Wave turbulence with applications to atmospheric and oceanic waves

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- Lectures 1/2. Basic ideas and methods of wave turbulence. Example: surface waves.
- Lecture 3. Weak turbulence of waves in the rotating shallow water model: inertia-gravity waves, waves in the equatorial wave-guide, Rossby waves.
- Lecture 4. Weak turbulence of internal gravity waves in a stratified fluid.

1 Lecture 1. Basic ideas and methods of wave (weak) turbulence. Example: surface waves.

1.1 The main hypotheses and ideas of the wave turbulence approach

Consider *ensemble* of *large number* of *weakly nonlinear* (i.e. small-amplitude) *harmonic* waves taken in the complex form:

$$a_{\mathbf{k}}^{(0)} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega(\mathbf{k})t)}. \tag{1}$$

The waves are solutions of the *linearized* dynamical equations describing some non-dissipative medium which occupies the whole space (no boundaries).

The properties of the medium determine the *dispersion relation*: dependence of wave-frequencies ω on wave-numbers $\mathbf{k}$: $\omega = \omega(\mathbf{k})$. This equation provides a *spectrum* of infinithesimal excitations of the medium. The equations (the model) of the medium are, generally, *nonlinear* and nonlinear terms engender *interactions* of harmonic waves. The nonlinearities of the system may be quadratic, cubic, etc in wave- amplitudes. In the *weak turbulence* approach the wave-amplitudes $a_{\mathbf{k}}$ are always supposed to be small, and hence only lower-order nonlinearities are taken into account. Nonlinear effects are considered as *small perturbations* to the linear wave field. The supposed weakness of wave interactions is essential.

If the dispersion relation (spectrum) is of *decay* type, i.e. the following equation:

$$\omega(\mathbf{k}_1 + \mathbf{k}_2) = \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) \tag{2}$$

has (non-zero) solutions, then it is sufficient to take into account only the lowest-order interactions. Otherwise (spectrum of *non-decay* type), the next-order interactions should be retained.

A systematic way to realize the weak turbulence approach is through *Hamiltinian description* of the medium. If all physical fields are Fourier-transformed in space, the linear equations for their Fourier-transforms $a_{\mathbf{k}}$ giving harmonic waves (1) are:

$$\dot{a}_{\mathbf{k}} + i\omega(\mathbf{k})a_{\mathbf{k}} = 0; \quad a_{\mathbf{k}} = a_{\mathbf{k}}^{(0)} e^{-i\omega(\mathbf{k})t}. \tag{3}$$

Here and below dot denotes time-derivative. These equations are Hamiltonian:

$$\dot{a}_{\mathbf{k}} = -i\frac{\delta H_0}{\delta a_{\mathbf{k}}^*}, \tag{4}$$

with

$$H_0 = \int d\mathbf{k}\omega(\mathbf{k})\, a_{\mathbf{k}} a_{\mathbf{k}}^*. \tag{5}$$

The Hamiltonian of the free waves H_0 is the energy of an ensemble of non-interacting harmonic oscillators with frequencies $\omega(\mathbf{k})$.

We limit ourselves in this section by a pair of real functions describing the medium and, hence, by a single complex amplitude $a_{\mathbf{k}}$. For instance, for surface waves in the fluid the two real functions are the free surface displacement and the value of the velocity potential at the surface. If more fields are necessary to describe a medium, a number of complex Fourier-amplitude should be introduced, with obvious modifications below.

The full Hamiltonian contains an interaction term: $H = H_0 + H_{int}$ with H_{int} which may be expanded in powers of $a_{\mathbf{k}}$ and its complex conjugate $a^*_{\mathbf{k}}$: $H_{int} = H_3 + H_4 + ...$ with (for stable homogeneous medium):

$$H_3 = \frac{1}{2} \int d\mathbf{k} d\mathbf{k}_1 d\mathbf{k}_2 \, \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) \, V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} a^*_{\mathbf{k}} a_{\mathbf{k}_1} a_{\mathbf{k}_2} + c.c. \, , \qquad (6)$$

$$H_4 = \frac{1}{2} \int d\mathbf{k} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \, \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \, W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} a^*_{\mathbf{k}} a^*_{\mathbf{k}_1} a_{\mathbf{k}_2} a_{\mathbf{k}_3} . \qquad (7)$$

The full equations of motion are, correspondingly,

$$\dot{a}_{\mathbf{k}} = -i\omega(\mathbf{k})a_{\mathbf{k}} - i\frac{\delta H_{int}}{\delta a^*_{\mathbf{k}}}. \tag{8}$$

Remember that amplitudes are always supposed to be small and, hence, the interaction Hamiltonian gives only small corrections to the linear solutions.

The main hypothesis of the weak turbulence is that weak interactions of a large number of waves lead to *phase randomization* (a central limit theorem-like argument) and *Gaussian statistics* of the wave field. As usual in statistical description, *ergodic hypothesis* is supposed to be valid. From the full dynamical description (8) one passes then to the *statistical* description, where the system is described by a set of correlation functions of complex amplitudes. Ensemble averaging will be denoted from now on by $\langle ... \rangle$.

Gaussianity for spatially uniform medium means that all *odd-order correlators vanish* and that all even-order correlators are expressed in terms of the (real) quadratic one:

$$\begin{aligned}
\langle a_{\mathbf{k}} a_{\mathbf{k}'} \rangle &= 0, \\
\langle a_{\mathbf{k}} a^*_{\mathbf{k}'} \rangle &= N(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}'), \\
\langle a^*_{\mathbf{k}_1} a^*_{\mathbf{k}_2} a_{\mathbf{k}_3} a_{\mathbf{k}_4} \rangle &= N(\mathbf{k}_1)N(\mathbf{k}_2)\left(\delta(\mathbf{k}_1-\mathbf{k}_3)\delta(\mathbf{k}_2-\mathbf{k}_4)\right. \\
&+ \left.\delta(\mathbf{k}_1-\mathbf{k}_4)\delta(\mathbf{k}_2-\mathbf{k}_3)\right). \qquad (9)
\end{aligned}$$

The first line follows from the hypothesis of phase randomization.

As all correlators are expressed in terms of the quadratic one, i. e. in terms of $N(\mathbf{k})$, the main goal of the statistical theory is to determine the evolution of this quantity in time which is given by *kinetic equation*:

$$\dot{N}(\mathbf{k}) = \mathcal{I}\left[N(\mathbf{k})\right], \tag{10}$$

where expression in the r.h.s. is called *collision integral.*

1.2 Kinetic equations for decay and non-decay dispersion laws

A quantum-mechanical analogy allows to establish heuristically such equation both for decay and non-decay spectra (for direct derivation see Section 4). Indeed, $N(\mathbf{k})$ is the distribution function for wave amplitudes in the phase-space, and may be interpreted as density of some quasi-particles in the phase-space. The changes of this density are due to the difference between inflow and outflow of quasi-particles in the element of the phase-space. The complex wave amplitudes $a_{\mathbf{k}}$, $a^*_{\mathbf{k}}$ are interpreted as annihilation and creation operators of quasi-particles, respectively.

The quantum-mechanical probability of transition between two given states ($i \to f$) is proportional to the matrix element of the interaction Hamiltonian between these states times delta-function corresponding to energy and other integrals of motion:

$$W = \frac{2\pi}{\hbar} \left|(f|H_{int}|i)\right|^2 \delta(E_f - E_i). \qquad (11)$$

(The Planck constant may be absorbed in what follows by proper renormalization of wave-amplitudes and frequencies.) The matrix elements of creation and annihilation operators are:

$$(N-1|a|N) = \sqrt{N}; \;\; (N+1|a|N) = \sqrt{N+1}. \qquad (12)$$

Take first a **decay spectrum case**, i.e. a cubic interaction Hamiltonian. Then the term $V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} a^*_{\mathbf{k}} a_{\mathbf{k}_1} a_{\mathbf{k}_2}$ correspond to the process of annihillation of two quasi-particles with momenta $\mathbf{k}_1$, $\mathbf{k}_2$, and creation of quasi-particle with momentum $\mathbf{k}$. Hence, this term will give an inflow probability $2\pi \left|V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}\right|^2 (N(\mathbf{k})+1)N(\mathbf{k}_1)N(\mathbf{k}_2)$. Its complex conjugate in H_3 (cf. (6)) will give an outflow term with a factor $N(\mathbf{k})(N(\mathbf{k}_1)+1)(N(\mathbf{k}_2)+1)$ with the same conservation laws and the same factor $|V|^2$. Hence, the occupation numbers N enter the quantum collision integral in the following combination:

$$f^{QM}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = (N(\mathbf{k})+1)N(\mathbf{k}_1)N(\mathbf{k}_2) - N(\mathbf{k})(N(\mathbf{k}_1)+1)(N(\mathbf{k}_2)+1) \quad (13)$$

The classical limit corresponds to $N \gg 1$, and the classical analog of (13) is:

$$f_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = N(\mathbf{k}_1)N(\mathbf{k}_2) - N(\mathbf{k})\left(N(\mathbf{k}_1) + N(\mathbf{k}_2)\right). \tag{14}$$

To get the full collision integral, the terms obtained by cyclic permutation of wave-vectors, with corresponding changes of sign as inflow and outflow change roles, should be added and we get:

$$\mathcal{I}^{(3)}\left[N(\mathbf{k})\right] = \int d\mathbf{k}_1 d\mathbf{k}_2 \left[W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} f_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} - W_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}} f_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}} - W_{\mathbf{k}_2\mathbf{k}\mathbf{k}_1} f_{\mathbf{k}_2\mathbf{k}\mathbf{k}_1}\right], \tag{15}$$

where

$$W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = 2\pi \left|V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}\right|^2 \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\delta(\omega(\mathbf{k}) - \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2)). \tag{16}$$

For the **non-decay spectrum case**, the interaction Hamiltonian, in general, contains both H_3 and H_4 contribution. The cubic terms in the Hamiltonian may be removed by the following canonical transformation:

$$\begin{aligned} a_{\mathbf{k}} \quad \rightarrow \quad & a_{\mathbf{k}} - \int d\mathbf{k}_1 d\mathbf{k}_2 \, \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) \frac{V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}}{\omega(\mathbf{k}) - \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2)} \, a_{\mathbf{k}_1} a_{\mathbf{k}_2} \\ & -2 \int d\mathbf{k}_1 d\mathbf{k}_2 \, \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2) \frac{V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}}{-\omega(\mathbf{k}) + \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2)} \, a^*_{\mathbf{k}_1} a_{\mathbf{k}_2} . \end{aligned} \tag{17}$$

The normal form of quartic Hamiltonian thus follows:

$$\begin{aligned} H \quad = \quad & \int d\mathbf{k} \omega(\mathbf{k}) \, a_{\mathbf{k}} a^*_{\mathbf{k}} \\ + \quad \frac{1}{2} \quad & \int d\mathbf{k} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \, \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \, T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \, a^*_{\mathbf{k}} a^*_{\mathbf{k}_1} a_{\mathbf{k}_2} a_{\mathbf{k}_3} , \end{aligned} \tag{18}$$

where

$$
\begin{aligned}
T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \quad = \quad & -2\frac{V_{\mathbf{k}+\mathbf{k}_1,\mathbf{k},\mathbf{k}_1} V^*_{\mathbf{k}_2+\mathbf{k}_3,\mathbf{k}_2,\mathbf{k}_3}}{\omega(\mathbf{k}+\mathbf{k}_1)-\omega(\mathbf{k})-\omega(\mathbf{k}_1)} \\
& -2\frac{V_{\mathbf{k},\mathbf{k}_2,\mathbf{k}-\mathbf{k}_2} V^*_{\mathbf{k}_3,\mathbf{k}_1,\mathbf{k}_3-\mathbf{k}_1}}{\omega(\mathbf{k}_3-\mathbf{k}_1)+\omega(\mathbf{k}_1)-\omega(\mathbf{k}_3)} \\
& -2\frac{V_{\mathbf{k},\mathbf{k}_3,\mathbf{k}-\mathbf{k}_3} V^*_{\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2-\mathbf{k}_1}}{\omega(\mathbf{k}_2-\mathbf{k}_1)+\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)} \\
& -2\frac{V_{\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_1-\mathbf{k}_3} V^*_{\mathbf{k}_2,\mathbf{k},\mathbf{k}_2-\mathbf{k}}}{\omega(\mathbf{k}_3-\mathbf{k}_1)+\omega(\mathbf{k}_1)-\omega(\mathbf{k}_3)} \\
& -2\frac{V_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1-\mathbf{k}_2} V^*_{\mathbf{k}_3,\mathbf{k},\mathbf{k}_3-\mathbf{k}}}{\omega(\mathbf{k}_2-\mathbf{k}_1)+\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)} \\
& +W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}. \qquad (19)
\end{aligned}
$$

In order to reconstruct the collision integral, the quantum analogy, again, works. Instead of $2 \to 1$ fusion and $1 \to 2$ decay processes of quasi-particles, only $2 \to 2$ processes are allowed by the Hamiltonian (18), either direct ones, given by the W - term in (19), or two-stage ones, given by VV^* - terms. By associating a factor $N+1$ to created and a factor N to annihilated in this process particles we get a counterpart of the quantum factor (13):

$$\begin{aligned} f^{QM}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} &= (N(\mathbf{k})+1)N(\mathbf{k}_1)+1)N(\mathbf{k}_2)N(\mathbf{k}_3) \\ &- N(\mathbf{k})N(\mathbf{k}_1)(N(\mathbf{k}_2)+1)(N(\mathbf{k}_3)+1) \end{aligned} \tag{20}$$

which gives a classical one at $N \gg 1$:

$$\begin{aligned} f_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} &= N(\mathbf{k}_1)N(\mathbf{k}_2)N(\mathbf{k}_3) + N(\mathbf{k})N(\mathbf{k}_2)N(\mathbf{k}_3) \\ &- N(\mathbf{k})N(\mathbf{k}_1)N(\mathbf{k}_2) - N(\mathbf{k})N(\mathbf{k}_1)N(\mathbf{k}_3). \end{aligned} \tag{21}$$

Hence, the collision integral is:

$$\mathcal{I}^{(4)}\left[N(\mathbf{k})\right] = \pi \int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \, W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \, f_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}, \tag{22}$$

where

$$W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} = \left|T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\right|^2 \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\delta(\omega(\mathbf{k})+\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)-\omega(\mathbf{k}_3)) \tag{23}$$

1.3 Exact solutions of kinetic equations

The beauty of the weak turbulence theory is that for wide classes of dispersion laws it allows to obtain *exact* stationary solutions of the kinetic equations in the form of power-law Kolmogorov-like distributions. Let us first consider the **decay dispersion** law.

