# Chapter 8: Vortex dynamics and wave emission by vortices.

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Course GFD M2 MOCIS

# Plan

Two-dimensional vortex dynamics

Dynamics of point vortices

Contour dynamics

Modons: non-singular analogs of point dipoles
- QG modons on the $f$-plane
- RSW modons on the $f$-plane

Vortices on the $\beta$- plane

Interaction of vortices with topography

Wave emission by vortex systems
- Wave emission by a vortex pair
- Back-reaction of the wave emission

# Archetype model: 2D Euler equations, incompressible fluid ($\rho = 1$)

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P, \quad \nabla \cdot \mathbf{v} = 0. \tag{1}$$

## Remark:

Same result for the 2D Euler equations for an incompressible fluid in rotating frame, as Coriolis terms disappear because $\nabla \cdot \mathbf{v} = 0$.

## Streamfunction:

$$\nabla \cdot \mathbf{v} = 0, \; \Rightarrow \mathbf{v} = \hat{\mathbf{z}} \wedge \nabla \psi \leftrightarrow u = -\partial_y \psi, \; v = \partial_x \psi \tag{2}$$

## Rewriting the equations:

$$\hat{\mathbf{z}} \wedge \nabla \partial_t \psi + (\hat{\mathbf{z}} \wedge \nabla \psi \cdot \nabla)(\hat{\mathbf{z}} \wedge \nabla \psi) = -\nabla P \Rightarrow \tag{3}$$

$$\partial_t \nabla^2 \psi + \mathcal{J}(\psi, \nabla^2 \psi) = 0 \tag{4}$$

# Vorticity

$$\zeta = \hat{\mathbf{z}} \cdot (\nabla \wedge \mathbf{v}) = \nabla^2 \psi, \Rightarrow \tag{5}$$

$$\partial_t \zeta + \mathcal{J}(\psi, \zeta) = \frac{d\zeta}{dt} = 0 \Rightarrow \tag{6}$$

Lagrangian conservation: 2D incompressible hydrodynamics $\equiv$ dynamics of vorticity

## Inversion problem

Poisson problem in the domain $\mathcal{D}$: $\nabla^2 \psi = \zeta$. Solution with the help of Green's function:

$$\psi = \int_{\mathcal{D}} \mathcal{G}(\mathbf{x}, \mathbf{x}') \, \zeta(\mathbf{x}') \, d\mathbf{x}', \;\; \nabla^2 \mathcal{G} = \delta(\mathbf{x} - \mathbf{x}'), \tag{7}$$

$\delta-$Dirac's delta: $\int_{\mathcal{D}} \delta(\mathbf{x}-\mathbf{x}') \, d\mathbf{x}' = 1, \; \int_{\mathcal{D}} f(\mathbf{x}') \delta(\mathbf{x}-\mathbf{x}') \, d\mathbf{x}' = f(\mathbf{x}).$ (8)

Whole plane: $\mathcal{G}(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi} \log |\mathbf{x} - \mathbf{x}'|.$

# Energy in terms of $\zeta$

$$
\begin{aligned}
E &= \frac{1}{2}\int_{\mathcal{D}} \mathbf{v}^2 \, d\mathbf{x} = \frac{1}{2}\int_{\mathcal{D}} (\hat{\mathbf{z}} \wedge \nabla\psi)^2 \, d\mathbf{x} = \frac{1}{2}\int_{\mathcal{D}} (\nabla\psi)^2 \, d\mathbf{x} \\
&= const - \frac{1}{2}\int_{\mathcal{D}} \zeta\psi \, d\mathbf{x}. \qquad (9)
\end{aligned}
$$

Green's formula (Stokes theorem in 2D):

$$
\oint_{\Gamma} (Pdx + Qdy) = \int_{\mathcal{D}_\Gamma} (\partial_x Q - \partial_y P) \, dxdy \Rightarrow \qquad (10)
$$

$$
E = -\frac{1}{2}\int_{\mathcal{D}}\int_{\mathcal{D}} d\mathbf{x}d\mathbf{x}' \zeta(\mathbf{x})\mathcal{G}(\mathbf{x},\mathbf{x}')\zeta(\mathbf{x}') \qquad (11)
$$

## Eulerian expression of the conservation of $\zeta$

$$\int_{\mathcal{D}} \mathcal{F}(\zeta)\, d\mathbf{x} = const, \;\; \mathcal{F} - \text{arbitrary function} \tag{12}$$

- $\mathcal{F}(\zeta) = \zeta$ - conservation of the mean vorticity,
- $\mathcal{F}(\zeta) = \zeta^2$ - conservation of enstrophy,
- $\mathcal{F}(\zeta) = 1$, si $\zeta_- < \zeta < \zeta_+$, otherwise zero $\Rightarrow$ conservation of area between the iso-lines of vorticity $\zeta = \zeta_-$ and $\zeta = \zeta_+$.

