Chapter 7. Non-linear wave phenomena in GFD

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Cours GFD M2 MOCIS

Plan

Archetype model of breaking: "simple wave"

Non-dispersive unidimensionalwave with advective non-linearity :

$$u_t + \epsilon u u_x + c_0 u_x = 0. \tag{1}$$

No dispersion ↔ phase velocity c_0 constant ⇒ breaking and shock formation.

Unidirectional waves with weak dispersion

Dispersion of long waves:

Phase velocity: $c = c(k)$, k - wavenumber → 0. Strictly non-dispersive waves: $c = c_0 =$ const.
Weak dispersion: $c = c_0 + c_1 k + c_2 k^2 + ...$
Uni-directional waves: $\omega = kc(k)$ - odd function ↔ c - even function:

$$\omega = k(c_0 + c_2 k^2 + ...).$$

Phase-space vs physical space:

Translation rules for linear systems:

$$u(k, \omega) \to u(x, t) \; \Rightarrow \; k \to -i\partial_x, \quad \omega \to i\partial_t$$

$$\omega = k(c_0 + c_2 k^2 + ...) \; \leftrightarrow \; i\partial_t u = -c_0 i\partial_x u + c_2 i\partial^3_{xxx} + ...$$

Weak advective non-linearity + weak dispersion:

$$\partial_t u + c_0 \partial_x u + \alpha u \partial_x u + \beta \partial^3_{xxx} u = 0. \qquad (2)$$

Korteweg - de Vries (KdV) equation

Remark:

c_0 can be eliminated by a change of reference frame.

Solitary waves (solitons)

Solution propagating without change of form:

$$u = u(x - Vt) \;\Rightarrow$$

$$(c_0 - V)u' + \alpha u u' + \beta u''' = 0 \qquad (3)$$

Integration (localised solution - solitary wave, soliton):

$$(c_0 - V)u + \alpha \frac{u^2}{2} + \beta u'' = 0 \qquad (4)$$

Multiplication by u' and one more integration:

$$(c_0 - V)\frac{u^2}{2} + \alpha \frac{u^3}{6} + \beta \frac{u'^2}{2} = 0 \qquad (5)$$

Solution $u(x - Vt) = \frac{3}{\alpha} \frac{V - c_0}{\cosh^2 \sqrt{\frac{V - c_0}{4\beta}}(x - Vt)}$. $V = c_0 + \frac{\alpha}{3} u_{max}$

- speed depends on amplitude

Multi-soliton solutions

Standard normalisation of the KdV equation:

$$\partial_t u + 6u\partial_x u + \partial^3_{xxx} u = 0. \tag{6}$$

Solutions: $u_N = 2\partial_x^2 F_N$, N - number of solitons.

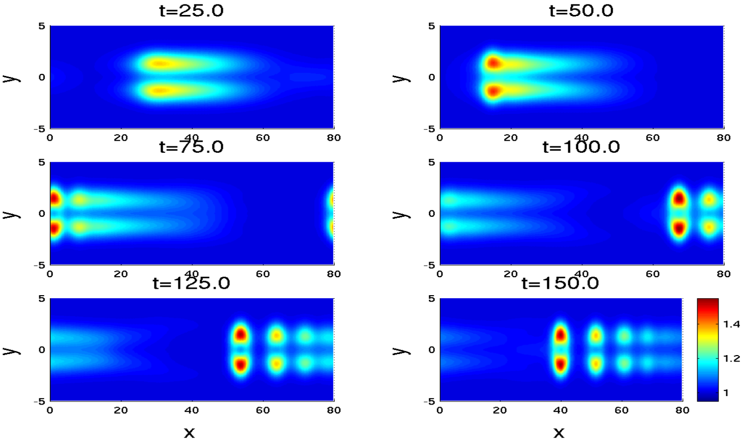
$$\begin{aligned} F_1 &= 1 + e^{\eta_1}, \\ F_2 &= 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+A_{12}}, \\ F_3 &= 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + e^{\eta_1+\eta_2+A_{12}}, \\ &+ e^{\eta_1+\eta_3+A_{13}} + e^{\eta_2+\eta_3+A_{23}} \\ &+ e^{\eta_1+\eta_2+\eta_3+A_{12}+A_{23}+A_{13}}, \\ F_N &= \ldots\ldots. \end{aligned} \tag{7}$$

where $\eta_i = k_i x - k_i^3 t - \eta_i^{(0)}$, $\quad A_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2$.
Arbitrary initial perturbation → series of solitons (complete integrability).

