Lecture 9: Wave - mean flow interactions.

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Course GFD M2 MOCIS

Plan

Interaction Rossby wave - mean current

- Mean fields, fluctuation fields, and their interactions
- Wave pseudomomentum
- Rossby waves over a constant mean flow
- Wave-mean interaction
- Critical levels

Gravity-wave meanflow interactions

Barotropic fluid on the β-plane

Equations of motion

$$\begin{aligned} u_t + uu_x + vu_y - f(y)v + \frac{1}{\rho}p_x &= 0\,, \\ v_t + uv_x + vv_y + f(y)u + \frac{1}{\rho}p_y &= 0\,, \\ u_x + v_y &= 0\,. \end{aligned} \tag{1}$$

Typical configuration- zonal channel: $-D \leq y \leq D$, b.c.: $v|_{\pm D} = 0$.

Decomposition: zonal mean/fluctuations

Zonal mean and fluctuations:

Zonal mean:

$$\bar{A} = \lim_{L\to\infty} \frac{1}{2L}\int_{-L}^{+L} dx\, A(x,y,t) \text{ou}$$

$$\bar{A} = \frac{1}{2L}\int_{-L}^{+L} dx\, A(x,y,t), \text{si}\, A(x+2L) = A(x). \quad (2)$$

Properties:

$$A = \overline{A} + \tilde{A};\ \tilde{A} \equiv A - \overline{A};\ \overline{A_x} \equiv 0;\ \overline{\tilde{A}} \equiv 0.$$

$$\overline{\overline{A}} = \overline{A};\ \overline{\overline{A}\tilde{B}} \equiv 0.$$

Interactions mean - fluctuations

Mean barotropic zonal current in a channel

$$(u, v) = (\overline{u} + \tilde{u}, \overline{v} + \tilde{v})$$

$$\overline{u_x + v_y} = 0, \Rightarrow \overline{v_y} = \overline{v}_y = 0, \Rightarrow \overline{v} = 0. \tag{3}$$

$$\tilde{u}_x + \tilde{v}_y = 0.$$

Therefore:

$$\overline{u_t + uu_x + vu_y - f(y)v + \frac{1}{\rho}p_x} = 0 \Rightarrow$$

$$\overline{u}_t + \overline{\tilde{v}\tilde{u}_y} = \overline{u}_t + \overline{(\tilde{v}\tilde{u})_y} = 0. \tag{4}$$

Physical meaning: change of the mean zonal momentum $\leftrightarrow$ momentum flux of the fluctuations.

Zonal flow + waves of small amplitude

Linearisation:

$\overline{u} = U(y)$, fluctuations: $||\tilde{u}, \tilde{v}|| << 1$.

Absolute (potential) vorticity:

$$q = v_x - u_y + f(y) \equiv Q(y) + \tilde{q}. \tag{5}$$

$$Q(y) = f_0 + \beta y - U'(y), \quad \tilde{q} = \tilde{v}_x - \tilde{u}_y. \tag{6}$$

Dynamics of the mean flow:

$$\overline{u}_t = -\overline{(\tilde{v}\tilde{u})_y} = \overline{(\tilde{q}\tilde{v})} \tag{7}$$

Change of zonal flow $\leftrightarrow$ potential vorticity flux

Pseudomomentum

Linearised vorticity equation :

$$\frac{D\tilde{q}}{Dt} + \tilde{v}Q'(y) = 0, \quad \frac{D}{Dt} = \partial_t + U(y)\partial_x. \tag{8}$$

Observation:

$$\overline{-\frac{\tilde{q}}{Q'(y)}\left[\tilde{q}_t + U(y)\tilde{q}_x + \tilde{v}Q'(y)\right]} = 0, \Rightarrow \tag{9}$$

Pseudomomentum:

$$\left[-\frac{\overline{\tilde{q}^2}}{2Q'(y)}\right]_t - \overline{\tilde{q}\tilde{v}} = 0. \tag{10}$$

pseudomomentum: $p(y,t) \equiv -\frac{\overline{\tilde{q}^2}}{2Q'(y)} \Rightarrow$ zonal flow:

$$\bar{u}_t = p_t + \ldots. \tag{11}$$

Alternative formulation:

$$\tilde{v} = \frac{D\tilde{Y}}{Dt}, \Rightarrow \tag{12}$$

$$\frac{D\tilde{q}}{Dt} + \frac{D\tilde{Y}}{Dt} Q'(y) = 0, \leftrightarrow \frac{D}{Dt}\left[\tilde{q} + \tilde{Y} Q'(y)\right] = 0, \Rightarrow \tag{13}$$

$$\tilde{Y} = -\frac{\tilde{q}}{Q'(y)} + \text{const}, \Rightarrow \tag{14}$$

$$p = -Q'(y)\frac{\overline{\tilde{Y}^2}}{2} = \frac{\overline{\tilde{Y}\tilde{q}}}{2} \tag{15}$$