## Point vortices

Vorticity concentrated in a point $\mathbf{x}_1$ of the $x, y$ plane, with circulation $\kappa$:

$$\zeta(\mathbf{x}) = \kappa\delta(\mathbf{x} - \mathbf{x}_1). \tag{13}$$

System of $N$ vortices:

$$\zeta_N(\mathbf{x}) = \sum_{i=1}^{N} \kappa_i \delta(\mathbf{x} - \mathbf{x}_i). \tag{14}$$

Corresponding streamfunction (entire plane):

$$\psi_N(\mathbf{x}) = \frac{1}{2\pi}\int d\mathbf{x}' \, \log|\mathbf{x}-\mathbf{x}'| \sum_{i=1}^{N} \kappa_i \delta(\mathbf{x}'-\mathbf{x}_i) = \frac{1}{2\pi}\sum_{i=1}^{N} \kappa_i \log|\mathbf{x}-\mathbf{x}_i| \tag{15}$$

- superposition principle.

# Evolution of vortices

- Vortex intensities do not change: $\dot{\kappa}_i = 0$, $\dot{...} \equiv \frac{d...}{dt}$ - Kelvin theorem.
- Each vortex is advected by the total velocity field:

$$\dot{\mathbf{x}}_i = \hat{\mathbf{z}} \wedge \nabla \psi_N|_{\mathbf{x}=\mathbf{x}_i} = \frac{1}{2\pi} \sum_{j=1}^{N} \kappa_i \frac{\hat{\mathbf{z}} \wedge \mathbf{x}_{ij}}{\mathbf{x}_{ij}^2}, \ i = 1, 2, ..., N, \tag{16}$$

where $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$.

# Conservation laws:

- Energy:

$$E_N = \frac{1}{2}\int \zeta_N(\mathbf{x})\psi_N(\mathbf{x})\, d\mathbf{x} = \frac{1}{4\pi}\sum_{i\neq j}^{N} \kappa_i\kappa_j \log|\mathbf{x}_{ij}| \quad (17)$$

- Centroid of vorticity:

$$\mathbf{R}_N = \frac{\int \mathbf{x}\zeta_N(\mathbf{x})\, d\mathbf{x}}{\int \zeta_N(\mathbf{x})\, d\mathbf{x}} = \frac{\sum_{i=1}^{N} \mathbf{x}_i\kappa_i}{\sum_{i=1}^{N} \kappa_i}, \quad (18)$$

- Angular momentum:

$$M_N = \frac{1}{2}\int \zeta_N(\mathbf{x})\mathbf{x}^2\, d\mathbf{x} = \frac{1}{2}\sum_{i=1}^{N} \kappa_i\mathbf{x}_i^2. \quad (19)$$

# Exercises:

1. Demonstrate the conservation laws for a system of $N$ vortices,
2. Demonstrate that a vortex pair rotates around its centre of vorticity with angular velocity $\Omega = \frac{\kappa_1 + \kappa_2}{2\pi \mathbf{x}_{12}^2}$,
3. Demonstrate that 3 vortices of equal intensity forming an equilateral triangle rotate with constant angular velocity.