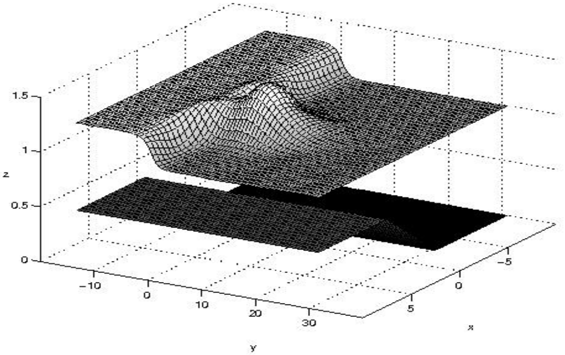
Exercise:

Check the formula for the bi-soliton solution $F2$.

Solitons of equatorial Rossby waves



Solitons of trapped topographic waves



Non-linear Rossby waves in the QG RSW model on the β-plane

Non-dimensional equations of motion :

$$\nabla^2\psi_t - \psi_t + \epsilon\mathcal{J}(\psi, \nabla^2\psi) + \psi_x = 0,$$

ϵ - non-linearity parametre, $\epsilon \to 0$. Asymptotic expansion in non-linearity parametre. Solution - asymtotic series:

$$\psi = \psi^{(0)} + \epsilon\psi^{(1)} + \ldots$$

Order zero: linear Rossby waves.

$$\nabla^2\psi_t^{(0)} - \psi_t^{(0)} + \psi_x^{(0)} = 0, \;\Rightarrow$$

$$\psi^{(0)} = \sum_i A_i e^{i(\mathbf{k}_i\cdot\mathbf{x} - \omega(\mathbf{k}_i)t)} + c.c., \quad \omega(\mathbf{k}) = -\frac{k}{k^2 + l^2 + 1}, \quad \mathbf{k} = (k, l). \tag{8}$$

Order one: first non-linear correction :

$$\nabla^2\psi_t^{(1)} - \psi_t^{(1)} + \psi_x^{(1)} = -\mathcal{J}(\psi^{(0)}, \nabla^2\psi^{(0)}), \qquad (9)$$

Term in the r.h.s.:

$$\sum_{i,j} A_i A_j \left[\left(k_i l_j - k_j l_i\right) \mathbf{k}_j^2\right] e^{i\left[(\mathbf{k}_i+\mathbf{k}_j)\cdot\mathbf{x}-(\omega(\mathbf{k}_i)+\omega(\mathbf{k}_j))t\right]} \qquad (10)$$

$$- \sum_{i,j} A_i A_j^* \left[\left(k_i l_j - k_j l_i\right) \mathbf{k}_j^2\right] e^{i\left[(\mathbf{k}_i-\mathbf{k}_j)\cdot\mathbf{x}-(\omega(\mathbf{k}_i)-\omega(\mathbf{k}_j))t\right]} + c.c.$$

Integrability conditions: solution $\psi^{(1)}$ should be bounded

Integrability conditions:

$$\begin{aligned} \forall \hat{\psi} : \nabla^2 \hat{\psi}_t - \hat{\psi}_t + \hat{\psi}_x &= 0, \\ \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \hat{\psi}^* \left(\nabla^2 \psi_t^{(1)} - \psi_t^{(1)} + \psi_x^{(1)} \right) &= 0. \end{aligned}$$

Therefore:

$$\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \hat{\psi}^* \left(\mathcal{J}(\psi, \nabla^2 \psi) \right) = 0. \qquad (11)$$

- orthogonality of the r.h.s. to the eigen-vectors of the zero-order linear operator.

