \,, \end{aligned} \tag{57}$$

Elimination of v:

Conservation of M and b. c.:

$$M(x) = v + fX = fx + v_I(x). \tag{58}$$

Elimination of ϕ by cross-differentiation:

$$\frac{\partial(X, \ddot{X} - fv_I - f^2 x)}{\partial(x, z)} + \frac{g}{\theta_0}\frac{\partial(\theta_I, Z)}{\partial(x, z)} = 0 \tag{59}$$

Stationary adjusted states:

$$\frac{\partial(X, -fv_I - f^2 x)}{\partial(x,z)} + \frac{g}{\theta_0}\frac{\partial(\theta_I, Z)}{\partial(x,z)} = 0 \qquad (60)$$

$$\frac{\partial(X, Z)}{\partial(x,z)} = 1 \qquad (61)$$

These equations can be solved analytically for configurations with constant PV, for example for a layer of the fluid between a flat bottom (at $z = 0$) and a rigid lid (at $z = H = 1$), with b. c.:

$$Z(x,0) = 0\,, Z(x,1) = 1\,. \qquad (62)$$

Localised fronts/jets correspond to $X|_{x\to\pm\infty} = x$.

Example: zero PV

Initial configuration:

Gradient of potential temperature (density) θ - purely horizontal, no vertical shear in v:

$$\theta_I = \theta_I(x), \quad v_I = v_I(x). \tag{63}$$

Horizontal momentum equation:

$$\frac{\partial X}{\partial z} f(v_I' + f) + \frac{\partial Z}{\partial z}\frac{g\theta_I'}{\theta_0} = 0\,, \tag{64}$$

where prime denotes differentiation with respect to x.

Integration in z:

$$X = \frac{\mathcal{F}(x)}{fv_I' + f^2} - \frac{g\theta_I'/\theta_0}{fv_I' + f^2} Z\,. \tag{65}$$

Using the incompressibility equation:

$$Z^2\left(\frac{g\theta_I'/\theta_0}{fv_I'+f^2}\right)'-2\left(\frac{\mathcal{F}}{fv_I'+f^2}\right)'Z+2(\mathcal{G}(x)+z)=0\ , \tag{66}$$

where $\mathcal{G}(x)$ - another integration "constant" after integration in z.

Applying b. c.

$$\mathcal{G}(x) = 0, \tag{67}$$

$$\left(\frac{\mathcal{F}}{fv_I'+f^2}\right)' = 1+\frac{1}{2}\left(\frac{g\theta_I'/\theta_0}{fv_I'+f^2}\right)' \equiv 1+\frac{1}{2}\mathcal{A}'(x). \tag{68}$$

Explicit form of stationary solutions:

$$X_s = x + \mathcal{A}(x)\left(\frac{1}{2} - Z\right), \; \mathcal{A} = \frac{g\theta_I'/\theta_0}{fv_I' + f^2} \tag{69}$$

$$Z_s = \frac{1}{\mathcal{A}'(x)}\left[1 + \frac{1}{2}\mathcal{A}'(x) - \sqrt{\left(1 + \frac{1}{2}\mathcal{A}'(x)\right)^2 - 2z\mathcal{A}'(x)}\right], \tag{70}$$

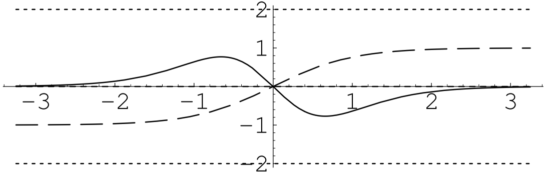
This mapping $(x, z) \rightarrow (X_s, Z_s)$ can be singular $\equiv$ not bijective, if $\exists (x, z)$: $\frac{\partial X_s}{\partial x} = 0$.
Singularity appears at the boundaries $\Rightarrow$ criterion: $1 \pm \frac{\mathcal{A}'}{2} = 0$, or:

$$\frac{g}{f\theta_0}\left(\frac{g\theta_I'/\theta_0}{f + v_I'}\right)' = \pm 2. \tag{71}$$

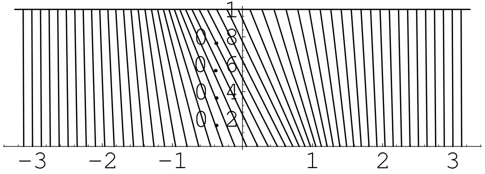
Illustrations: zero PV, localised anomaly of θ without initial jet $v_I \equiv 0$.