# "Patch" of constant vorticity

Domain of constant vorticity $\zeta = \zeta_0 = const$, with the boundary $\Gamma$; zero vorticity elsewhere. Each point of the contour $\Gamma$ is advected:

$$\dot{\mathbf{x}}_\Gamma = \mathbf{v}_\Gamma = \hat{\mathbf{z}} \wedge \nabla \, \psi|_{\mathbf{x}_\Gamma}, \quad \psi = \frac{\zeta_0}{2\pi} \int_{S_\Gamma} d\mathbf{x}' \, \log|\mathbf{x} - \mathbf{x}'| \;\Rightarrow \quad (20)$$

$$\begin{aligned}
\dot{x}_\Gamma &= -\frac{\zeta_0}{2\pi} \int_{S_\Gamma} d\mathbf{x}' \, \partial_y \log|\mathbf{x} - \mathbf{x}'| = \frac{\zeta_0}{2\pi} \int_{S_\Gamma} d\mathbf{x}' \, \partial_{y'} \log|\mathbf{x} - \mathbf{x}'| \\
&= -\frac{\zeta_0}{2\pi} \oint_\Gamma dx' \, \log|\mathbf{x} - \mathbf{x}'| \\
\dot{y}_\Gamma &= \frac{\zeta_0}{2\pi} \int_{S_\Gamma} d\mathbf{x}' \, \partial_x \log|\mathbf{x} - \mathbf{x}'| = -\frac{\zeta_0}{2\pi} \int_{S_\Gamma} d\mathbf{x}' \, \partial_{x'} \log|\mathbf{x} - \mathbf{x}'| \\
&= -\frac{\zeta_0}{2\pi} \oint_\Gamma dy' \, \log|\mathbf{x} - \mathbf{x}'| \qquad (21)
\end{aligned}$$

## Piecewise-constant distribution of vorticity

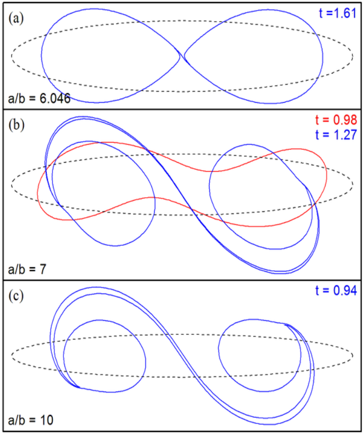
Single contour

$$\dot{\mathbf{x}}_\Gamma = -\frac{\zeta_0}{2\pi} \oint_\Gamma d\mathbf{x}' \log|\mathbf{x} - \mathbf{x}'| \tag{22}$$

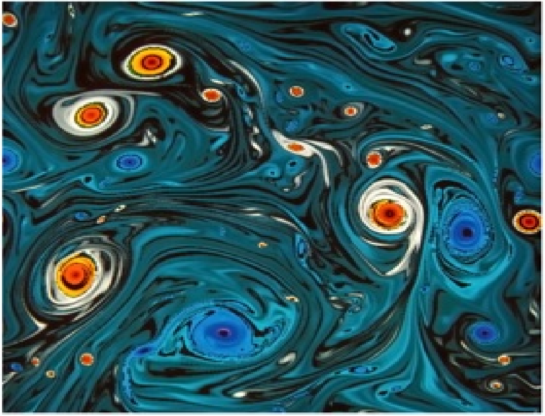
Several superimposed contours:

$$\dot{\mathbf{x}}_{\Gamma_i} = -\frac{1}{2\pi} \sum_j^N \zeta_j \oint_{\Gamma_j} d\mathbf{x}' \log|\mathbf{x}_i - \mathbf{x}_j'| \tag{23}$$

# Example of simulations with contour dynamics: destabilisation of the Kirchhoff vortex and formation of vortex filaments



# Vortices and vortex filaments in 2D turbulence as seen in the vorticity field



# Geostrophic modon on the $f$-plane

QG equation on the $f$-plane:

$$\partial_t(\nabla^2 h - \gamma^{-1} h) + \mathcal{J}(h, \nabla^2 h) = 0, \qquad (24)$$

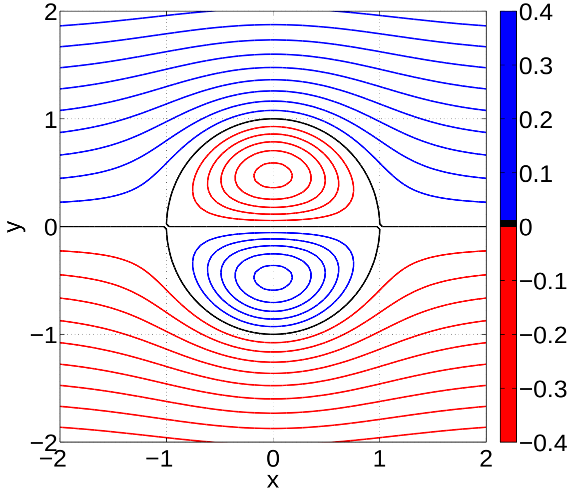
$\gamma$ Burger number.
Modon: dipolar localised solution of (24). Steady propagation along the $x$-axis with constant velocity $U_0$.
Continuous solution in terms of streamfunction $\Psi = h + y$:

$$\begin{aligned} \Psi = & \quad U_0 \left[ \frac{J_1(\Pi r/L)}{J_1(\Pi)} - r \right] \left(\frac{\Lambda}{\Pi}\right)^2 \sin\theta & \quad r < L \\ = & \quad U_0 \left[ r - \frac{K_1(\Lambda r/L)}{K_1(\Lambda)} \right] \sin\theta & \quad r \geq L \end{aligned} \qquad (25)$$

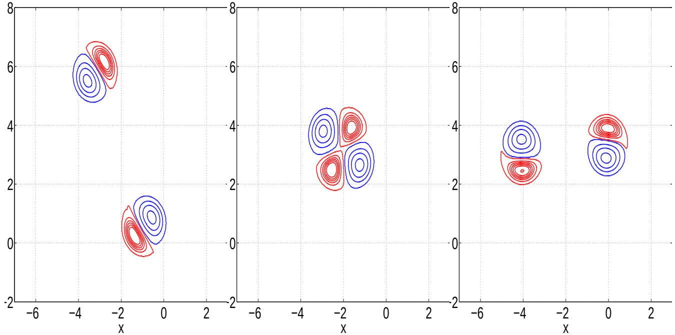
$(r, \theta)$ polar coordinates in the plane $(x - U_0 t, y)$; $J_1$, $K_1$ - Bessel and modified Bessel functions,
$\Lambda = \frac{L}{R_d} = \gamma^{-1/2}$, $\Pi = \sqrt{A^2 - \Lambda^2}$.

# Obtained under assumptions:

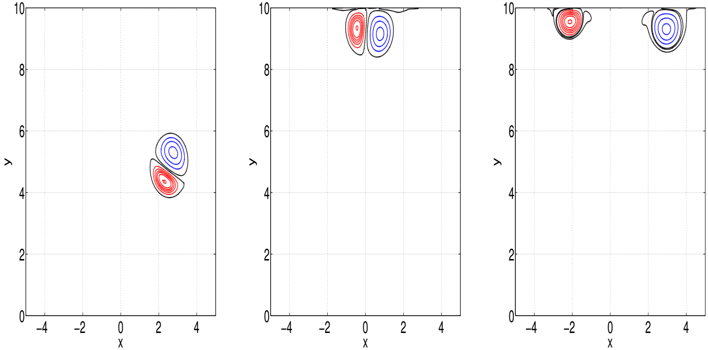
Geostrophic PV: $Q = \nabla^2\psi - \Lambda^2\psi$ supposed zero in the exterior of a circular separatrix of radius $L$, and proportional to $\Psi$ in the interior, with proportionality constant $-A^2$.



# Elastic modon-modon collisions as seen in the PV field



# Modon collision with a wall at the upper boundary as seen in the PV field



# Why vortex goes along the wall?

Explanation with a point vortex dipole:

$$\psi = -\frac{\kappa}{2\pi}\left(\log\sqrt{(y-Vt)^2+(x-\frac{d}{2})^2} - \log\sqrt{(y-Vt)^2+(x+\frac{d}{2})^2}\right), \quad V = -\frac{\kappa}{2\pi d}$$

Transverse velocity at the middle line:

$$u|_{x=0} = -\frac{\kappa}{2\pi}\left[\frac{y-Vt}{(y-Vt)^2+(x-\frac{d}{2})^2} - \frac{y-Vt}{(y-Vt)^2+(x-\frac{d}{2})^2}\right] \equiv 0, \tag{26}$$

Streamline $x = 0$ may be replaced by the boundary. The boundary $\equiv$ a mirror. Vortex dynamics near the wall(coast): "real" + image vortices.