Resonances

Necessarily: $\hat{\psi} \propto e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega(\hat{\mathbf{k}})t)}$, and (11) becomes:

$$\begin{aligned}
&\sum_{i,j} A_i A_j \left[\left(k_i l_j - k_j l_i\right)\mathbf{k}_j^2\right]\cdot \\
&\int_{-\infty}^{\infty} dt\,dx\,dy e^{i\left[(\mathbf{k}_i+\mathbf{k}_j-\hat{\mathbf{k}})\cdot\mathbf{x}-(\omega(\mathbf{k}_i)+\omega(\mathbf{k}_j)-\omega(\hat{\mathbf{k}}))t\right]} \quad - \\
&\sum_{i,j} A_i A_j^* \left[\left(k_i l_j - k_j l_i\right)\mathbf{k}_j^2\right]\cdot \\
&\int_{-\infty}^{\infty} dt\,dx\,dy e^{i\left[(\mathbf{k}_i-\mathbf{k}_j-\hat{\mathbf{k}})\cdot\mathbf{x}-(\omega(\mathbf{k}_i)-\omega(\mathbf{k}_j)-\omega(\hat{\mathbf{k}}))t\right]} + c.c. \quad = \quad 0
\end{aligned}$$

Calculus of generalised functions:

$$\int_{-\infty}^{\infty} dx e^{ikx} = \delta(k) - \text{Dirac's delta-function.} \qquad (12)$$

Generalisation of

$$\int_{0}^{2\pi} dx e^{ikx} = \delta_{k0} - \text{tensor delta of Kronecker} \qquad (13)$$

for periodic boundary conditions.

Resonances:

Non-zero contributions:

$$\mathbf{k}_i \pm \mathbf{k}_j = \hat{\mathbf{k}}, \quad \omega(\mathbf{k}_i) \pm \omega(\mathbf{k}_j) = \omega(\hat{\mathbf{k}}). \qquad (14)$$

three-wave resonances, resonant triads.

Elimination of resonances

If $\exists\,\hat{\mathbf{k}}$ which verifies (14) the first non-linear correction is not bounded $\Rightarrow$ asymptotic procedure is not self-consistent: resonances should be "killed".

Introducing slow evolution of the amplitudes:

$$\begin{aligned} \partial_t \to \partial_t + \epsilon \partial_T \quad & \Rightarrow \qquad (15) \\ \nabla^2 \psi_t^{(1)} - \psi_t^{(1)} + \psi_x^{(1)} \quad &= \quad -\nabla^2 \psi_T^{(0)} - \psi_T^{(0)} - \mathcal{J}(\psi^{(0)}, \nabla^2 \psi^{(0)}) \end{aligned}$$

New contribution in the r.h.s.:

$$\sum_i A_{i_T} e^{i(\mathbf{k}_i \cdot \mathbf{x} - \omega(\mathbf{k}_i) t)} + c.c. \quad \Rightarrow \qquad (16)$$

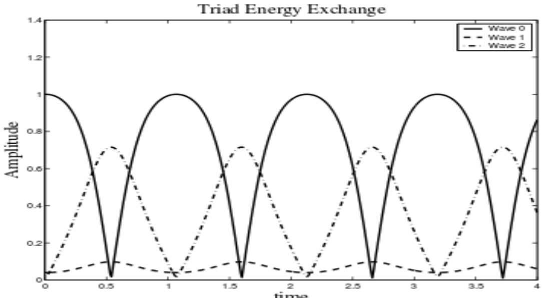
Possibility of compensation of resonant contributions by slow evolution of amplitudes.

A resonant triad:

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3, \quad \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) = \omega(\mathbf{k}_3), \tag{17}$$

$$\begin{aligned} \dot{A}_3 &= c(\mathbf{k}_1, \mathbf{k}_2) A_1 A_2, \\ \dot{A}_2 &= c(\mathbf{k}_3, -\mathbf{k}_1) A_3 A_1^*, \\ \dot{A}_1 &= c(\mathbf{k}_3, -\mathbf{k}_2) A_3 A_2^*, \end{aligned} \tag{18}$$

where $c(\mathbf{k}_1, \mathbf{k}_2) = \hat{\mathbf{z}} \cdot (\mathbf{k}_1 \wedge \mathbf{k}_2) \mathbf{k}_2^2$ - interaction coefficients . This is an integrable system (in elliptic fonctions). Energy is conserved and redistributed among three waves.