The first observation is that energy equipartition, i.e. Rayleigh - Jeans distribution

$$N^{RJ}(\mathbf{k}) \propto \omega(\mathbf{k})^{-1}, \tag{24}$$

is always solution. It is sufficient to rewrite $f_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2}$ in the form:

$$f_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2} = N(\mathbf{k})N(\mathbf{k}_1)N(\mathbf{k}_2)\left(N(\mathbf{k})^{-1} - N(\mathbf{k}_1)^{-1} - N(\mathbf{k}_2)^{-1}\right), \tag{25}$$

to see that due to the delta-function in ω in the collision integral $N^{RJ}(\mathbf{k})$ annihilates it as $f_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2}$ becomes proportional to $\omega(\mathbf{k}) - \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2)$.

Let us remind that the Rayleigh - Jeans distribution $N = \frac{T}{\omega}$ corresponds to thermodynamic equilibrium with temperature T. It is equipartition in energy because in zeroth order the mean energy density of each mode is $\epsilon(\mathbf{k}) = \omega(\mathbf{k})N(\mathbf{k})$. A generalization of this solution is a drift Rayleigh - Jeans distribution

$$N^{DRJ}(\mathbf{k}) \propto (\omega(\mathbf{k}) - \mathbf{k} \cdot \mathbf{u})^{-1}, \tag{26}$$

which comes from both energy and momentum conservation. Here $\mathbf{u}$ is a vector proportional to the overall momentum of the system.

However, apart from equilibrium solutions, there are other, nonequilibrium ones with $f_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2} \neq 0$, at least for certain classes of dispersion laws. Let us suppose now that dispersion is isotropic and has the form

$$\omega(\mathbf{k}) = |\mathbf{k}|^{\beta} \tag{27}$$

(this is the case of e.g. capillary waves, see below). It is easy to see that $\beta > 1$ corresponds to a decay dispersion law. The existence of nonequilibrium solutions is based on *symmetries* of the collision integral. Indeed, dispersion law is invariant with respect to rotations in $\mathbf{k}$- space. If we denote by $\hat{r}$ rotation operator, it means that $\omega(\hat{r}\mathbf{k}) = \omega(\mathbf{k})$. The interaction coefficient $V_{\mathbf{k},\mathbf{k}_1\mathbf{k}_2}$ is a function of scalar products of the vectors $\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2$ and, hence is rotation - invariant. The delta-functions are also invariant with respect to rotations. Hence, $W_{\hat{r}\mathbf{k},\hat{r}\mathbf{k}_1\hat{r}\mathbf{k}_2} = W_{\mathbf{k},\mathbf{k}_1\mathbf{k}_2}$.

Another symmetry of the dispersion law (27) is *scale invariance*. We have $\omega(\lambda\mathbf{k}) = \lambda^{\beta}\omega(\mathbf{k})$. Hence, delta function of frequencies scales with a factor $\lambda^{-\beta}$, while delta-function of wave-numbers scales, as usual, as λ^{-d}, where d is space (and $\mathbf{k}$-space) dimension. Normally, the squares of interaction coefficients $|V|^2$ entering the collision integral are also homogeneous functions with index m. Hence

$$W_{\lambda\mathbf{k},\lambda\mathbf{k}_1\lambda\mathbf{k}_2} = \lambda^{m-\beta-d} W_{\mathbf{k},\mathbf{k}_1\mathbf{k}_2}. \tag{28}$$

The triads of wave-vectors entering each of three terms in the collision integral (15) form three triangles with one of the sides fixed by external argument $\mathbf{k}$ ($\mathbf{k}_1, \mathbf{k}_2$ are dumb integration variables). The *crucial observation* is that by appropriate rotations and rescalings the second and the third triangles may be transformed into the first one. However, while integration measure is invariant under rotations, rescalings provide an extra (Jacobian) factor λ^{3d} which will add up to the rescalings of W (cf. (28)).

In this way the collision integral may be brought to the following form:

$$\mathcal{I}^{(3)}\left[N(\mathbf{k})\right] = \int d\mathbf{k}_1 d\mathbf{k}_2\, \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\delta(\omega(\mathbf{k})-\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2))\, W_{\mathbf{k},\mathbf{k}_1\mathbf{k}_2} \cdot \left(f_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2} - \lambda_1^{\alpha} f_{\hat{G}_1^2\mathbf{k}_1,\mathbf{k},\hat{G}_1\mathbf{k}_2} - \lambda_2^{\alpha} f_{\hat{G}_2^2\mathbf{k}_2,\hat{G}_2\mathbf{k}_1,\mathbf{k}} \right). \quad (29)$$

Here the scaling factors are $\lambda_{1,2} = \frac{|\mathbf{k}|}{|\mathbf{k}_{1,2}|}$, and the transformations $\hat{G}_{1,2}$ are defined by rotations and rescalings with $\lambda_{1,2}$ such that $\hat{G}_{1,2}\mathbf{k}_{1,2} = \mathbf{k}$. The scaling factor: $\alpha = m + 2d - \beta$.

If one looks for power-law solutions $N(\mathbf{k}) \propto \omega^s(\mathbf{k})$, then they transform under rescaling as $N(\lambda\mathbf{k}) = \lambda^{\beta s}$, and hence $f_{\lambda\mathbf{k},\lambda\mathbf{k}_1,\lambda\mathbf{k}_2} = \lambda^{2\beta s} f_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2}$. In addition, f is invariant with respect to rotations in this case. Inverting the dispersion relation $|\mathbf{k}| = \omega^{\frac{1}{\beta}}$ and introducing this expression into the definitions of $\lambda_{1,2}$ one reduces the collision integral to the following expresion:

$$\mathcal{I}^{(3)}\left[N(\mathbf{k})\right] = \omega^{\nu}(\mathbf{k}) \int d\mathbf{k}_1 d\mathbf{k}_2\, W_{\mathbf{k},\mathbf{k}_1\mathbf{k}_2}\, f_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2} \cdot \left(\omega^{-\nu}(\mathbf{k}) - \omega^{-\nu}(\mathbf{k}_1) - \omega^{-\nu}(\mathbf{k}_2)\right), \qquad (30)$$

where

$$\nu = 2s - 1 + \frac{2d+m}{\beta}. \qquad (31)$$

Note that

$$f_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2} = \left(\omega(\mathbf{k})\omega(\mathbf{k}_1)\omega(\mathbf{k}_2)\right)^s \left(\omega^{-s}(\mathbf{k}) - \omega^{-s}(\mathbf{k}_1) - \omega^{-s}(\mathbf{k}_2)\right). \qquad (32)$$

There are two solutions for s which annihilate the collision integral and, hence, give stationary distributions. At $s=-1$ $f_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2}$ vanishes: this is an equilibrium solution. But there is another one: $\nu=-1$ and $s=-\frac{2d+m}{2\beta}$ giving

$$N(\mathbf{k}) = |\mathbf{k}|^{-\frac{2d+m}{2}}. \qquad (33)$$

In distinction with the equilibrium spectrum it is non-universal and is determined by the scaling exponent of the interaction coefficients in the Hamiltonian.

The same approach may be applied to the **non-decay case**. The resonant wave *quadrangle* $(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ defining the integration measure in the collision integral (22) may be transformed, with the help of symmetry transformations $\hat{G}_i : \hat{G}_i \mathbf{k}_i = \mathbf{k},\ i = 1, 2, 3$ including rotations and dilatations, into another resonant quadrangle $(\mathbf{k}, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ in three different ways:

$$\mathbf{q}_1 = \hat{G}_1{}^2 \mathbf{k}_1,\ \mathbf{q}_2 = \hat{G}_1 \mathbf{k}_2,\ \mathbf{q}_3 = \hat{G}_1 \mathbf{k}_3; \tag{34}$$

$$\mathbf{q}_1 = \hat{G}_2 \mathbf{k}_3,\ \mathbf{q}_2 = \hat{G}_2{}^2 \mathbf{k}_2,\ \mathbf{q}_3 = \hat{G}_2 \mathbf{k}_1; \tag{35}$$

$$\mathbf{q}_1 = \hat{G}_3 \mathbf{k}_2,\ \mathbf{q}_2 = \hat{G}_3 \mathbf{k}_1,\ \mathbf{q}_3 = \hat{G}_3{}^2 \mathbf{k}_3. \tag{36}$$

For power-law dispersion $\omega(\mathbf{k}) = |\mathbf{k}|^\beta$ the interaction coefficients and delta-function are scale-invariant, similar to the decay-spectrum case and the collision integral may be represented as a sum of four replicas of itself in the form:

$$\mathcal{I}^{(4)}\left[N(\mathbf{k})\right] = \frac{1}{4}\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3\, W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\left(f_{\mathbf{k}} + \lambda_1^\alpha f_{\hat{G}_1\mathbf{k}} + \lambda_2^\alpha f_{\hat{G}_2\mathbf{k}} + \lambda_3^\alpha f_{\hat{G}_3\mathbf{k}}\right), \tag{37}$$

where the scale factors are $\lambda_i = \frac{|\mathbf{k}|}{|\mathbf{k}_i|}$, $i = 1, 2, 3$, and $\alpha = m + 3d - \beta$.

As a result, the collision integral, for isotropic and homogeneous distributions $N(\mathbf{k}) \propto \omega^s(\mathbf{k})$, is factorized:

$$\begin{aligned}\mathcal{I}^{(4)}\left[N(\mathbf{k})\right] &= \frac{\omega^\nu(\mathbf{k})}{4}\int d\mathbf{k}_1 d\mathbf{k}_2\ d\mathbf{k}_3\, W_{\mathbf{k},\mathbf{k}_1\mathbf{k}_2,\mathbf{k}_3}\, f_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}\cdot \\ &\quad \left(\omega^{-\nu}(\mathbf{k})+\omega^{-\nu}(\mathbf{k}_1)-\omega^{-\nu}(\mathbf{k}_2)\omega^{-\nu}(\mathbf{k}_3)\right), \qquad (38)\end{aligned}$$

where

$$\nu = 3s - 1 + \frac{3d+m}{\beta} \qquad (39)$$

and

$$\begin{aligned}f_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} &= \left(\omega(\mathbf{k})\omega(\mathbf{k}_1)\omega(\mathbf{k}_2)\omega(\mathbf{k}_2)\right)^s \\ &\quad \left(\omega^{-s}(\mathbf{k})+\omega^{-s}(\mathbf{k}_1)-\omega^{-s}(\mathbf{k}_2)-\omega^{-s}(\mathbf{k}_2)\right). \qquad (40)\end{aligned}$$

Hence, the collision integral vanishes, i.e. stationary distribution results, either when f vanishes, or the last factor vanishes. The first case corresponds to two limiting cases $\mu \gg \omega$ and $\mu \ll \omega$ of the equilibrium distribution with "chemical potential" μ:

$$N^{RJ}(\mathbf{k}) \propto (\omega(\mathbf{k}) - \mu)^{-1}\,, \tag{41}$$

which, as in the decay case, may be extended to include a drift:

$$N^{DRJ}(\mathbf{k}) \propto (\omega(\mathbf{k}) - \mu - \mathbf{k}\cdot\mathbf{u})^{-1}\,. \tag{42}$$

The appearance of chemical potential is related to additional conservation law of the number of "particles" in non-decay kinetic equations, see below.

The non-equilibrium stationary distributions correspond to $\nu = 0, -1$ and are related to conservation of the number of waves and energy, respectively, in an elementary collision proccess. Correspondingly, $s = \frac{1}{3} - \frac{3d+m}{3\beta}$, and $s = -\frac{3d+m}{3\beta}$ which gives two solutions for N:

$$N(\mathbf{k}) = |\mathbf{k}|^{\frac{\beta - 3d+m}{3}}, \tag{43}$$

$$N(\mathbf{k}) = |\mathbf{k}|^{-\frac{3d+m}{3}} \tag{44}$$

1.4 Conservation laws and dimensional estimates

The *mean energy* of the medium in the lowest order (cf. (5)) is $E = \int d\mathbf{k}\omega(\mathbf{k})N(\mathbf{k})$, whence $\epsilon = \omega(\mathbf{k})N(\mathbf{k})$ is the mean energy density. Multiplying both sides of (10) by $\omega(\mathbf{k})$, integrating by $\mathbf{k}$, and using the symmetry properties of interaction coefficients we get, e.g. in the decay spectrum case:

$$\begin{aligned} \dot{E} &= \int d\mathbf{k}d\mathbf{k}_1 d\mathbf{k}_2 \, (\omega(\mathbf{k}) - \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2)) \cdot \\ &2\pi \quad |V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}|^2 \, \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\delta(\omega(\mathbf{k}) - \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2)) \equiv 0, \end{aligned} \quad (45)$$

and similarly in the decay case. Thus, the total energy is conserved by the kinetic equation. It is clear from the above expression that total energy conservation is ensured by energy conservation in each elementary interaction act.