# Vortices on the $\beta$- plane

2D Euler (barotropic) equations on the $\beta$- plane:

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + f(y)\hat{\mathbf{z}} \wedge \mathbf{v} = -\nabla P, \quad \nabla \cdot \mathbf{v} = 0. \Rightarrow \tag{27}$$

$$\partial_t \nabla^2 \psi + \mathcal{J}(\psi, \nabla^2 \psi) + \beta \partial_x \psi = 0 \leftrightarrow \tag{28}$$

Conservation of absolute vorticity:

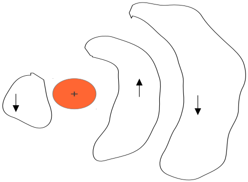
$$\frac{d\zeta_a}{dt} = 0, \quad \zeta_a = \nabla^2 \psi + f(y) \tag{29}$$

Vortex and wave (Rossby waves) regimes depending on the value of the Rhines parameter

$$Rh = L \left( \frac{\beta}{U} \right)^{\frac{1}{2}},$$

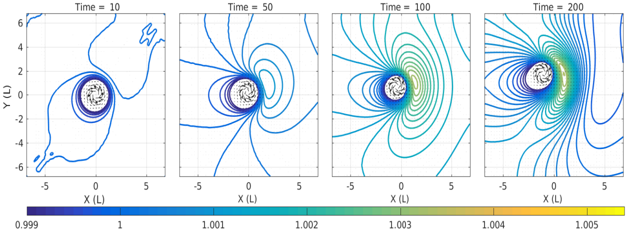
where $L, U$ - typical spatial and velocity scales.
Dynamical regimes: $Rh \to \infty$ - waves, $Rh \to 0$ - vortices.

# Qualitative evolution of the monopolar vortex on the $\beta$- plane



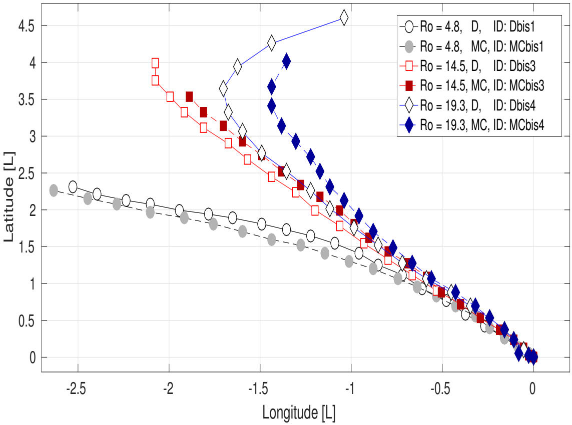

Red: initial relative vorticity. .

# Development of beta-gyres by localised vortices $\equiv$ emission of Rossby waves

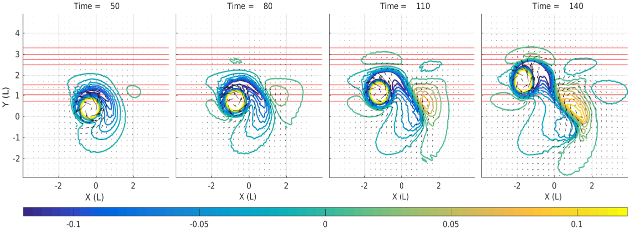



Pressure (colors) and velocity (arrows) fields of a hurricane-like vortex

# Deflection of trajectories of intense vortices due to beta-gyres

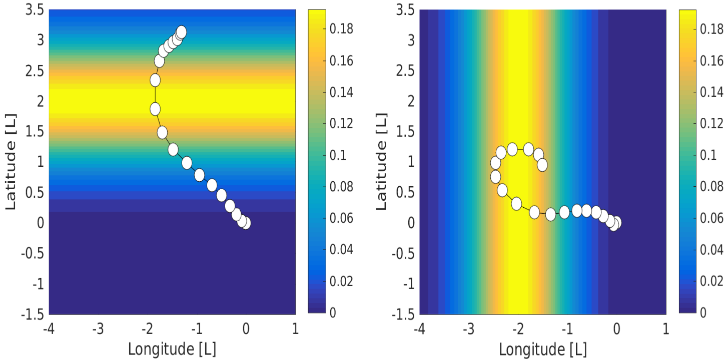


# Development of topographic gyres by a vortex encountering a zonal ridge




Pressure and velocity fields of an intense vortex crossing a ridge. Topography levels - red.