Initial configuration

Profiles of $\theta_I = tanh(x)$ (dashed), of $\mathcal{A}'$ (continuous), and discontinuity thresholds $\mathcal{A}' = \pm 2$ (dotted):

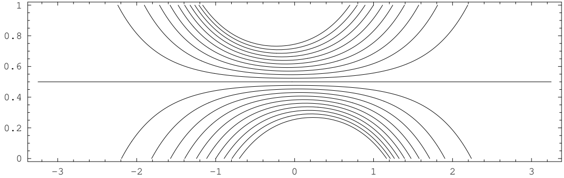


Isentropes in the adjusted state

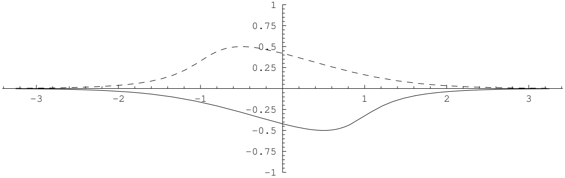


Adjusted state

Isotachs of the velocity field



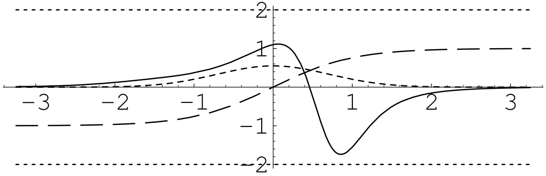
Velocity as a function of X on the vertical boundaries



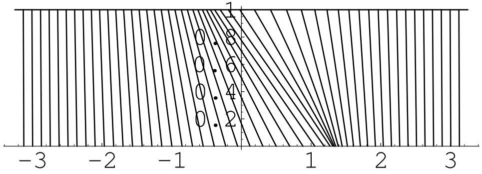
Illustrations: zero PV, localised anomaly of θ with an initial jet $v_I = 0.55e^{-x^2}$.

Initial configuration

Profiles of $\theta_I = tanh(x)$ (dashed) and of $\mathcal{A}'$ (continuous), and discontinuity thresholds $\mathcal{A}' = \pm 2$ (dotted):

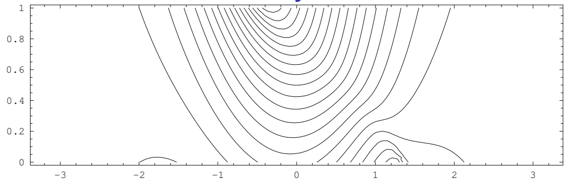


Isentropes in the adjusted state

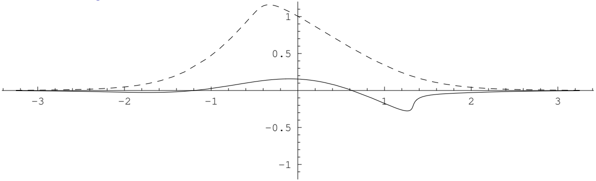


Adjusted state

Isotachs of the velocity field

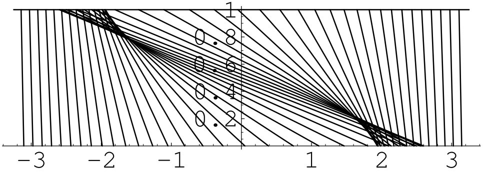


Velocity as a function of *X* at the vertical boundaries

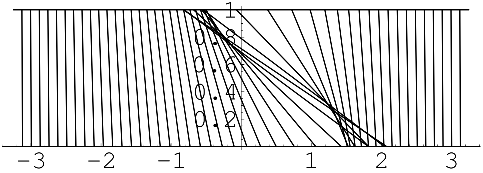


Beyond the singularity:

Configuration with $v_l = 0$, isentropes:



Configuration with $v_l \neq 0$, isentropes:



Exercises:

- Linearise the primitive equations (40) - (44) about a stationary state $v = V(x)$ (barotropic jet) with constant stratification $N^2 = \text{const}$. Decompose the solutions in vertical modes (b.c.: flat bottom, rigid lid), and demonstrate the existence of symmetric inertial instability for sufficiently strong shears $V'(x)$. Analyse the instability at various vertical wavenumbers.
- What is the necessary condition for existence of this instability? What is a link between existence of the instability and frontogenesis?