Rossby waves over a mean current

Linear waves, dispersion:

Wave solutions $\propto e^{i(kx+ly-\omega t)}$. Absolute frequency:

$$\omega = Uk - \beta \frac{k}{k^2 + l^2}. \tag{16}$$

Intrinsic frequency: $\hat{\omega} = -\beta \frac{k}{k^2+l^2}$. Phase velocity: $\vec{c} = \frac{\omega}{k^2+l^2}\vec{k}$. Group velocity:

$$\vec{c}_g = (u_g, v_g) = (\partial_k \omega, \partial_l \omega) = \left(U + \beta \frac{k^2 - l^2}{\vec{k}^4}, 2\beta \frac{kl}{\vec{k}^4} \right). \tag{17}$$

Momentum flux produced by waves:

$$\overline{\tilde{u}\tilde{v}} = -\frac{l}{k}\overline{\tilde{v}^2} \equiv -\frac{kl}{k^2}, \Rightarrow$$

$$-\text{sign}\left(\overline{\tilde{u}\tilde{v}}\right) = \text{sign}\, kl = \text{sign}\, v_g.$$

Pseudomomentum:

$$U(y) = \text{const} \Rightarrow \beta > 0, \Rightarrow p = -\beta \frac{\overline{\tilde{Y}^2}}{2} \leq 0.$$

Physical meaning of pseudomomentum for a wave system:

Mean energy of a wave sysyem:

$$\overline{E} = \frac{1}{2}\left(\overline{\tilde{u}^2 + \tilde{v}^2}\right) = \frac{\hat{\omega}}{k} p, \leftrightarrow p = \frac{k}{\hat{\omega}} \overline{E}, \tag{18}$$

because $\tilde{u} = -\frac{l}{k}\tilde{v}$ and $-\hat{\omega}^2 \overline{\tilde{Y}^2} = \overline{\tilde{v}^2}$.

Conservation of pseudomomentum:

$$\overline{\tilde{u}\tilde{v}} = -\frac{l}{k}\overline{\tilde{v}^2} = +\frac{l\hat{\omega}^2}{k}\overline{\tilde{Y}^2} = p v_g, \Rightarrow \tag{19}$$

$$p_t + \left(p v_g\right)_y = 0. \tag{20}$$

Generation of Rossby waves by undulating boundary

Ondulating boundary

Channel with a constant mean flow and undulating boundary:

$$-D + h_0 \cos k_0 x \leq y \leq +D, \quad |h_0| << 1, \; k_0 > 0. \Rightarrow \quad (21)$$

Linearised b.c.:

$$\tilde{v}\big|_{y=-D} = \frac{D(h_0 \cos kx)}{Dt} = U \left(h_0 \cos k_0 x\right)_x \quad (22)$$

$$\leftrightarrow \quad \tilde{Y}\Big|_{y=-D} = h_0 \cos k_0 x.$$

Emitted waves:

$k = k_0, \omega = Uk_0 + \hat{\omega}$ - fixed by the b. c. $\Rightarrow$

$$\hat{\omega} = -Uk_0, \Rightarrow U = \frac{\beta}{k_0^2 + l^2}, \Rightarrow l^2 = \frac{\beta}{U} - k_0^2. \tag{23}$$

- Propagative waves (Charney - Drazin):

$$l^2 > 0, \Rightarrow 0 < U < \frac{\beta}{k_0^2} \Rightarrow \tag{24}$$

- Evanescent (trapped) waves:

$$l^2 < 0.$$

Propagation:

Out of the source (to the North):

$$v_g > 0, \Rightarrow l > 0 (k_0 > 0) \Rightarrow l = \sqrt{\frac{\beta}{U} - k_0^2}$$

Charney - Drazin theorem

Pseudomomentum:

$$\tilde{Y} = h_0 \cos(k_0 x + ly), \Rightarrow p = -\frac{\beta}{4} h_0^2 = \text{const} \qquad (25)$$

Stationary vs non-stationary waves:

- Stationary configuration:

$$p = \text{const} \Rightarrow p_t = 0 \Rightarrow \overline{u}_t = 0.$$

  stationary waves can not change the mean flow (Charney-Drazin)

- Non-stationary configuration : Example: h_0 increases from zero to $h_0 \Rightarrow$ net result:

$$\overline{u} = U + p = U - \frac{\beta}{4} h_0^2 \rightarrow$$

  Deceleration

Critical levels

Mean flow with a shear:

$U = U(y)$, critical level: if $\exists y_c : \; U(y_c) = 0$.