The conservation law may rewritten in (pseudo-) local form:

$$\frac{\partial \epsilon}{\partial t} + \frac{\partial P}{\partial \mathbf{k}} = 0, \tag{46}$$

where P is (the density of) the energy flux in the **k**- space: $\frac{\partial P}{\partial \mathbf{k}} = -\omega(\mathbf{k})\mathcal{I}\left[N(\mathbf{k})\right]$. The true locality is ensured if integrals defining P are convergent (*locality criterion*).

It may be shown in analogous way that the three components of the *total momentum* of the medium $\mathbf{K} = \int d\mathbf{k}\,\mathbf{k}\,N(\mathbf{k})$ are conserved by the kinetic equation, as they are conserved in each elementary interaction.

Moreover, in the non-decay case the total *wave action*, or the total number of waves (quasi-particles) $N = \int d\mathbf{k} N(\mathbf{k})$ is also conserved, as it is conserved in elementary interactions. It may be again rewritten in the (pseudo-) local form:

$$\frac{\partial N(\mathbf{k})}{\partial t} + \frac{\partial Q}{\partial \mathbf{k}} = 0, \tag{47}$$

where Q is (the density of) the wave-action flux in the $\mathbf{k}$- space: $\frac{\partial Q}{\partial \mathbf{k}} = -\mathcal{I}^{(4)}\left[N(\mathbf{k})\right]$, the locality ensured if integrals converge.

Like the famous Kolmogorov - Obukhov spectrum of (strong) hydrodynamical turbulence, energy spectra corresponding to constant fluxes of conserved quantities may be constructed by dimensional considerations. The difference between isotropic and homogeneous developed turbulence and wave turbulence is that in the last case additional parameter - frequency of the waves which constitute the turbulent field - is present. Hence, pure dimensional considerations are insufficient and some dynamical information should be added. It is provided by the very structure of the kinetic equation with either $\mathcal{I}^{(3)} \propto N^2$, or $\mathcal{I}^{(4)} \propto N^3$. In its turn the collision integral defines the conserved quantity flux, as just explained. The dimensions of N, P, and Q are:

$$[N] = L^5 T^{-1}, \ \ [P] = L^{5-d} T^{-3}, \ \ [Q] = L^{5-d} T^{-2}. \qquad (48)$$

In the decay-spectrum case $N \sim P^{\frac{1}{2}}$ and looking for the spectrum of the form $N = P^{\frac{1}{2}} \omega^a |\mathbf{k}|^b$ we get by comparing dimensions:

$$N = P^{\frac{1}{2}} \omega^{-\frac{1}{2}} |\mathbf{k}|^{-\frac{d+5}{2}}. \tag{49}$$

Under hypothesis of full self-similarity (necessary for applying dimensional arguments), the interaction coefficients should scale as some combination of wavenumber and frequency (no P). Hence $|V_{\mathbf{k}\mathbf{k}_1\ bk_2}|^2 \sim |\mathbf{k}|^{5-d}\omega \Rightarrow m = 5 - d + \beta$ and we thus recover the earlier result (33).

In the non-decay spectrum case $N \sim P^{\frac{1}{3}}$ or $N \sim Q^{\frac{1}{3}}$ and we get a spectrum with constant energy flux:

$$N = P^{\frac{1}{3}} |\mathbf{k}|^{-\frac{d+10}{3}}, \tag{50}$$

and that with constant wave-action flux:

$$N = Q^{\frac{1}{3}} \omega^{\frac{1}{3}} |\mathbf{k}|^{-\frac{d+10}{3}}. \tag{51}$$

In case of full self-similarity these spectra correspond to those found earlier.

1.5 Application to surface waves

Waves on the free surface of the fluid subject to capillary forces are described in terms of the position of fluid surface $\eta(\mathbf{x}, t)$ and the value of the fluid velocity potential $\phi(\mathbf{x}, t)$ (assumption of potential flow is made) at the free surface: $\psi = \phi|_{z=\eta}$. The Hamiltonian of the system is given by energy per unit mass:

$$H = \frac{1}{2}\int d\mathbf{x} \int_{-\infty}^{\eta} dz \; (\nabla\phi)^2 + \frac{g}{2}\int d\mathbf{x}\, \eta^2 + \frac{\alpha}{\rho}\int d\mathbf{x} \left(\sqrt{1+(\nabla\eta)^2} - 1\right). \tag{52}$$

Here g is gravity acceleration, α is the surface tension coefficient, ρ is the fluid density.

Normal variables a, a^* are introduced via the Fourier transforms of η and ψ:

$$\eta(\mathbf{k}) = \frac{1}{2\pi}\sqrt{\frac{|\mathbf{k}|}{2\omega(\mathbf{k})}}\left(a_{\mathbf{k}} + a^*_{-\mathbf{k}}\right), \; \psi(\mathbf{k}) = -\frac{i}{2\pi}\sqrt{\frac{\omega(\mathbf{k})}{2|\mathbf{k}|}}\left(a_{\mathbf{k}} - a^*_{-\mathbf{k}}\right). \tag{53}$$

The dispersion relation for capillary-gravity waves is:

$$\omega(\mathbf{k}) = \sqrt{g|\mathbf{k}| + \frac{\alpha}{\rho}|\mathbf{k}|^3}. \tag{54}$$

The derivative of the dispersion curve changes sign at $k_0 = \sqrt{\frac{\rho\, g}{\alpha}}$. The whole dispersion law is not self-similar, so we will consider separately capillary waves, $k \ll k_0$, and gravity waves (deep fluid), $k \gg k_0$.

For **capillary waves** $\omega(\mathbf{k}) \propto |\mathbf{k}|^{\frac{3}{2}}$, hence $\beta = \frac{3}{2}$ and $d = 2$.
A stationary spectrum thus results:

$$N = P^{\frac{1}{2}} |\mathbf{k}|^{-\frac{17}{4}}. \tag{55}$$

For **gravity waves** $\omega(\mathbf{k}) \propto |\mathbf{k}|^{\frac{1}{2}}$, hence $\beta = \frac{1}{2}$ and $d = 2$.
A stationary spectrum with constant P is

$$N = P^{\frac{1}{3}} |\mathbf{k}|^{-4}, \tag{56}$$

and that with constant Q is

$$N = Q^{\frac{1}{3}} |\mathbf{k}|^{-\frac{23}{6}}. \tag{57}$$

1.6 Historical comments and biblography

The idea of weak turbulence appeared first in the context of plasma physics. The whole group led by Sagdeev who was, probably, the first to promote it, worked on different applications of this idea in early and mid-sixties in Novosibirsk (Galeev and Karpman, 1963, Zaslavsky and Sagdeev, 1967, Sagdeev and Galeev, 1969). Zakharov (1985, and references therein) was first to apply these ideas to hydrodynamical waves. Above-displayed power spectra of capillary-gravity waves were obtained in the pioneering paper by Zakharov and Filonenko, 1966. At the same time K. Hasselmann (1967) developed statistical approach to water waves.

Random-wave closures in the hydrodynamical context were obtained also by Benney and Saffman, 1966, and Benney and Newell, 1969.

Above, we followed heuristic quantum-mechanical short-cut in order to derive the kinetic equation; For straightforward derivation see, e.g. Zakharov, L'vov, Falkovich, 1992. For more systematic derivation via phase averaging and coarse-graining procedure in action-angle variables see Zaslavsky and Sagdeev, 1967.

The presentation above is standard. There is a number of books and reviews on the subject (Kadomtsev and Kontorovich, 1974, Zakharov, 1985, Zakharov, L'vov and Falkovich, 1992. Zakharov (1985) used transformations of frequency variables in the collision integral in order to obtain exact stationary solutions. We followed above an alternative approach developed by Kats and Kontorovich, 1973, 1974, which is also applicable for non strictly self-similar distributions (see below). For discussion of applicability criteria of kinetic equations, see Zakharov, L'vov and Falkovich, 1992, as well as for extensive discussion of stability and locality of exact power-law solutions.

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2 Weak turbulence of waves in the rotating shallow water model

2.1 Rotating shallow water model (RSW), a reminder

Physical meaning: horizontal momentum and mass conservation for shallow water layer in hydrostatic approximation on the tangent plane to a rotating planet. Rotation: Coriolis force. Centrifugal force included in effective gravity g. Stratification is rudimentary: dynamics of a single isopycnal surface $z = h(x, y, t)$, the free surface. Fluid at rest $h = h_0$. Equations of motion (non-dissipative):

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + f(y)\hat{\mathbf{z}} \wedge \mathbf{v} + g\nabla h = 0\,, \tag{58}$$

$$\partial_t h + \nabla \cdot (\mathbf{v}h) = 0\,, \tag{59}$$

Coriolis parameter: $f(y)$ (the meridional dependence of the normal component of the planet's angular velocity is the only remnant of the planet's sphericity).

- Mid-latiude tangent plane: $f = f_0 + \beta y$.
- Equatorial tangent plane: $f = \beta y$.

Sphericity neglected: f - plane approximation, $f = f_0 = const.$

Key quantity: potential vorticity (PV):

$$q = \frac{v_x - u_y + f(y)}{h}, \tag{60}$$

Here $v_x - u_y$ is relative vorticity, $v_x - u_y + f(y)$ is total (relative plus planetary $f(y)$) vorticity. PV is a Lagrangian invariant:

$$\frac{dq}{dt} = 0, \quad \frac{d}{dt} = \partial_t + \mathbf{v} \cdot \nabla. \tag{61}$$

Waves in the RSW are obtained by linearization. Linearization of (58) about the rest state on the f - *plane* results in a system of linear PDE with constant coefficients. looking for solution $\propto e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ results in the dispersion relation:

$$\omega\left(\omega^2 - gh_0\mathbf{k}^2 - f^2\right) = 0. \tag{62}$$

The root $\omega = 0$ corresponds to linearized PV - *slow variable*. Two other roots correspond to *fast variables*, surface inertia-gravity waves (SIGW). Specifics of SIGW: 1) spatial isotropy, 2) spectral gap (no waves with $\omega < f$, 3) absence of dispersion for short waves ($k \gg 1$).

The presence of both slow and fast variables in the system is typical for ocean and atmosphere motions.

For small *Rossby numbers* $Ro = \frac{U}{fL}$, where U and L are typical velocity and horizontal scale of the motion, slow and fast variables are *dynamically decoupled* and, for not too long times, may be treated independently. A closed equation arises for slow evolution of PV which, for small Ro, may be expressed in terms of h via *geostrophic balance* between the pressure force and the Coriolis force:

$$\partial_t \left(h - \nabla^2 h\right) - \mathcal{J}\left(h, \nabla^2 h\right) = 0. \tag{63}$$

Here the Rossby deformation radius $R_d = \frac{\sqrt{gh_0}}{f^2}$, an intrinsic length-scale present in th system, is taken as a length unit, and the time - scale is L/U in contradistinction with the fast time-scale f^{-1} of SIGW.

There are no waves in (63). They, however, appear if the slow dynamics is considered on the β - *plane*. In this case, supposing non-dimensional β to be of the same order of magnitude as Ro, one obtains instead of (63) the following equation:

$$\partial_t \left(h - \nabla^2 h\right) - \beta\partial_x h - \mathcal{J}\left(h, \nabla^2 h\right) = 0. \tag{64}$$

(we keep β to indicate the origin of the last term). This equation, if linearized, produces *Rossby waves* with dispersion relation:

$$\omega = -\beta \frac{k_x}{\mathbf{k}^2 + 1}, \tag{65}$$

where k_x is the wave vector component in x- direction. The most characteristic features of Rossby waves are: unique sign of phase velocity in the x- direction, strong spatial anisotropy and non-monotonicity of the dispersion curve.

Finally, on the *equatorial β - plane* with boundary conditions of exponential decay far from the equator (equatorial wave-guide) a specific hybrid wave motions appear. They are obtained from linearization of (58) and decomposition of all fields in parabolic cylinder functions of the form

$$\phi_n(y) = \frac{H_n(y)e^{-\frac{y^2}{2}}}{\sqrt{2^n n! \sqrt{\pi}}}, \tag{66}$$

where H_n are Hermite polynomials. They are (in non-dimensional units):

- Kelvin waves with linear dispersion $\omega = k$,
- Yanai waves with the dispersion law $\omega^2 - k\omega - 1 = 0$,
- Rossby and inerta-gravity waves with the dispersion law $\omega^3 - (k^2 + (2n+1))\omega - k = 0$ (lower frequency - Rossby wave, higher frequencies - inertia-gravity waves).