Remark:

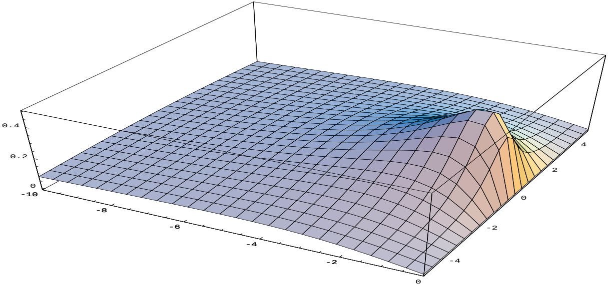
If resonances of three waves are not possible → resonant quartets etc.

Wave turbulence:

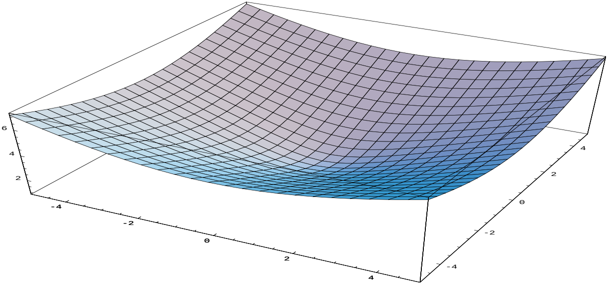
Ensemble of waves with random phases ⇒ Gaussian statistics ⇒ kinetic equation for the wave density, entierely determined by resonant triads (quartets) ⇒ energy spectra.
Successful applications

- spectra of the surface wind waves
- spectra of the internal waves in the ocean (Garret - Munk)

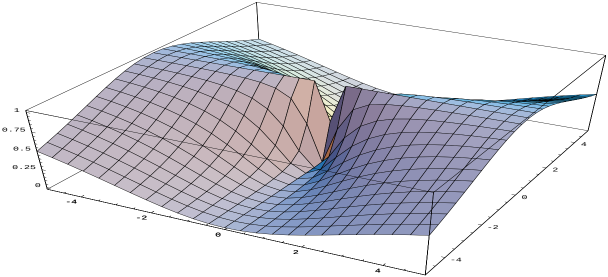
Rossby waves - strong dispersion



Barotropic inertia - gravity waves - weak dispersion



Internal gravity waves - strong dispersion



Exercice:

Demonstrate that resonant triads

- of Rossby waves in the QG RSW model on the beta-plane exist
- of inertia-gravity waves in the RSW model on the f-plane do not exist

Essentially non-linear Rossby waves

Equations of motion

$$\nabla^2\psi_t - \psi_t + \mathcal{J}(\psi, \nabla^2\psi) + \psi_x = 0, \tag{19}$$

Solutions - waves propagating without change of form:

$$\psi(x,y,t) = \psi(x - Ut, y), \Rightarrow \mathcal{J}(\psi + Uy, \nabla^2\psi + (1+U)y) = 0 \Rightarrow$$

$$\nabla^2\psi + (1+U)y = F(\psi + Uy), \tag{20}$$

F - arbitrary function. Physical meaning: potential vorticity is constant along the streamlines.

Remark:

F is not neccessarily the same over the whole (x, y) plane. Domains with different $F \Rightarrow$ matching.