Stationary perturbations: (for ex. generated by ondulating boundary)

Streamfunction of perturbations $\tilde{\psi} = \hat{\psi}(y) e^{ikx}$, $\Rightarrow$ potential vorticity:

$$\tilde{q} = \left(\hat{\psi}''(y) - k^2 \hat{\psi}(y) \right) e^{ikx}.$$

Vorticity equation:

$$U(y)\hat{\psi}''(y) + \left[\beta - U''(y) - k^2 U(y) \right] \hat{\psi}(y) = 0 \qquad (26)$$

Ray tracing and critical levels

WKB method:

Solution is represented as:

$$\tilde{\psi} = \hat{\psi}(y) e^{ikx} e^{i \int_{-D}^{y} dy' \, l(y')}, \tag{27}$$

with

$$l(y) = \sqrt{\frac{\beta}{U} - k^2} \tag{28}$$

- slow (adiabatic) change of wave parametres.

Condition of applicability of the method: $\left(\frac{U'(y)}{U(y)}\right)^2 << l^2$.

Critical level:

$$y \to y_c, \Rightarrow U(y) \to 0, \; l \to \infty.$$

In the vicinity of the critical level:

$$U(y) \propto y_c - y, \Rightarrow l(y) \propto (y_c - y)^{-\frac{1}{2}}.$$

Hence:

$$v_g = 2\beta \frac{kl}{(k^2 + l^2)^2} \approx l^{-3} \propto (y_c - y)^{\frac{3}{2}} \to 0$$

wave-front stops.
Estimate of approach time:

$$t = \int dt' = \int \frac{dy'}{v_g(y')} \propto \int \frac{dy'}{(y_c - y')^{\frac{3}{2}}} \propto (y_c - y)^{-\frac{1}{2}} \to \infty \tag{29}$$

- accumulation below $y_c \Rightarrow$ critical layer.

Breaking near the critical level:

Potential vorticity:

$$q = f_0 + \beta y + \beta \tilde{Y} + ... \Rightarrow$$

$$q_y = \beta \left(1 + \tilde{Y}_y\right)$$

- Quasi-linear waves: $\left|\tilde{Y}_y\right| << 1$,
- Breaking: $\left|\tilde{Y}_y\right| >> 1$.

Estimate: $\overline{\tilde{Y}_y^2} \approx l^2 \overline{\tilde{Y}^2} \approx l^2 p$. Conservation of pseudomomentum for stationary waves:

$$\left(p v_g\right)_y = 0 \Rightarrow p(y) \propto \frac{1}{v_g} \Rightarrow \overline{\tilde{Y}_y^2} \propto \frac{l^2}{v_g} \propto l^5 \propto (y_c - y)^{-\frac{5}{2}} \Rightarrow$$

Breaking approaching y_c

Exercise

System: stratified fluid with Brunt - Väisälä frequency $N = const$, in the vertical plane $x - z$ without rotation in Boussinesq approximation.

- Consider internal gravity waves over a mean flow $\vec{U} = U\hat{\mathbf{x}}$, $U =$ const. Determine absolute and intrinsic frequencies and group velocity,
- Introduce the pseudomomentum $p = -\overline{\tilde{Z}\,(\tilde{u}_z - \tilde{w}_x)}$, where u, w - horizontal and vertical velocity components , $\tilde{Z}$ - vertical displacements of fluid parcels, $\overline{...}$ - horizontal mean, and demonstrate that $p_t + \overline{\tilde{u}\tilde{w}}_z = 0$,
- Demonstrate that $p = k\frac{E}{\hat{\omega}}$, where k is the horizontal wavenumber, $E = \frac{1}{2}\overline{\tilde{u}^2 + \tilde{w}^2 + \frac{\tilde{\sigma}^2}{N^2}}$ - wave energy. Demonstrate that the pseudomomentum flux is $pw_g = \overline{\tilde{u}\tilde{w}}$,

Exercices

- Demonstrate that $\overline{u}_t = \overline{(\tilde{u}\tilde{w})}_z$, and that for the waves of small amplitude $\overline{u}_t = p_t + ...$,
- Consider mountain waves produced by the relief $h(x) = H_0 \cos k_0 x$. Demonstrate that the propagative waves (in vertical direction) exist if $0 < U^2 < \frac{N^2}{k_0^2}$. Determine corresponding w_g and p,
- Calculate the mountain drag $D = -\overline{(\tilde{u}\tilde{w})}$.