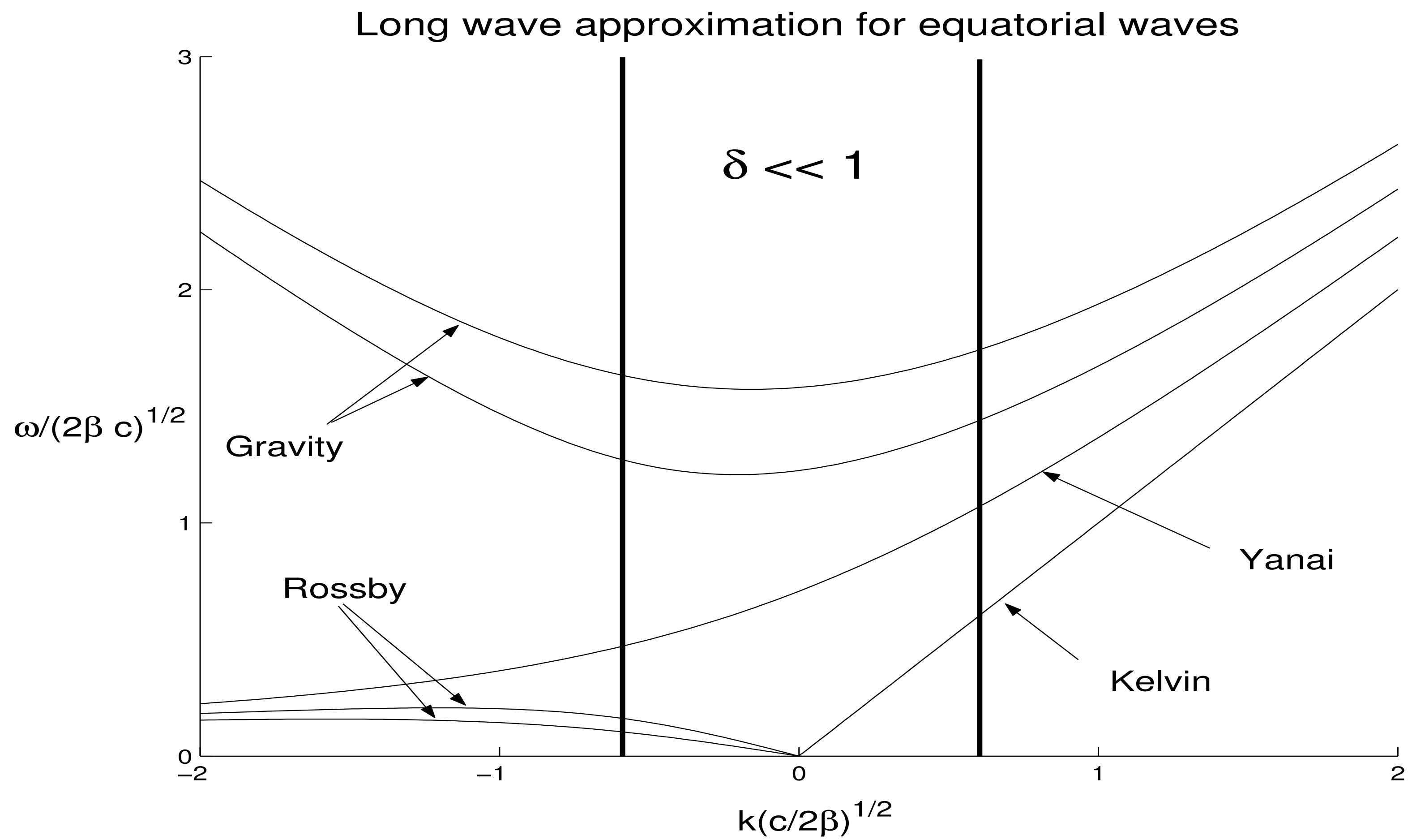

Figure 1: Dispersion curves for equatorial waves.

Thus, fast SIGW, both at mid-latitudes and at equator, are strongly dispersive at long wave-lengths and weakly dispersive at short wave-lengths. Rossby waves are strongly-dispersive and strongly spatially anisotropic. Equatorial Kelvin waves are rigorously non-dispersive, Short Yanai waves join Rossby-waves family for negative k_x and SIGW family for positive k_x (see figure). Below we will show how the weak turbulence approach may be applied to waves in RSW. We will treat each type of waves separately (SIGW on the f- plane, Rossby waves on the β- plane, equatorial SIGW).

2.2 Factorization of the collision integral for almost non-dispersive waves

The dispersion relation (62) is, obviously, not scale invariant, so the standard recipies of the weak turbulence do not apply. However, for dispersion laws of the form:

$$\omega(\mathbf{k}) = c|\mathbf{k}| + \gamma(\mathbf{k}), \tag{67}$$

considered in the domains of $\mathbf{k}$ such that $|\gamma| \ll c|\mathbf{k}|$, the method of conformal transformations of the integration domain in the collision integral may be extended to get, under some additional hypotheses, factorization. Indeed, the main spectrum being of non-decay type, the collision integral is of the form (22). The smallness of dispersion means that wave scattering is dominated by small-angle processes.

For angles between the wave-vectors

$$\theta_i = (\widehat{\mathbf{k}, \mathbf{k}_i}) \leq \sqrt{\frac{\gamma}{c|\mathbf{k}|}} \tag{68}$$

For the four-wave interaction coefficient, cf (19), the following estimate then holds: $T \sim \frac{V^2}{c|\mathbf{k}|(\theta^2 + |\gamma/c|\mathbf{k}||)}$ and everywhere, except for resonance denominators and energy conservation, one may take $\mathbf{k}_i$ parallel to $\mathbf{k}$ (except for a degenarate case when V vanish at such values of $\mathbf{k}_i$).

Introducing the variables $\kappa_i = \frac{\mathbf{k}_{i_\perp}}{|\mathbf{k}_{i_\perp}|}$ the product of δ- functions in the collision integral (22), (23) may be then rewritten as

$$\delta(|\mathbf{k}| + |\mathbf{k}_1| - |\mathbf{k}_2| - |\mathbf{k}_3|)\,\delta(\kappa_1|\mathbf{k}|_1\theta_1 - \kappa_2|\mathbf{k}|_2\theta_2 - \kappa_3|\mathbf{k}|_3\theta_3)$$
$$\delta\left(\gamma + \gamma_1 - \gamma_2 - \gamma_3 + \frac{c}{2}\left(|\mathbf{k}|_1\theta_1^2 - |\mathbf{k}|_2\theta_2^2 - |\mathbf{k}|_3\theta_3^2\right)\right). \qquad (69)$$

Collision integral is not conformal-invariant. Nevertheless, if

$$\gamma(\lambda\mathbf{k}) = \lambda^\beta\gamma(\mathbf{k}),\ V_{\lambda\mathbf{k},\lambda\mathbf{k}_1,\lambda\mathbf{k}_2} = \lambda^\mu V_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2}, \qquad (70)$$

then the collision integral may be factorized using rescaling of angles and conformal transformations

Indeed, in this case the conservation laws and resonant denominators are mapped onto themselves under the transformation $\mathbf{k}_i \to \lambda \mathbf{k}_i$, $\theta_i \to \lambda^{\frac{\beta-1}{2}} \theta_i$. Correspondingly:

$$W_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \to \lambda^w W_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}, \; w = 4\mu - 3\beta - 1 - \frac{1}{2}(\beta+1)(d-1). \quad (71)$$

If one supposes that distribution functions are homogeneous in $|\mathbf{k}|$ (with possible slow angle dependence): $N(\mathbf{k}) \propto |\mathbf{k}|^s$, then the collision integral may be factorized by the following transformations (the scaling factors λ_i are defined as before and non-bold notation is used for wave-vector moduli):

$$k_2' = \lambda_2 k = \lambda_2^2 k_2, \; k_1' = \lambda_2 k_3, \; k_3' = \lambda_2 k_1, \quad (72)$$

$$\theta_2' = -\lambda_2^{\frac{\beta-1}{2}} \theta_2, \; \theta_1' = \lambda_2^{\frac{\beta-1}{2}} (\theta_3 - \theta_2), \; \theta_3' = \lambda_2^{\frac{\beta-1}{2}} (\theta_1 - \theta_2) \quad (73)$$

The collision integral becomes:

$$\mathcal{I}^{(4)}\left[N(\mathbf{k})\right] = \frac{k^{\nu}}{4}\int d\mathbf{k}_1 d\mathbf{k}_2\; d\mathbf{k}_3\, W_{\mathbf{k},\mathbf{k}_1\mathbf{k}_2,\mathbf{k}_3}\; f_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}\;\cdot$$
$$\left(k^{-\nu} + k_1^{-\nu} - k_2^{-\nu} - k_3^{-\nu}\right), \tag{74}$$

where

$$\nu = 3s + w + 4d + \frac{3}{2}(d-1)\,(\beta - 1) \tag{75}$$

Hence, there are two non-equilibrium power-law spectra corresponding to $\nu = 0$ (constant wave- action flux solution), and to $\nu = -1$ (constant energy flux solution).

2.3 Weak turbulence of short SIGW on the f-plane

As was already mentioned, RSW equations describe both slow vortical (zero frequency in the linear approximation) and fast SIGW motions. An invariant criterion of their separation, at least at weak nonlinearities, is provided by PV. Indeed, SIGW do not bear PV - anomaly, i.e. deviation of PV from the background value $\frac{f}{h_0}$. For zero PV-anomaly, i.e. for *constant PV*, the motion is described by a pair of canonical Hamiltonian variables: velocity potential Φ and h. Namely for two components of velocity in this case we have:

$$u = \frac{\partial \Phi}{\partial x} - \frac{f}{h_0}\frac{\partial}{\partial y}\Delta^{-1}(h - h_0), \; v = \frac{\partial \Phi}{\partial y} + \frac{f}{h_0}\frac{\partial}{\partial x}\Delta^{-1}(h - h_0), \quad (76)$$

where Δ^{-1} denotes inverse Laplacian. Thus reduced system is Hamiltonian

$$\dot{h} = \frac{\delta H}{\delta \Phi}, \quad \dot{\Phi} = -\frac{\delta H}{\delta h}, \tag{77}$$

with a Hamiltonian given by the full (kinetic + potential) energy:

$$H = \frac{1}{2} \int dx\, dy \; \left[h(u^2 + v^2) + g h^2 \right]. \tag{78}$$

Introducing normal wave amplitudes $b_{\mathbf{k}}$ such that

$$h_{\mathbf{k}} = \sqrt{\frac{k^2 h_0}{2\omega(\mathbf{k})}}(b^*_{-\mathbf{k}} - b_{-\mathbf{k}}), \quad \Phi_{\mathbf{k}} = \sqrt{\frac{\omega(\mathbf{k})}{2k^2 h_0}}(b^*_{-\mathbf{k}} - b_{-\mathbf{k}}), \tag{79}$$

we get the interaction Hamiltonian of the form:

$$\begin{aligned} H_3 &= \frac{1}{2}\int d\mathbf{k}d\mathbf{k}_1 d\mathbf{k}_2 \, V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} b^*_{\mathbf{k}} b_{\mathbf{k}_1} b_{\mathbf{k}_2} + c.c. \\ &+ \frac{1}{3}\int d\mathbf{k}d\mathbf{k}_1 d\mathbf{k}_2 \, U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} b_{\mathbf{k}} b_{\mathbf{k}_1} b_{\mathbf{k}_2} + c.c. \end{aligned} \tag{80}$$

The interaction coefficients are:

$$V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = 2U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = \sqrt{18g}\frac{\mathbf{k}_2 \cdot \mathbf{k}_3 \, \omega(\mathbf{k}_1) + \mathbf{k}_1 \cdot \mathbf{k}_3 \, \omega(\mathbf{k}_2) + \mathbf{k}_1 \cdot \mathbf{k}_2 \, \omega(\mathbf{k}_3)}{\sqrt{\omega(\mathbf{k}_1)\omega(\mathbf{k}_2)\omega(\mathbf{k}_3)}}. \tag{81}$$

The presence of "vacuum-to-three-particles" decay terms with coefficients U in (80) is not dangerous as *there are no three-wave resonances in RSW on the f- plane*. By this reason further canonical transformation

$$
\begin{aligned}
b_{\mathbf{k}} \quad \rightarrow \quad & b_{\mathbf{k}} - \int d\mathbf{k}_1 d\mathbf{k}_2\, \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2) \frac{V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}}{\omega(\mathbf{k})-\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)}\, b_{\mathbf{k}_1} b_{\mathbf{k}_2} \\
& -2\int d\mathbf{k}_1 d\mathbf{k}_2\, \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2) \frac{V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}}{-\omega(\mathbf{k})+\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)}\, b^*_{\mathbf{k}_1} b_{\mathbf{k}_2} \\
& -\int d\mathbf{k}_1 d\mathbf{k}_2\, \delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2) \frac{U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}}{\omega(\mathbf{k})+\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)}\, b_{\mathbf{k}_1} b_{\mathbf{k}_2} .
\end{aligned}
\tag{82}
$$

is necessary to bring the Hamiltonian to the normal form:

$$\begin{aligned} H &= \int d\mathbf{k}\omega(\mathbf{k})\, a_{\mathbf{k}} a^*_{\mathbf{k}} \\ &+ \frac{1}{2} \int d\mathbf{k} d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3\, \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\, T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\, a^*_{\mathbf{k}} a^*_{\mathbf{k}_1} a_{\mathbf{k}_2} a_{\mathbf{k}_3}, \end{aligned} \tag{83}$$

There was no generality loss up to now. However, the dispersion law (62) is not scale-invariant and finding exact power-law solutions of the four-wave kinetic equation is therefore impossible. We consider, hence, the short SIGW with $|\mathbf{k}| \to \infty$ and retain only the leading term in the expansion of the dispersion law in powers of $|\mathbf{k}|^{-1}$:

$$\omega(\mathbf{k}) = \sqrt{gh_0}|\mathbf{k}| \left(1 + \frac{1}{2\left(|\mathbf{k}|R_d\right)^2}\right). \tag{84}$$

This dispersion law is exactly of the form (67) with scale-invariant addition to the "acoustic" dispersion law $\omega = ck$. Hence, the theory of the previous subsection may be applied, which gives two power-law spectra:

$$N(\mathbf{k}) \propto |\mathbf{k}|^{-\frac{14}{3}}, \;\; N(\mathbf{k}) \propto |\mathbf{k}|^{-\frac{13}{3}}, \tag{85}$$

corresponding to constant energy and wave-action fluxes through the spectrum, respectively.

2.4 Weak turbulence of short SIGW on the equatorial β- plane

The non-dimensional equations for velocities u, v and deviation of the height field from the rest value $z = \frac{h-h_0}{h_0}$ on the equatorial beta - plane are:

$$u_t + uu_x + vu_y - \beta yv + z_x = 0, \tag{86}$$

$$v_t + uv_x + vv_y + \beta yu + z_y = 0, \tag{87}$$

$$z_t + u_x + v_y + (uz)_x + (vz)_y = o. \tag{88}$$

For simplicity we suppose that there is unique small parameter ϵ, and assume a weak nonlinearity $u, v, z \sim \epsilon$ and a weak inhomogeneity $\beta \sim \epsilon$. Smallness of the β - term means, in fact, that we are considering motions with a characteristic scale small with respect to the equatorial deformation radius $R_e = \frac{(gh_0)^{\frac{1}{4}}}{\sqrt{\beta}}$.