# Deflection of trajectories of vortices by topography




*Left (Right) :* vortex encountering zonal (meridional) ridge. White dots: successive positions of the vortex center.

# Exercise

1. Demonstrate that a pair of point vortices of opposite intensity (a vortex dipole) which moves with constant velocity $U$ is a solution of the Euler equations for the incompressible fluid on the $\beta$- plane. Find the spectrum of possibles values of $U$ and analyse the resulting velocity field.
   *Hint: use modified Bessel functions to solve the equation for the streamfunction.*
2. Give qualitative explanation of formation of the topographic gyres in the case of 1) meridional, 2) zonal gradiant of bottom topography.

# 2d hydrodynamics in complex notation

## Complex velocity and potential :

Complex coordonates: $z = x + iy$, $\overline{z} = x - iy$. Complex velocity: $V = u - iv$. Potential motion $\rightarrow$ complex potential $\phi(z)$: $V = \partial_z \phi$. Example: point vortex of intensity $\kappa$ at location $z_0$:

$$V = \partial_z \phi, \quad \phi_1 = i\frac{\kappa}{2\pi}\log(z - z_0). \tag{30}$$

## Vortex pair:

$$\phi_2 = i\frac{\kappa_1}{2\pi}\log(z - a_1 e^{i\Omega t}) + i\frac{\kappa_2}{2\pi}\log(z - a_2 e^{i\Omega t}), \tag{31}$$

where $\Omega = \frac{\kappa_1+\kappa_2}{2\pi a^2}$, $a$ - distance between the vortices, and

$$a_1 = a\frac{\kappa_2}{\kappa_1 + \kappa_2}, \quad a_1 = -a\frac{\kappa_1}{\kappa_1 + \kappa_2}. \tag{32}$$

## Asymptotics $r \to \infty$:

Polar oordinates $z = re^{i\theta}$ +multipolar expansion:

$$\phi_2|_{r\to\infty} = i\frac{\kappa_1+\kappa_2}{2\pi}\log\left(re^{i\theta}\right) - \frac{i}{4\pi}\frac{\kappa_1\kappa_2}{\kappa_1+\kappa_2}\left(\frac{a}{r}\right)^2 e^{2i(\Omega t-\theta)} + \dots . \tag{33}$$

Note that dipolar term $\propto e^{i(\Omega t-\theta)}$ is absent, as it is proportional to $a_1\kappa_1 + a_2\kappa_2 = 0$.

# Llinear gravity waves in SW

$$\partial_{tt} h - gH_0 \nabla^2 h = 0. \quad (34)$$

Energy balance to the leading order :

$$\partial_t \left( H_0 \frac{\mathbf{v}^2}{2} + g\frac{h^2}{2} \right) = -H_0 \nabla \cdot (\mathbf{v} gh) . \quad (35)$$

Total energy flux far from the wave sources:

$$I_E = H_0 \int d\mathbf{l} \cdot \hat{\mathbf{z}} \wedge \mathbf{v} gh. \quad (36)$$

## Lighthill radiation:

Motion if the vortex pair: frequency $\Omega \Rightarrow$ emission of gravity waves of the same frequency. Characteristic wave-length: $\lambda = \frac{\sqrt{gH_0}}{\Omega}$. Characteristic Froude (Mach) number : $M = \frac{a}{\lambda} = \frac{a\Omega}{\sqrt{gH_0}}$. If $M \to 0$, near field - incompressible regime; far field - wave regime.

## Wave equation in polar coordinates:

Solution $h$ of the wave equation:

$$h(r,\theta,t) = \sum_n h_n(r) e^{in(\Omega t - \theta)}. \qquad (37)$$

New variable: $\rho = \frac{n\Omega}{\sqrt{gH_0}} r \Rightarrow$ Bessel equation:

$$h_n'' + \frac{1}{\rho} h_n' + \left(1 - \frac{n^2}{\rho^2}\right) h_n = 0. \qquad (38)$$

## Emitted waves:

Complex combination of fundamental solutions verifying radiation b. c. at $r \to \infty$ - Hankel function:
$H_n^{(2)} = J_n - iN_n$, where $J_n$ - Bessel function, $N_n$ - Neumann function.