Localised solutions:

$$r = \sqrt{x^2 + y^2} \to \infty \Rightarrow \psi \to 0 \tag{21}$$

$\Rightarrow$ "external" F - linear function:

$$F(\psi + Uy) = p^2(\psi + Uy), \; p^2 = \frac{1+U}{U} > 0. \tag{22}$$

Modons

Solutions with F linear outside and inside the circle $r = a$:

$$\begin{aligned} \nabla^2\psi &= p^2(\psi + Uy) - (1+U)y, \; r > a, \\ \nabla^2\psi &= -k^2(\psi + Uy) - (1+U)y, \; r < a, \end{aligned} \qquad (23)$$

Solutions in polar coordinates $x = r\cos\phi$, $y = r\sin\phi \rightarrow$ Bessel functions:

$$\begin{aligned} \psi &= BK_1(pr)\sin\phi, \; r > a, \\ \psi &= \left[AJ_1(kr) - \frac{r}{k^2}(1 + U + Uk^2)\right]\sin\phi, \; r < a \end{aligned} \quad (24)$$

where J_1 - Bessel function (oscillating), K_1 - modified Bessel function (decaying), A, B - constants to be determined from the conditions of matching and b.c..

Matching

Continuity of $\psi, \partial_r\psi, \partial^2_{rr}\psi$:

$$\begin{aligned} \psi + Uy|_{a-} &= \psi + Uy|_{a+} = 0, \\ \partial_r\psi|_{a-} &= \partial_r\psi|_{a+} . \end{aligned} \tag{25}$$

2 first conditions giving A, B:

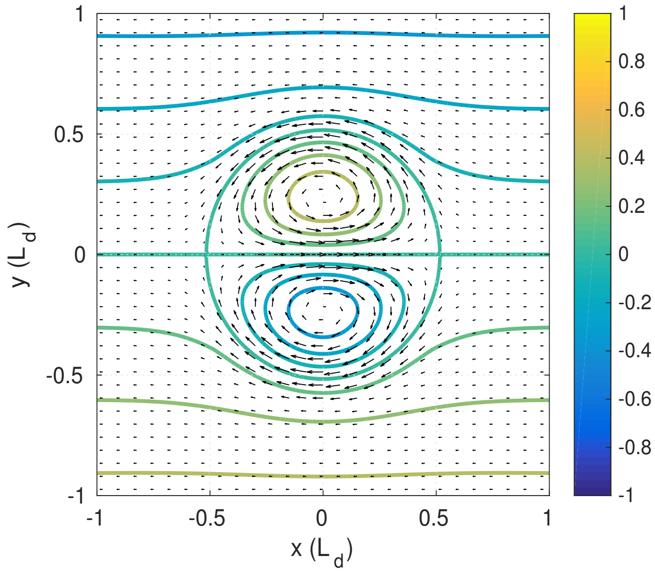
$$A = \frac{a(1+U)}{k^2 J_1(ka)}, \quad B = -\frac{Ua}{K_1(pa)}. \tag{26}$$

3rd condition: determining k:

$$\frac{J_1'(ka)}{J_1(ka)} = \frac{1}{ka}\left(1 + \frac{k^2}{p^2}\right) - \frac{k}{p}\frac{K_1'(pa)}{K_1(pa)}. \tag{27}$$

$\forall(a, p)$ infinite series of solutions for k. Minimal value of k - dipolar structure.

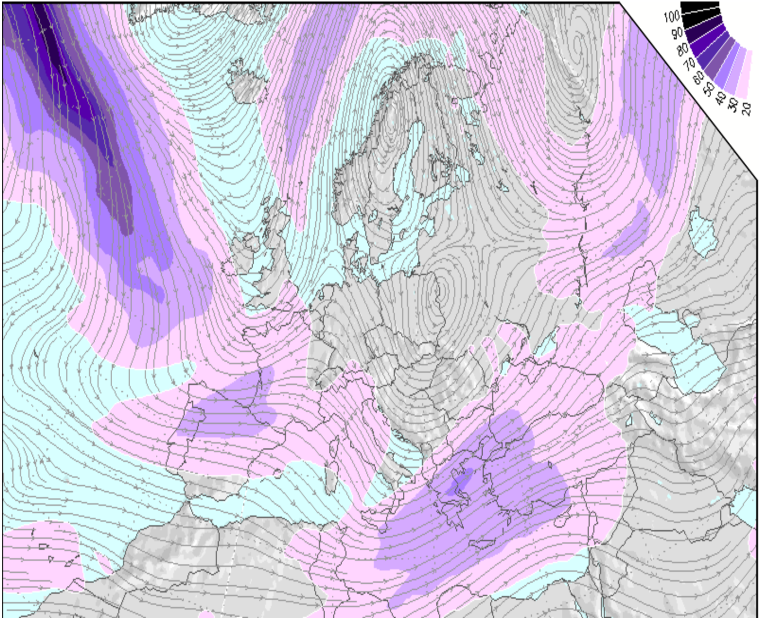
Streamfunction of a modon



Exercice:

Obtain the formulas (26), (27).