Thus the leading- order part of the system (86)-(88) is a system with constant coefficients

$$u_t + z_x = 0, \tag{89}$$

$$v_t + z_y = 0, \tag{90}$$

$$z_t + u_x + v_y = o. \tag{91}$$

Introducing the Fourier - transforms:

$$(u(\mathbf{r}), v(\mathbf{r}), z(\mathbf{r})) = \int (u_{\mathbf{k}}, v_{\mathbf{k}}, z_{\mathbf{k}}) \exp(i\mathbf{kr}) d\mathbf{k}, \qquad (92)$$

where $\mathbf{r} = (x, y)$ and $\mathbf{k} = (k_1, k_2)$, and integration is over the whole plane, we rewrite the equations (86)-(88) for Fourier-transforms in a symmetric form:

$$\begin{pmatrix} \partial u_{\mathbf{k}}/\partial t \\ \partial v_{\mathbf{k}}/\partial t \\ \partial z_{\mathbf{k}}/\partial t \end{pmatrix} + \begin{pmatrix} 0 & -i\beta\frac{\partial}{\partial k_2} & ik_1 \\ i\beta\frac{\partial}{\partial k_2} & 0 & ik_2 \\ ik_1 & ik_2 & 0 \end{pmatrix} \begin{pmatrix} u_{\mathbf{k}} \\ v_{\mathbf{k}} \\ z_{\mathbf{k}} \end{pmatrix} + \qquad (93)$$

$$+ \begin{pmatrix} ik_1/2 \int (u_{\mathbf{l}} u_{\mathbf{m}} + v_{\mathbf{l}} v_{\mathbf{m}}) d\lambda - \int \Omega_{\mathbf{l}} v_{\mathbf{m}} d\lambda \\ ik_2/2 \int (u_{\mathbf{l}} u_{\mathbf{m}} + v_{\mathbf{l}} v_{\mathbf{m}}) d\lambda + \int \Omega_{\mathbf{l}} u_{\mathbf{m}} d\lambda \\ ik_1 \int z_{\mathbf{l}} u_{\mathbf{m}} d\lambda + ik_2 \int z_{\mathbf{l}} v_{\mathbf{m}} d\lambda \end{pmatrix} = 0$$

where $\Omega = v_x - u_y$, $\Omega_{\mathbf{l}} = il_1 v_{\mathbf{l}} - il_2 u_{\mathbf{l}}$, $d\lambda = \delta(\mathbf{k} - \mathbf{l} - \mathbf{m}) d\mathbf{l} d\mathbf{m}$.

We diagonalize the main part by a change of variables

$$\begin{pmatrix} u_{\mathbf{k}} \\ v_{\mathbf{k}} \\ z_{\mathbf{k}} \end{pmatrix} = \begin{pmatrix} \frac{-ik_2}{|\mathbf{k}|} & \frac{k_1}{\sqrt{2}|\mathbf{k}|} & \frac{-k_1}{\sqrt{2}|\mathbf{k}|} \\ \frac{ik_1}{|\mathbf{k}|} & \frac{k_2}{\sqrt{2}|\mathbf{k}|} & \frac{-k_2}{\sqrt{2}|\mathbf{k}|} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} a_{\mathbf{k}} \\ b_{\mathbf{k}} \\ c_{\mathbf{k}} \end{pmatrix}, \tag{94}$$

>From (94) for real-valued u, v, z we have

$$a_{\mathbf{k}} = \bar{a}_{-\mathbf{k}}, \quad c_{\mathbf{k}} = \bar{b}_{-\mathbf{k}} \tag{95}$$

As a result of the diagonalization we have

$$\begin{pmatrix} \partial a_{\mathbf{k}}/\partial t \\ \partial b_{\mathbf{k}}/\partial t \\ \partial c_{\mathbf{k}}/\partial t \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & i \mid \mathbf{k} \mid & 0 \\ 0 & 0 & -i \mid \mathbf{k} \mid \end{pmatrix} \begin{pmatrix} a_{\mathbf{k}} \\ b_{\mathbf{k}} \\ c_{\mathbf{k}} \end{pmatrix} +$$

$$+\beta \begin{pmatrix} -\frac{ik_1}{|\mathbf{k}|^2} & \frac{1}{\sqrt{2}}\frac{\partial}{\partial k_2} & -\frac{1}{\sqrt{2}}\frac{\partial}{\partial k_2} \\ \frac{1}{\sqrt{2}}\frac{\partial}{\partial k_2} & -\frac{ik_1}{2|\mathbf{k}|^2} & \frac{ik_1}{2|\mathbf{k}|^2} \\ -\frac{1}{\sqrt{2}}\frac{\partial}{\partial k_2} & \frac{ik_1}{2|\mathbf{k}|^2} & -\frac{ik_1}{2|\mathbf{k}|^2} \end{pmatrix} \begin{pmatrix} a_{\mathbf{k}} \\ b_{\mathbf{k}} \\ c_{\mathbf{k}} \end{pmatrix} + NL = 0 \quad .(96)$$

Here nonlinear terms NL have the form:

$$\begin{pmatrix} \int (U^{(0)}_{\mathbf{klm}} a_{\mathbf{l}} a_{\mathbf{m}} + U^{(1)}_{\mathbf{klm}} a_{\mathbf{l}} b_{\mathbf{m}} + U^{(2)}_{\mathbf{klm}} a_{\mathbf{l}} c_{\mathbf{m}}) d\lambda \\ \int (V^{(0)}_{\mathbf{klm}} a_{\mathbf{l}} b_{\mathbf{m}} + V^{(1)}_{\mathbf{klm}} a_{\mathbf{l}} a_{\mathbf{m}} + V^{(2)}_{\mathbf{klm}} a_{\mathbf{l}} c_{\mathbf{m}}) d\lambda \\ \int (W^{(0)}_{\mathbf{klm}} a_{\mathbf{l}} c_{\mathbf{m}} + W^{(1)}_{\mathbf{klm}} a_{\mathbf{l}} a_{\mathbf{m}} + W^{(2)}_{\mathbf{klm}} a_{\mathbf{l}} b_{\mathbf{m}}) d\lambda \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 \\ \int (V^{(3)}_{\mathbf{klm}} b_{\mathbf{l}} b_{\mathbf{m}} + V^{(4)}_{\mathbf{klm}} c_{\mathbf{l}} c_{\mathbf{m}} + V^{(5)}_{\mathbf{klm}} b_{\mathbf{l}} c_{\mathbf{m}}) d\lambda \\ \int (W^{(3)}_{\mathbf{klm}} c_{\mathbf{l}} c_{\mathbf{m}} + W^{(4)}_{\mathbf{klm}} b_{\mathbf{l}} b_{\mathbf{m}} + W^{(5)}_{\mathbf{klm}} c_{\mathbf{l}} b_{\mathbf{m}}) d\lambda \end{pmatrix} = 0, \qquad (97)$$

with interaction coefficients which can be easily found from (94).

Thus, the variable $a_{\mathbf{k}}$ describes the short equatorial Rossby waves with a dispersion law $\Omega_{\mathbf{k}} = -\frac{\beta k_1}{|\mathbf{k}|^2}$ and the variables $b_{\mathbf{k}}$, $c_{\mathbf{k}}$ describe the short inertia-gravity waves with a dispersion law $\omega_{\mathbf{k}} =| \mathbf{k} | -\frac{\beta k_1}{2|\mathbf{k}|^2}$.

The linear part of (96) may be diagonalized by further transforming the variables:

$$\begin{pmatrix} a_{\mathbf{k}} \\ b_{\mathbf{k}} \\ c_{\mathbf{k}} \end{pmatrix} \to \begin{pmatrix} a_{\mathbf{k}} \\ b_{\mathbf{k}} \\ c_{\mathbf{k}} \end{pmatrix} + \beta \begin{pmatrix} 0 & -\frac{\partial}{\partial k_2}\frac{1}{\sqrt{2}|\mathbf{k}|} & -\frac{\partial}{\partial k_2}\frac{1}{\sqrt{2}|\mathbf{k}|} \\ \frac{i}{\sqrt{2}|\mathbf{k}|}\frac{\partial}{\partial k_2} & 0 & -\frac{k_1}{4|\mathbf{k}|^3} \\ \frac{i}{\sqrt{2}|\mathbf{k}|}\frac{\partial}{\partial k_2} & \frac{k_1}{4|\mathbf{k}|^3} & 0 \end{pmatrix} \begin{pmatrix} a_{\mathbf{k}} \\ b_{\mathbf{k}} \\ c_{\mathbf{k}} \end{pmatrix}. \tag{98}$$

Then the system in the leading order takes the form:

$$\begin{pmatrix} \partial a_{\mathbf{k}}/\partial t \\ \partial b_{\mathbf{k}}/\partial t \\ \partial c_{\mathbf{k}}/\partial t \end{pmatrix} + \begin{pmatrix} i\Omega_{\mathbf{k}} & 0 & 0 \\ 0 & i\omega_{\mathbf{k}} & 0 \\ 0 & 0 & -i\omega_{-\mathbf{k}} \end{pmatrix} \begin{pmatrix} a_{\mathbf{k}} \\ b_{\mathbf{k}} \\ c_{\mathbf{k}} \end{pmatrix} +$$

$$+ \begin{pmatrix} \int (U^{(0)}_{\mathbf{klm}} a_{\mathbf{l}} a_{\mathbf{m}} + U^{(1)}_{\mathbf{klm}} a_{\mathbf{l}} b_{\mathbf{m}} + U^{(2)}_{\mathbf{klm}} a_{\mathbf{l}} c_{\mathbf{m}}) d\lambda \\ \int (V^{(0)}_{\mathbf{klm}} a_{\mathbf{l}} b_{\mathbf{m}} + V^{(1)}_{\mathbf{klm}} a_{\mathbf{l}} a_{\mathbf{m}} + V^{(2)}_{\mathbf{klm}} a_{\mathbf{l}} c_{\mathbf{m}}) d\lambda \\ \int (W^{(0)}_{\mathbf{klm}} a_{\mathbf{l}} c_{\mathbf{m}} + W^{(1)}_{\mathbf{klm}} a_{\mathbf{l}} a_{\mathbf{m}} + W^{(2)}_{\mathbf{klm}} a_{\mathbf{l}} b_{\mathbf{m}}) d\lambda \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 \\ \int (V^{(3)}_{\mathbf{klm}} b_{\mathbf{l}} b_{\mathbf{m}} + V^{(4)}_{\mathbf{klm}} c_{\mathbf{l}} c_{\mathbf{m}} + V^{(5)}_{\mathbf{klm}} b_{\mathbf{l}} c_{\mathbf{m}}) d\lambda \\ \int (W^{(3)}_{\mathbf{klm}} c_{\mathbf{l}} c_{\mathbf{m}} + W^{(4)}_{\mathbf{klm}} b_{\mathbf{l}} b_{\mathbf{m}} + W^{(5)}_{\mathbf{klm}} c_{\mathbf{l}} b_{\mathbf{m}}) d\lambda \end{pmatrix} = 0, \qquad (99)$$

So Rossby waves split out , i. e. if $a_{\mathbf{k}} = 0$ at the initial moment, then $a_{\mathbf{k}} = 0$ for all times while these equation are applicable. As a result we obtain separate equations for short inertia-gravity waves

$$\begin{pmatrix} \partial b_{\mathbf{k}}/\partial t \\ \partial c_{\mathbf{k}}/\partial t \end{pmatrix} + \begin{pmatrix} i\omega_{\mathbf{k}} & 0 \\ 0 & -i\omega_{-\mathbf{k}} \end{pmatrix} \begin{pmatrix} b_{\mathbf{k}} \\ c_{\mathbf{k}} \end{pmatrix} +$$

$$+ \begin{pmatrix} \int (V^{(3)}_{\mathbf{klm}} b_{\mathbf{l}} b_{\mathbf{m}} + V^{(4)}_{\mathbf{klm}} c_{\mathbf{l}} c_{\mathbf{m}} + V^{(5)}_{\mathbf{klm}} b_{\mathbf{l}} c_{\mathbf{m}}) d\lambda \\ \int (W^{(3)}_{\mathbf{klm}} c_{\mathbf{l}} c_{\mathbf{m}} + W^{(4)}_{\mathbf{klm}} b_{\mathbf{l}} b_{\mathbf{m}} + W^{(5)}_{\mathbf{klm}} c_{\mathbf{l}} b_{\mathbf{m}}) d\lambda \end{pmatrix} = 0. \quad (100)$$

These equations are equivalent to the following Hamiltonian equations:

$$\begin{pmatrix} \dot{\varphi} \\ \dot{z} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta H/\delta \varphi \\ \delta H/\delta z \end{pmatrix} = 0, \quad (101)$$

with Hamiltonian

$$H = \int \left(\frac{1}{2}(1+z)(\varphi_x^2 + \varphi_y^2) + \frac{1}{2} z^2 + \beta \varphi \Delta^{-1} z_x \right) dxdy, \tag{102}$$

in terms of two space-time variables $\phi(\mathbf{r}, t)$, $z(\mathbf{r}, t)$ with Fourier - transforms:

$$\begin{pmatrix} \varphi_{\mathbf{k}} \\ z_{\mathbf{k}} \end{pmatrix} = \begin{pmatrix} \frac{i}{\sqrt{2}k} & -\frac{i}{\sqrt{2}k} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} b_{\mathbf{k}} \\ c_{\mathbf{k}} \end{pmatrix}. \tag{103}$$

For real initial variables we have $c_{\mathbf{k}} = b^*_{-\mathbf{k}}$ therefore $b^*_{\mathbf{k}}$ can be considered as an indepedent variable.