# Asymptotics:

$$H_n^{(2)}(\rho)\Big|_{\rho\to 0} \to \frac{i}{\pi} n! \left(\frac{\rho}{2}\right)^{-n}, \tag{39}$$

$$H_n^{(2)}(\rho)\Big|_{\rho\to\infty} \to \sqrt{\frac{2}{\pi\rho}} e^{\left(-i(\rho - n\frac{\pi}{2} - \frac{\pi}{4}\right)}. \tag{40}$$

# Matching of near and far fields

## Matching:

Conditions of matching: continuity of pressure. Pressure - time- derivative of the potential (Bernoulli) $\Rightarrow$

$$gh^{(ondes)}\Big|_{r\to 0} = \partial_t \phi_2^{(vortex)}\Big|_{r\to\infty}. \qquad (41)$$

Uniquely $n = 2$ component possible in the wave-field:

$$gh^{(ondes)} = -\frac{i}{8\pi^3 c_0}\frac{\kappa_1\kappa_2(\kappa_1+\kappa_2)^2}{a^4} H_2^{(2)}\left(\frac{2\Omega}{c_0} r\right) e^{2i(\Omega t - \theta)}, \qquad (42)$$

$c_0 = \sqrt{gH_0}$.

## Asymptotics at infinity:

$$gh^{(ondes)}\Big|_{r\to\infty} \propto \frac{\kappa_1\kappa_2|\kappa_1+\kappa_2|^{\frac{3}{2}}}{a^3 r^{\frac{1}{2}}} e^{2i(\Omega t - \frac{\Omega r}{c_0} - \theta - \frac{3}{8}\pi)} \qquad (43)$$

# Back-reaction of wave emission

## Energy flux of waves:

$$I_E = H_0 \int d\mathbf{l} \cdot \hat{\mathbf{z}} \wedge \mathbf{v} gh = H_0 \int dl\, ghv_r. \qquad (44)$$

$v_r \propto \frac{gh}{c_0} \;\Rightarrow$

$$I_E \propto \frac{(\kappa_1 \kappa_2)^2 |\kappa_1 + \kappa_2|^3}{a^6} \geq 0. \qquad (45)$$

Quadrupolar emission, $I_E \propto M^4$.

## Energy balance:

$E_{paire} \propto \kappa_1 \kappa_2 \log a$, $\kappa_{1,2}$, $H_0, c_0$ - constant $\Rightarrow$

$$\dot{E} = -I_E \Rightarrow -\frac{\text{const}}{a}\dot{a} = -\frac{\text{const}}{a^6} \Rightarrow a^5 \dot{a} = \text{const} \propto \kappa_1 \kappa_2 |\kappa_1 + \kappa_2|^3. \qquad (46)$$

## Adiabatic evolution:

$$a(t) = a(0)\left(1+\frac{t}{\tau}\right)^{\frac{1}{6}}, \quad \tau \propto \left(\kappa_1\kappa_2|\kappa_1+\kappa_2|^3\right)^{-1}. \quad (47)$$

Different cases:

- $\kappa_1\kappa_2 > 0 \Rightarrow$ increase of $a$,
- $\kappa_1\kappa_2 < 0 \Rightarrow$ decrease of $a \Rightarrow$ collapse in finite time $\tau$

Even slow $\left(T \propto M^{-4}\right)$, Lighthill radiation can lead to catastrophic consequences.

# Comments on the rotating case

Rossby number of the vortex pair:

$$Ro_2 = \frac{2\Omega}{f}. \tag{48}$$

Wave equation in the presence of rotation:

$$-\frac{\partial^2 \eta}{\partial t^2} - \frac{1}{R_d^2}\eta + \nabla^2 \eta = 0, \tag{49}$$

$R_d$ - deformation radius. Modification of radial variable $\rho$d:

$$\rho = \frac{n\Omega}{c_0} r \sqrt{Ro_n^2 - 1}; \quad Ro_n = \frac{n\Omega}{f}, \tag{50}$$

$Ro_n$ - equivalent Rossby number of the mode $n \Rightarrow$ same equation (38).
Immediate consequence: propagative waves exist only at $Ro_n > 1$. Matching: $a << r << R_d \Rightarrow Bu >> 1$.
Condition $M << 1 \Rightarrow Bu >> Ro_2^2$.