Modons and atmospheric blockings



Stratified non-rotating fluid

2d Boussinesq equations:

$$\begin{aligned} u_t + uu_x + ww_z + \phi_x &= 0, \\ w_t + uw_x + ww_z + \Xi + \phi_z &= 0, \\ u_x + w_z = 0, \quad \Xi_t + u\Xi_x + w\Xi_z &= 0. \end{aligned} \tag{28}$$

$\Xi = \frac{g(\rho(z)+\sigma)}{\rho_0}$ - buoyancy variable, including the effects of background stratification $\rho(z)$, $\phi = \frac{P}{\rho_0}$ - geopotential, ρ_0 - constant normalisation density, σ - density perturbation.

Equations in streamfunction - buoyancy variables

$$\begin{aligned} \Delta\psi_t + \mathcal{J}(\psi, \Delta\psi) + \Xi_x &= 0, \\ \Xi_t + \mathcal{J}(\psi, \Xi) &= 0. \end{aligned} \tag{29}$$

ψ - streamfunction, $\zeta = -\Delta\psi$ - horizontal vorticity, Δ - Laplacian, $\mathcal{J}$ - Jacobian.

Hydrostatic limit (long waves)

Replacement of the equation for *w* by hydrostatic equation $-\Xi = \phi_z$:

$$\begin{aligned} \psi_{zzt} + \mathcal{J}(\psi, \psi_{zz}) + \Xi_x &= 0, \\ \Xi_t + \mathcal{J}(\psi, \Xi) &= 0. \end{aligned} \tag{30}$$

Stationary solutions (change of reference frame ⇒ propagation at constant speed):

$$\begin{aligned} \mathcal{J}(\psi, \Delta\psi) + \Xi_x &= 0, \\ \mathcal{J}(\psi, \Xi) &= 0. \end{aligned} \tag{31}$$

Therefore $\Xi = \Xi(\psi)$ and

$$\begin{aligned} \mathcal{J}(\psi, \Delta\psi) + \Xi'(\psi)\psi_x = 0 \quad &\Rightarrow \\ \mathcal{J}(\psi, \Delta\psi + \Xi'(\psi)z) = 0 \quad &\Rightarrow \\ \Delta\psi + \Xi'(\psi)z = F(\psi), & \end{aligned} \tag{32}$$

where $\Xi(\psi)$ and $F(\psi)$ - arbitrary functions.

Long's waves

Upstream ($x \to \infty$): b.c. of constant velocity $\to \psi = cz$, and of a given stratification $\Xi = \Xi_0(z) \Rightarrow$

$$\Xi(\psi) = \Xi_0(\frac{\psi}{c}), \quad F(\psi) = \Xi'(\psi)\frac{\psi}{c} = \frac{\psi}{c^2}\Xi_0'(\frac{\psi}{c}) \tag{33}$$

Example: linear stratification upstream: $\Xi_0 = \text{const} + \alpha z$ $\to$ Long's equation (linear!) for a non-linear stationary wave:

$$\Delta\psi + \frac{\alpha}{c}(z - \frac{\psi}{c}) = 0. \tag{34}$$

New variable - deviation of streamlines:

$$\phi = \psi - cz, \Rightarrow \Delta\phi - \frac{\alpha}{c^2}\phi = 0. \tag{35}$$

B.c. in z: $\phi|_{z=h(x)} = h(x)$, h - topography.

Exercice:

- Consider hydrostatic version of (34). Find its particuliar solution verifyi ng automaticaly the b. c. at $z = h(x)$.
- In a half-bounded domain in z, what should be a b. c. at $z \to \infty$ to be used?