The equations have a stardard Hamiltonian form

$$\frac{\partial b_{\mathbf{k}}}{\partial t} + i\frac{\delta H}{\delta b^*_{\mathbf{k}}} = 0 \tag{104}$$

with Hamiltonian $H = H_2 + H_3$.

$$H_2 = \int \omega_{\mathbf{k}} \mid b_{\mathbf{k}} \mid^2 d\mathbf{k}, \quad \omega_{\mathbf{k}} = k - \frac{\beta}{2}\frac{k_x}{k^2}, \quad k = \sqrt{k_x^2 + k_y^2}, \tag{105}$$

where the frequency $\omega_{\mathbf{k}}$ is positive for the short waves.

$$
\begin{aligned}
H_3 = & \quad \frac{1}{2}\int V_{123}\left(b_1 b_2^* b_3^* + b_1^* b_2 b_3\right)\delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 + \\
+ & \quad \frac{1}{3}\int U_{123}\left(b_1 b_2 b_3 + b_1^* b_2^* b_3^*\right)\delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3, \qquad (106)
\end{aligned}
$$

where in the leading order in β

$$
2U_{123} = V_{123} = \sqrt{18}\frac{k_1(\mathbf{k}_2, \mathbf{k}_3) + k_2(\mathbf{k}_3, \mathbf{k}_1) + k_3(\mathbf{k}_1, \mathbf{k}_2)}{\sqrt{k_1 k_2 k_3}}
$$

Therefore, the standard methods of weak turbulence may be applied to equatorial SIGW. The results will, however, depend on the decay or non-decay character of the dispersion law (105). The following analysis shows that the dispersion law *changes its type* depending on orientation of the wave-vectors triad. We consider the synchronism conditions

$$\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2, \quad \omega_{\mathbf{k}} = \omega_1 + \omega_2, \tag{107}$$

and redirect the coordinate system along the vector $\mathbf{k}$. Then

$$k = k_1 \cos\theta_1 + k_2 \cos\theta_2, \quad 0 = k_1 \sin\theta_1 + k_2 \sin\theta_2, \tag{108}$$

$$k_1 + k_2 - k - \frac{\beta}{2}\left(\frac{\cos(\alpha+\theta_1)}{k_1} + \frac{\cos(\alpha+\theta_2)}{k_2} - \frac{\cos\alpha}{k}\right) = 0, \tag{109}$$

where α is the angle between $\mathbf{k}$ and the direction $\mathbf{e}_x$.

The frequency equation is rewritten as

$$\left(2k_1 \sin^2 \frac{\theta_1}{2} + 2k_2 \sin^2 \frac{\theta_2}{2}\right) - \\ - \frac{\beta \cos \alpha}{2} \left(\frac{\cos \theta_1}{k_1} + \frac{\cos \theta_2}{k_2} - \frac{1}{k}\right) - \frac{\beta \sin \alpha}{2} \left(\frac{\sin \theta_1}{k_1} + \frac{\sin \theta_2}{k_2}\right) = 0 \quad (110)$$

The scales are

$$k_1 \sim k_2 \sim 1 \gg \beta > 0$$

therefore angles are small

$$\theta_1, \theta_2 \ll 1$$

Here two situations are possible. First, assume that the third bracket is small: $\cos\alpha \gg \theta_j \sin\alpha$. Then the first and the second are in balance $\theta_j^2 \sim \beta\cos\alpha$. From these relations it is easy to find a validity condition

$$\frac{\cos\alpha}{\sin^2\alpha} \gg \beta. \tag{111}$$

For example, for $\alpha = \pi/4$ we have $\frac{\cos\alpha}{\sin^2\alpha} \sim 1$.

Second, if the second brackets are small $\cos\alpha \ll \theta_j \sin\alpha$ then $\theta_j \sim \beta\sin\alpha$ and therefore

$$\frac{\cos\alpha}{\sin^2\alpha} \ll \beta. \tag{112}$$

For $\alpha = 1.55$ (i.e. 89ř) we have $\frac{\cos\alpha}{\sin^2\alpha} \sim 0.016$.

In the first case we have the following balance in the main order

$$2k_1 \sin^2 \frac{\theta_1}{2} + 2k_2 \sin^2 \frac{\theta_2}{2} = \frac{\beta \cos\alpha}{2} \left(\frac{1}{k_1} + \frac{1}{k_2} - \frac{1}{k} \right) \tag{113}$$

and since

$$\frac{1}{k_1} + \frac{1}{k_2} - \frac{1}{k_1 + k_2} = \frac{(k_1 + k_2/2)^2 + 3k_2^2/4}{k_1 k_2 (k_1 + k_2)} > 0$$

a solition exists only for $\cos\alpha > 0$!

In the second case we have

$$2k_1 \sin^2 \frac{\theta_1}{2} + 2k_2 \sin^2 \frac{\theta_2}{2} = \frac{\beta \sin\alpha}{2} \left(\frac{\sin\theta_1}{k_1} + \frac{\sin\theta_2}{k_2} \right) \tag{114}$$

and

$$\theta_1, \theta_2 \sim \beta \ll 1$$

An existence of solution is independent of the sign of $\sin\alpha$ because there always exist a trivial solution $\theta_1 = \theta_2 = 0$.

Hence, ***resonant triads exist***

1. *in a relatively wide segment around* x *-axis in the right half-plane in the* $\mathbf{k}$ *- space;*
2. *in two very narrow segments around the* y *- axis.*

Apart from the narrow regions around the y *- axis, there are no resonant triads in the left half-plane in the* $\mathbf{k}$ *- space.*

The kinetic equations are, therefore, different in the regions of the phase-space with allowed and forbidden resonant triads. In the first case the standard three-wave kinetic equation applies.

$$\begin{aligned}\frac{\partial N(\mathbf{k},t)}{\partial t} &= \pi \int [\ | V_{k12} |^2 f_{k12} \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\delta(\omega_k-\omega_1-\omega_2) \\ &- | V_{12k} |^2 f_{12k} \delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k})\delta(\omega_1-\omega_2-\omega_k) \\ &- | V_{2k1} |^2 f_{2k1} \delta(\mathbf{k}_2-\mathbf{k}_k-\mathbf{k}_1)\delta(\omega_2-\omega_k-\omega_1)\ ]\, d\mathbf{k}_1 d\mathbf{k}_2,\end{aligned} \tag{115}$$

where $f_{k12} = N_1 N_2 - N N_1 - N N_2$ and $N_j = N(\mathbf{k}_j, t)$.

Dispersion is weak and applying techniques similar to those used to factorize the four-wave collision integral in the previous subsections, the collision integral in the case $\cos\alpha \gg \beta\sin\alpha$, may be factorized for $N(\mathbf{k}) = k^s$:

$$\begin{aligned} I(\mathbf{k}) &= \pi k^r \int \mid V_{k12} \mid^2 f_{k12}\delta(k-k_1-k_2)\delta(k_1\theta_1+k_2\theta_2) && (116)\\ & \delta\left(k_1\theta_1^2/2+k_2\theta_2^2/2-\frac{\beta\cos\alpha}{2}\left(\frac{1}{k_1}+\frac{1}{k_2}-\frac{1}{k}\right)\right) && (117)\\ & \left(k^{-r}-k_1^{-r}-k_2^{-r}\right) d\mathbf{k}_1 d\mathbf{k}_2. && (118)\end{aligned}$$

Here $r = 2s+5$ is a sum of powers coming from: f which give $2s$, the interaction coefficient which gives 3, the delta-functions which give $-1+0+1=0$, and two Jacobians which give 2.

As a result for $r = -1$ we have a Kolmogorov spectrum with $s = -3$ ($N = k^{-3}$). The spectral energy density per k is $\varepsilon_k = \omega_k k N_k = k^{-1}$. An equilibrium solution corresponds to $s = -1$ and equipartition of energy.

Curiously, the case of almost vertical resonant triads $\cos\alpha \ll \beta \sin\alpha$ does not allow such treatment

A four wave collision integral may be factorized as well in the non-decay regions of the phase - space giving:

$$I = k^r \int |T_{k123}|^2 \, \delta(k + k_1 - k_2 - k_3)$$

$$\delta\left(k_1\theta_1^2/2 - k_2\theta_2^2/2 - k_3\theta_3^2/2 - \frac{\beta\cos\alpha}{2}\left(k^{-1} + k_1^{-1} - k_2^{-1} - k_3^{-1}\right)\right)$$

$$\left(N_k^{-1} + N_1^{-1} - N_2^{-1} - N_3^{-1}\right)\left(k^{-r} + k_1^{-r} - k_2^{-r} - k_3^{-r}\right) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3$$

with $r = 3s + 11$. We thus get two stationary solutions: one with a constant energy flux: $r = -1$ and $N = k^{-4}$, and one with a constant wave-action: $r = 0$ and $N = k^{-11/3}$.

2.5 Weak turbulence of the Rossby waves on the mid-latitude β - plane

By supposing that nonlinearity parameter ϵ is small in the quasi-geostrophic equation:

$$\partial_t \left(h - \nabla^2 h\right) - \epsilon \mathcal{J}\left(h, \nabla^2 h\right) = 0, \tag{119}$$

and introducing Gaussian statistics for an ensemble of linear Rossby waves a following kinetic equation for the spectral density $F_{\mathbf{k}}$ of the height field h may be easily established by taking the Fourier-transform of (119), multiplying by the complex conjugate of $h_{\mathbf{k}}$, averaging and using the expansion of h in ϵ:

$$\dot{F}_{\mathbf{k}} = 4\pi \int d\mathbf{k}_1 d\mathbf{k}_2 \; \delta(\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2)\delta(\omega(\mathbf{k}) + \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2)) \cdot$$
$$\frac{D_{\mathbf{k}_1\mathbf{k}_2}}{(k^2+1)(k_1^2+1)(k_2^2+1)} \left(D_{\mathbf{k}_1\mathbf{k}_2} F_{\mathbf{k}_1} F_{\mathbf{k}_2} + D_{\mathbf{k}\mathbf{k}_2} F_{\mathbf{k}} F_{\mathbf{k}_1} + D_{\mathbf{k}\mathbf{k}_2} F_{\mathbf{k}} F_{\mathbf{k}_2}\right). \tag{120}$$

Here

$$D_{\mathbf{k}_1\mathbf{k}_2} = \frac{1}{4\pi} \mathbf{k}_1 \times \mathbf{k}_2 \left(\mathbf{k}_2^2 - \mathbf{k}_1^2\right), \tag{121}$$

and $\omega = -\beta \frac{k_x}{\mathbf{k}^2+1}$. The derivation is analogous to that for internal gravity waves in the stratified fluid given in the next Section and is not repeated here.

The specificity of Rossby waves is that equation (119) is Hamiltonian, but not canonical. This is reflected, in particular, in the fact that it possesses an *infinity of integrals of motion*. Indeed, rewriting it in the form:

$$\partial_t \left(h - \nabla^2 h\right) + \epsilon \mathcal{J}\left(h, h - \nabla^2 h - \beta y\right) = 0, \tag{122}$$

one sees that any function of $h - \nabla^2 h - \beta y$ integrated over the domain occupied by the flow is conserved. The Hamiltonian structure of the flow is given by the following Lie - Poisson bracket defined for any pair A and B of functionals of the *quasi-geostrophic relative potential vorticity* $q_{QG} = h - \nabla^2 h$:

$$\{A[q], B[q]\} = -\int dxdy\,(q - \beta y)\mathcal{J}\left(\frac{\delta A}{\delta q}, \frac{\delta B}{\delta q}\right). \tag{123}$$

The Hamiltonian is quadratic: $H = \frac{1}{2}\int dxdy\, h(h - \nabla^2 h)$.

For configurations with *no closed lines of* $q_{QG} - \beta y$ (regions with closed contours of correspond to *vortices* while *waves* have isolines of $q_{QG} - \beta y$ slightly deviating from the straight lines $y = const$) there exist a change of independent variables which allows to transform the non-canonical Poisson bracket (123) into the canonical one. (This transformation, in fact, straightens the isolines of $q_{QG} - \beta y$ in the wave case). The transformation adds nonlinear terms to the initial quadratic Hamiltonian. The standard Hamiltonian weak-turbulence approach may be then applied. It may be shown, however, that the resulting kinetic equation is identical to (120).

Following the same lines as in the subsequent section, the collision integral for Rossby waves may be represented as a sum of three integrals and, for short waves with $\omega = -\beta \frac{k_x}{|\mathbf{k}|^2}$, such that $k_x \ll k_y$, the integration domains may be transformed one into another using the integration variables $s_j = \frac{k_{jx}}{k_x}$ and $t_j = \frac{\omega(k_j)}{\omega(k_)}$. Supposing a self-similar solution $F(k_x, k_y) \propto k_x^{\alpha} k_y^{\beta}$, the whole collision integral then is factorized and the following stationary solutions are obtained:

$$F_k \propto k_x^{-\frac{3}{2}}, \;\; F_k \propto k_x^{-\frac{3}{2}} k_y^{-1}. \tag{124}$$

It may be checked by direct sustitution into (120) that generalized thermodynamic equilibria of the form:

$$F(\mathbf{k}) = \frac{1}{a + b\mathbf{k}^2} + \Phi(k_y)\delta(k_x), \tag{125}$$

where a, b arbitrary constants, and Φ - arbitrary function, annihilate the collision integral.

The first part of the solution corresponds to energy equipartition, while the second gives an equilibrium with arbitrary zonal current, which is a peculiar property of the Rossby waves dynamics

2.6 Historical remarks and bibliography

The RSW model is a standard tool in geophysical fluid dynamics, for separation in slow and fast components and their non-interaction cf, e.g. Reznik, Zeitlin, and Ben Jelloul, 2001, and references therein.

Factorisation of the collision integral in the non-decay case close to the acoustic law follows the paper of Volotsky, Kats, and Kontorovich, 1980, which is ill-known.

Kinetic equation and weak turbulence spectra of short SIGW on the f - plane were obtained by Falkovich and Medvedev (1992), following the ideas of Volotsky, Kats, and Kontorovich (1980).

For a recent review of equatorial waves in the framework of RSW model see Le Sommer, Reznik, and Zeitlin (2004). The results on weak turbulence of short equatorial waves are obtained recently by Medvedev and Zeitlin (2004).

Weak turbulence of the Rossby waves has a long history. Kinetic equation was obtained by Kenyon (1964) and Longuet-Higgins and Gill (1967). Equilibrium spectra were derived and investigated in Reznik and Soomere (1983), and Reznik (1984). The canonical variables for the equation (63) were obtained by Zakharov and Piterbarg (1987) and used to construct the kinetic equation. Relation between canonical and non-canonical Hamiltonian structure for nonlinear Rossby waves was established by Zeitlin (1992). Equivalence of thus obtained kinetic equation and the old one of Longuet-Higgins and Gill (1967) was established by Monin and Piterbarg (1987), and the nonequilibrium Kolmogorov-type spectra were found following the method of Kuznetsov (1972). This method is also used below for internal gravity waves in the stratified fluid. For stability of non-equilibrium power-law solutions of the kinetic equation for Rossby waves, see Balk and Nazarenko, 1990.

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3 Lecture 4. Weak turbulence of internal gravity waves: example of strongly anisotropic medium.

Starting point: stratified ideal fluid equations in the Boussinesq approximation written for velocity $\mathbf{v} = (u, v, w)$ and buoyancy $\xi = -g\frac{\rho-\rho_0}{\rho_0}$. $\rho_o(x_3)$ is a background vertical density stratification, $N^2 = -g\frac{d\rho_0}{dx_3}/\rho_0$ is the square of the Brunt-Väisälä frequency which is supposed to be constant, g is acceleration due to gravity (ξ should be replaced by potential temperature in atmospheric applications)

$$\begin{aligned} \dot{\mathbf{v}} + \mathbf{v}\nabla\mathbf{v} + \nabla P - \xi\,\hat{\mathbf{x}}_3 &= 0 \\ \dot{\xi} + \mathbf{v}\nabla\xi + N^2 w &= 0 \\ \nabla\cdot\mathbf{v} &= 0. \end{aligned} \tag{126}$$

3.1 Kinetic equation and energy spectra for plane-parallel internal gravity waves

We start by considering 2d situation with no dependence on one of the horizontal coordinates, x_2. In this case the Boussinesq equations for the (vertical) streamfunction ψ and buoyancy ξ (potential temperature in the atmospheric context) variables are

$$\Delta\dot{\psi} + J(\Delta\psi,\, \psi) + \xi_{x_1} = 0, \qquad \dot{\xi} + J(\xi,\, \psi) - N^2\psi_{x_1} = 0\,, \tag{127}$$

where the subscripts denote partial derivatives with respect to the corresponding argument. The velocity components are

$$u = \frac{\partial\psi}{\partial x_3}, \qquad v = -\frac{\partial\psi}{\partial x_1}.$$

The approximation is valid for short waves such that wave- amplitude growth with altitude due to decreasing density may be neglected.

We assume that the wave field is weakly nonlinear, introduce a non-dimensional small amplitude ϵ and expand ψ and ξ:

$$\psi = \epsilon\,\psi^{(0)} + \epsilon^2\,\psi^{(1)} + \cdots\,, \qquad \xi = \epsilon\,\xi^{(0)} + \epsilon^2\,\xi^{(1)} + \cdots\,.$$

In the first order we get

$$\dot{\psi}^{(0)} = -\Delta^{-1}\xi^{(0)}_{x_1},\ \dot{\xi}^{(0)} = N^2\psi^{(0)}_{x_1}, \tag{128}$$

which gives the dispersion relation for internal gravity waves:

$$\omega^2 = N^2\frac{p_H^2}{p^2}. \tag{129}$$

We chose to split waves in a standard way, according to their frequency sign, i.e. $\omega = \pm\Omega_p$ with $\Omega_p = N\frac{|p_1|}{p} = N\frac{\hat{p_1}}{p}$, $\mathbf{p} = (p_1, p_3)$ and a hat over any function of p means multiplication by sign$[p_1]$.

Solution of (128) in Fourier-space may be written as

$$\psi_{\mathbf{p}}^{(0)} = a_{\mathbf{p}} + \overline{a}_{-\mathbf{p}},\ \xi_{\mathbf{p}}^{(0)} = -N\hat{p}\,(a_{\mathbf{p}} - \overline{a}_{-\mathbf{p}})\,. \tag{130}$$

where $a_{\mathbf{p}} = \varphi^{+}(\mathbf{p})\, e^{i\Omega_p t}$ and $\varphi^{-} = \overline{\varphi}^{+}$ (the overbar denotes complex conjugation). In principle, mean flow and buoyancy corrections proportional to $\delta(p_1)$ may be added. However, as total (spatially integrated) buoyancy and vorticity are exactly conserved we may exclude these corrections in what follows by limiting ourselves by the states with zero total buoyancy and vorticity.

The energy of the system is a sum of kinetic and available potential energies

$$E = \frac{1}{2}\int d\mathbf{x}\,(-\psi\Delta\psi + N^{-2}\xi^2) = \frac{1}{2}\int d\mathbf{p}\,(p^2\,\psi_{\mathbf{p}}\,\psi_{-\mathbf{p}} + N^{-2}\xi_{\mathbf{p}}\,\xi_{-\mathbf{p}})$$

and for linear waves equipartition of energy between kinetic and potential parts takes place:

$$E = \int d\mathbf{p}\,p^2(a_{\mathbf{p}}\,\overline{a}_{\mathbf{p}} + a_{-\mathbf{p}}\,\overline{a}_{-\mathbf{p}}). \tag{131}$$

We calculate then the first nonlinear corrections by using the retarded Green's function (an infinitesimal damping of the linear waves δ is introduced shifting the poles of this function which is a solution of (158) upward from the real ω-axis). As a result, in the second order we get

$$\psi^{(1)}(\mathbf{p},\, t)_{p_1\neq 0} = -i\int d\mathbf{k}\, d\mathbf{l}\ \delta(\mathbf{k}+\mathbf{l}-\mathbf{p}) \sum_{\pm} A^{\pm}_{kl,p}$$
$$[\frac{a_{\mathbf{k}}a_{\mathbf{l}}}{\Omega_k+\Omega_l\mp\Omega_p-i\delta}+\frac{a_{\mathbf{k}}\overline{a}_{-\mathbf{l}}}{\Omega_k-\Omega_l\mp\Omega_p-i\delta}$$
$$-\frac{\overline{a}_{-\mathbf{k}}a_{\mathbf{l}}}{\Omega_k-\Omega_l\mp\Omega_p+i\delta}-\frac{\overline{a}_{-\mathbf{k}}\overline{a}_{-\mathbf{l}}}{\Omega_k+\Omega_l\mp\Omega_p+i\delta}], \quad (132)$$

$$\xi^{(1)}(\mathbf{p},\, t)_{p_1\neq 0} = iN\hat{p}\int d\mathbf{k}\, d\mathbf{l}\ \delta(\mathbf{k}+\mathbf{l}-\mathbf{p})\sum_{\pm}(\pm)A^{\pm}_{kl,p}$$
$$[\frac{a_{\mathbf{k}}a_{\mathbf{l}}}{\Omega_k+\Omega_l\mp\Omega_p-i\delta}+\frac{a_{\mathbf{k}}\overline{a}_{-\mathbf{l}}}{\Omega_k-\Omega_l\mp\Omega_p-i\delta}$$
$$+\frac{\overline{a}_{-\mathbf{k}}a_{\mathbf{l}}}{\Omega_k-\Omega_l\mp\Omega_p+i\delta}+\frac{\overline{a}_{-\mathbf{k}}\overline{a}_{-\mathbf{l}}}{\Omega_k+\Omega_l\mp\Omega_p+i\delta}], \quad (133)$$

where the interaction coefficients are defined as

$$A^{\pm}_{kl,p} = \mathbf{k}\times\mathbf{l}\ \frac{k^2\pm\hat{p}\hat{k}}{2p^2}, \qquad \mathbf{k}\times\mathbf{l}\equiv k_1l_3-k_3l_1. \quad (134)$$

We next make a standard random phase approximation by supposing that due to the presence of a large amount of weakly interacting waves the complex wave amplitudes $a_{\mathbf{k}}$ assume Gaussian statistics

$$\begin{aligned} < a_{\mathbf{k}}\, a_{\mathbf{l}} > &= 0\,, \\ < a_{\mathbf{k}}\, \overline{a}_{\mathbf{l}} > &= N_{\mathbf{k}}\ \delta(\mathbf{k}-\mathbf{l})\,, \\ < a_{\mathbf{k}}\, a_{\mathbf{l}}\, \overline{a}_{\mathbf{m}}\, \overline{a}_{\mathbf{n}} > &= N_{\mathbf{k}}\ N_{\mathbf{l}}(\delta(\mathbf{k}-\mathbf{m})\delta(\mathbf{l}-\mathbf{n}) + \delta(\mathbf{k}-\mathbf{n})\delta(\mathbf{l}-\mathbf{m}))\,, \\ N_{\mathbf{k}} &= N_{-\mathbf{k}}. \end{aligned} \tag{135}$$

We denote abusively the quadratic mean N_k even though this is not a function of the modulus of $\mathbf{k}$. From (127) we obtain the following (exact) equations for averages of the Fourier components of the wave field:

$$\begin{aligned} p^2 < \dot{\psi}_{\mathbf{p}} \, \psi_{\mathbf{q}} > \quad &= \quad \int d\mathbf{k} \, d\mathbf{l} \; V_{kl} \;\; < \psi_{\mathbf{k}} \, \psi_{\mathbf{l}} \, \psi_{\mathbf{q}} > \delta(\mathbf{k} + \mathbf{l} - \mathbf{p}) \\ & \qquad\qquad\qquad -ip_1 \; < \xi_{\mathbf{p}} \, \psi_{\mathbf{q}} > , \\ < \dot{\xi}_{\mathbf{p}} \, \xi_{\mathbf{q}} > \quad &= \quad - \int d\mathbf{k} \, d\mathbf{l} \; W_{kl} \;\; < \xi_{\mathbf{k}} \, \psi_{\mathbf{l}} \, \xi_{\mathbf{q}} > \delta(\mathbf{k} + \mathbf{l} - \mathbf{p}) \\ & \qquad\qquad\qquad -ip_1 N^2 < \psi_{\mathbf{p}} \, \xi_{\mathbf{q}} > , \end{aligned} \tag{136}$$

with $V_{kl} = k^2 \mathbf{k} \times \mathbf{l}$, $W_{kl} = -\mathbf{k} \times \mathbf{l}$.

By adding these two equations and their complex conjugates, we get the following kinetic equation for the spectral density of energy $\epsilon_{\mathbf{p}} = p^2(a_{\mathbf{p}}\,\overline{a}_{\mathbf{p}} + a_{-\mathbf{p}}\,\overline{a}_{-\mathbf{p}})$

$$< \dot{\epsilon}_{\mathbf{p}} >= \int d\mathbf{k}\, d\mathbf{l}\ \delta(\mathbf{k}+\mathbf{l}-\mathbf{p})V_{kl}[< \psi_{\mathbf{k}}\,\psi_{\mathbf{l}}\,\psi_{-\mathbf{p}} > + < \psi_{-\mathbf{k}}\,\psi_{-\mathbf{l}}\,\psi_{\mathbf{p}} >]$$
$$-N^{-2}\int d\mathbf{k}\, d\mathbf{l}\ \delta(\mathbf{k}+\mathbf{l}-\mathbf{p})W_{kl}\ [< \quad \xi_{\mathbf{k}}\,\psi_{\mathbf{l}}\,\xi_{-\mathbf{p}} > + < \xi_{-\mathbf{k}}\,\psi_{-\mathbf{l}}\,\xi_{\mathbf{p}} >] \tag{137}$$

and calculate the r.h.s. of this equation perturbatively using (164) and Sokhotsky formula

$$\frac{1}{A\ \pm\ i\delta} = P\frac{1}{A}\ \mp\ i\pi\delta(A). \tag{138}$$

Non-resonant contributions to (137) cancel out pairwise and only the resonant terms remain. Using (132) , (133) and (138), we get,

$$< \dot{\epsilon}_{\mathbf{p}} >= 2\pi p^2 \int d\mathbf{k}\, d\mathbf{l}\; \delta(\mathbf{k}+\mathbf{l}-\mathbf{p})\{$$

$$A^{+}_{\{kl\},p}\; [A^{+}_{\{kl\},p} N_k N_l - A^{+}_{\{lp\},k} N_p N_l - A^{+}_{\{kp\},l} N_p N_k]\; \delta(\Omega_k+\Omega_l-\Omega_p)$$
$$+ A^{\star}_{kl,p}\; [A^{\star}_{kl,p} N_k N_l - A^{-}_{\{kp\},l} N_p N_k - A^{\star}_{pl,k} N_p N_l]\; \delta(\Omega_l+\Omega_p-\Omega_k)$$
$$+ A^{\star}_{lk,p}\; [A^{\star}_{lk,p} N_k N_l - A^{-}_{\{lp\},k} N_p N_l - A^{\star}_{pk,l} N_p N_k]\; \delta(\Omega_k+\Omega_p-\Omega_l)\}, \qquad (139)$$

where we defined $A^{\pm}_{\{kl\},p} = A^{\pm}_{kl,p} + A^{\pm}_{lk,p}$ and $A^{\star}_{kl,p} = A^{+}_{kl,p} + A^{-}_{lk,p}$.

This is a typical kinetic equation with two δ-functions assuring that the collision process conserves energy and momentum. However, some transformations are necessary in order to cast it to the standard form. A change of variables should be made in order to have a standard form *frequency*$\times$*wavenumber density* for $\epsilon_{\mathbf{p}}$

$$N_p \ \rightarrow N\frac{\hat{p_1}}{p^3}N_p \tag{140}$$

so that $\epsilon_{\mathbf{p}} = 2\Omega_p N_p$, N_p is the wavenumber density and is an adiabatic invariant in wave dynamics. We correspondingly modify the interaction coefficients

$$\mathcal{A}^{\pm}_{kl,p} = A^{\pm}_{\{kl\},p}(\frac{p}{kl})^{\frac{3}{2}}, \quad \mathcal{A}^{\star}_{kl,p} = A^{\star}_{kl,p}(\frac{p}{kl})^{\frac{3}{2}} \tag{141}$$

and get

$$< \dot{N}_{\mathbf{p}} > \sim \pi N \int d\mathbf{k}\, d\mathbf{l}\ \delta(\mathbf{k}+\mathbf{l}-\mathbf{p})\{$$

$$\mathcal{A}^{+}_{kl,p}\left[|\frac{k_1\, l_1}{q_1}|\mathcal{A}^{+}_{kl,p}N_k N_l - |l_1|\mathcal{A}^{+}_{lp,k}N_p N_l - |k_1|\mathcal{A}^{+}_{kp,l}N_p N_k\right]\delta(\Omega_k+\Omega_l-\Omega_p)+$$

$$\mathcal{A}^{\star}_{kl,p}\left[|\frac{k_1\, l_1}{q_1}|\mathcal{A}^{\star}_{kl,p}N_k N_l - |k_1|\mathcal{A}^{-}_{kp,l}N_p N_k - |l_1|\mathcal{A}^{\star}_{pl,k}N_p N_l\right]\delta(\Omega_l+\Omega_p-\Omega_k)+$$

$$\mathcal{A}^{\star}_{lk,p}\left[|\frac{k_1\, l_1}{q_1}|\mathcal{A}^{\star}_{lk,p}N_k N_l - |l_1|\mathcal{A}^{-}_{lp,k}N_p N_l - |k_1|\mathcal{A}^{\star}_{pk,l}N_p N_k\right]\delta(\Omega_k+\Omega_p-\Omega_l)\}. \quad (142)$$

Taking into account the symmetry properties and the fact that the above expressions are to be calculated on the resonant surfaces defined by the arguments of the δ-functions, we have

$$|\frac{k_1\, l_1}{q_1}|\mathcal{A}^{+}_{kl,p} = |l_1|\mathcal{A}^{+}_{lp,k} = |k_1|\mathcal{A}^{+}_{kp,l}\,,\,|\frac{k_1\, l_1}{q_1}|\mathcal{A}^{\star}_{kl,p} = |l_1|\mathcal{A}^{\star}_{pl,k} = -|k_1|\mathcal{A}^{-}_{kp,l} \tag{143}$$

and the equation (142) becomes

$$\begin{aligned}\dot{N}_p \quad \sim \quad & \pi N \int d\mathbf{k}\, d\mathbf{l}\ \delta(\mathbf{k}+\mathbf{l}-\mathbf{p})\{ \\ & Z^{+}_{kl,p}\ (N_k N_l - N_p N_k - N_p N_l)\ \delta(\Omega_k + \Omega_l - \Omega_p) + \\ & Z^{-}_{kp,l}\ (N_k N_l - N_p N_l + N_p N_k)\ \delta(\Omega_l + \Omega_p - \Omega_k) + \\ & Z^{-}_{lp,k}\ (N_k N_l - N_p N_k + N_p N_l)\ \delta(\Omega_k + \Omega_p - \Omega_l)\}\end{aligned} \tag{144}$$

with

$$Z^{\pm}_{kl,p} = \frac{1}{4}\frac{\hat{k_1}\,\hat{l_1}\,\hat{p_1}}{klp}(\hat{k}+\hat{l}\pm\hat{p})^2\ (\frac{\hat{k_3}}{k} \mp \frac{\hat{l_3}}{l} \mp \frac{\hat{p_3}}{p})^2\,. \tag{145}$$

The sign in the first bracket of this expression corresponds to the superscript, the sign of the second bracket should be taken as that of the subsequent δ-function in (144).

Each integrand will be split in three to reduce the integration domain to $k_1 \geq 0$ and $l_1 \geq 0$ (we suppose then that $p_1 \geq 0$ as well):

$$
\begin{aligned}
\dot{N}_p \quad \sim \quad & \pi N \int d\mathbf{k}\, d\mathbf{l}\ \delta(\mathbf{k}+\mathbf{l}-\mathbf{p})\{ \\
& \quad Z^{+}_{kl,p}\ (N_k N_l - N_p N_k - N_p N_l)\ \delta(\Omega_k + \Omega_l - \Omega_p) + \\
& \quad Z^{-}_{kp,l}\ (N_k N_l - N_p N_l + N_p N_k)\ \delta(\Omega_l + \Omega_p - \Omega_k) + \\
& \quad Z^{-}_{lp,k}\ (N_k N_l - N_p N_k + N_p N_l)\ \delta(\Omega_k + \Omega_p - \Omega_l)\} \\
& + 2\pi N \int d\mathbf{k}\, d\mathbf{l}\ \delta(\mathbf{k}+\mathbf{p}-\mathbf{l})\{ \\
& \quad Z^{-}_{lp,k}\ (N_k N_l - N_p N_k - N_p N_l)\ \delta(\Omega_k + \Omega_l - \Omega_p) + \\
& \quad Z^{-}_{lk,p}\ (N_k N_l - N_p N_l + N_p N_k)\ \delta(\Omega_l + \Omega_p - \Omega_k) + \\
& \quad Z^{+}_{lp,k}\ (N_k N_l - N_p N_k + N_p N_l)\ \delta(\Omega_k + \Omega_p - \Omega_l)\}
\end{aligned}
\tag{146}
$$

Due to the complex form of the function Z in (145) it is impossible to find exact solutions of (146) without making additional assumptions. We will try to find an analytic power law solution by limiting ourselves by almost vertically propagating waves: $k_3 \gg k_1$, a hypothesis frequently used in geophysics. In this case $Z^{\pm}_{kl,p}$ greatly simplifies:

$$Z^{\pm}_{kl,p} = \frac{1}{4} \frac{|k_1\, l_1\, p_1|}{|k_3\, l_3\, p_3|} (\widehat{|k_3|} + \widehat{|l_3|} \pm \widehat{|p_3|})^2. \tag{147}$$

We are then looking for spectra averaged in the vertical direction, i.e. our spectral density N_p will depend only on $|p_3|$. This allows us to reduce all integrations to the domain k_3, $l_3 \geq 0$ for $p_3 \geq 0$. In this way, each integral splits in four and for each sub-integral we may transform the integration variables according to

$$k_1 = s\, p_1, \qquad k_3 = \frac{s}{t}\, p_3, \qquad l_1 = u\, p_1, \qquad l_3 = \frac{u}{v}\, p_3. \tag{148}$$

$(s > 0, t > 0, u > 0, v > 0)$ and look for a scaling solution

$$N_k \sim s^\alpha t^\beta N_p, \qquad N_l \sim u^\alpha v^\beta N_p, \qquad N_k \sim k_1^\alpha \Omega_k^\beta, \qquad N_k \sim k_1^{\alpha+\beta} k_3^{-\beta}. \tag{149}$$

Performing frequency and the horizontal wavevector integration by using the corresponding δ-functions and making, when necessary a change of variables $s \to s/(1-s), t \to t/(1-t)$ in order to have a common bounded integration domain ((0,1),(0,1)) we arrive to the expression (up to a constant) containing two different integrals:

$$
\begin{aligned}
\dot{N}_p \quad &\sim \quad \int_0^1 ds \int_0^1 dt\ \delta(1 \pm \frac{s}{t} \pm \frac{1-s}{1-t}) U(s,t) \times \\
&\qquad [s^\alpha t^\beta (1-s)^\alpha (1-t)^\beta - s^\alpha t^\beta - (1-s)^\alpha (1-t)^\beta] \times \\
&\qquad [1 - (1-s)^{-2\alpha-6}(1-t)^{-2\beta+2} - s^{-2\alpha-6} t^{-2\beta+2}] \\
&+ \quad 2 \int_0^1 ds \int_0^1 dt\ \delta(\frac{1}{t} \pm s \pm \frac{1-s}{1-t}) V(s,t) \times \\
&\qquad [s^\alpha t^\beta + s^\alpha (1-s)^\alpha (1-t)^\beta - t^\beta (1-s)^\alpha (1-t)^\beta] \times \\
&\qquad [t^{-2\beta+2} - s^{-2\alpha-6} + (1-s)^{-2\alpha-6}(1-t)^{-2\beta+2}]. \quad (150)
\end{aligned}
$$

Here $U(s,t) = \frac{s(1-s)}{t(1-t)}(1 + \frac{s}{t} + \frac{1-s}{1-t})^2$ and $V(s,t) = \frac{s(1-s)}{t(1-t)}(\frac{1}{t} + s - \frac{1-s}{1-t})^2$. Note a difference with similar expression for *unidirectional* Rossby waves, where a single integral arises.

The solutions are a statistical equilibrium spectrum

$$N_k = 1/\Omega_k, \qquad \epsilon_k = const, \tag{151}$$

and a single non-equilibrium spectrum

$$N_k \sim k_1^{-\frac{5}{2}} k_3^{-\frac{1}{2}}, \qquad \epsilon_k \sim k_1^{-\frac{3}{2}} k_3^{-\frac{3}{2}} \qquad \epsilon_k \sim \Omega_k^{-\frac{3}{2}} k_3^{-3}. \tag{152}$$

The first one is energy equipartition, i.e. the Rayleigh-Jeans spectrum while the second one is a Kolmogorov-like spectrum corresponding to a constant energy flux through the spectrum. Indeed, let P be an energy flux. Then by dimensional reasons a spectrum constructed from P and the square of the Brunt-Väisälä frequency must be proportional to the wavenumber in the -3 power which corresponds to (152)

$$\epsilon_k = C_2 \, N^{\frac{1}{2}} \, P^{\frac{1}{2}} \, k_1^{-\frac{3}{2}} \, k_3^{-\frac{3}{2}} \tag{153}$$

3.2 Kinetic equation and horizontally isotropic stationary energy spectra for 3D internal gravity waves in the absence of mean potential vorticity

In the 3d case it is convenient to use the Craya-Herring decomposition for velocity:

$$\mathbf{v} = \boldsymbol{\nabla}_H \times \psi\, \hat{\mathbf{x}}_3 + \boldsymbol{\nabla}_H \phi + w\, \hat{\mathbf{x}}_3 \tag{154}$$

which in the linear approximations gives a separation of the flow in a vortical (ψ) and wave (ϕ, w) parts.

$$
\begin{aligned}
\Delta_H \phi + \partial_3 w &= 0\,, \\
\Delta P - \partial_3 \xi + \nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) &= 0\,, \\
\dot{w} + \mathbf{v} \cdot \nabla w + \partial_3 P - \xi &= 0\,, \\
\dot{\xi} + \mathbf{v} \cdot \nabla \xi + N^2 w &= 0\,,
\end{aligned} \tag{155}
$$

$$
\begin{aligned}
&- \Delta_H \dot{\psi} - J(\Delta_H \psi, \psi) - \nabla_H \cdot (\nabla_H \phi \Delta_H \psi) - w \partial_3 \Delta_H \psi \\
&\qquad - \nabla_H w \cdot \nabla_H \partial_3 \psi + J(w, \partial_3 \phi) = 0\,,
\end{aligned} \tag{156}
$$

where the subscripts H and 3, denote the horizontal and vertical parts, respectively and

$$
\mathbf{v} \cdot \nabla ... = \nabla_H \phi \cdot \nabla ... + w \partial_3 + J(... \,, \psi). \tag{157}
$$

by introducing Fourier transforms $\psi_{\mathbf{p}}$, $\xi_{\mathbf{p}}$, $\phi_{\mathbf{p}}$ and $w_{\mathbf{p}}$ and linearizing the system (126) one gets internal gravity waves solutions wi th a dispersion relation

$$\omega^2 = N^2 \frac{\mathbf{p}_H^2}{\mathbf{p}^2} \tag{158}$$

where ω is the wave frequency, $\mathbf{p} = (p_1, p_2, p_3)$ is the wavenumber, $p \equiv |\mathbf{p}|$ and $\mathbf{p}_H = (p_1, p_2)$ is its projection on the horizonta l plane. Note that $\psi_{\mathbf{p}}$ represents a zero mode of the linear system, i.e. is time-independent in the first approximation, and that mean flow/stratification c orrections proportional to $\delta(p_H)$ must be added to get a general solution of the linear system. We shall discard this contributions altogether by the reasons explained below. Note also that the wave part of $\phi_{\mathbf{p}}$ may be always eliminated in favor of $w_{\mathbf{p}}$.

By introducing wave amplitudes $b_{\mathbf{p}} = e^{i\Omega_p t}\chi^+_{\mathbf{p}}$ corresponding to the two branches of the dispersion equation (158), where Ω_p is the positive-sign root of (158), we get

$$w_{\mathbf{p}} = b_{\mathbf{p}} + \overline{b}_{-\mathbf{p}}\,,\ \xi_{\mathbf{p}} = iN\frac{p}{p_H}(b_{\mathbf{p}} - \overline{b}_{-\mathbf{p}}) \tag{159}$$

The energy of the system in the absence of vortical part and mean buoyancy is

$$\begin{aligned} E &= \frac{1}{2}\int d\mathbf{x}\,\left(-\psi\Delta_H\psi - \phi\Delta_H\phi + w^2 + N^{-2}\xi^2\right) \\ &= \frac{1}{2}\int d\mathbf{p}\,\left(p_H^2\,\phi_{\mathbf{p}}\,\phi_{-\mathbf{p}} + w_{\mathbf{p}}w_{-\mathbf{p}} + N^{-2}\xi_{\mathbf{p}}\,\xi_{-\mathbf{p}}\right) \\ &= \frac{1}{2}\int d\mathbf{p}\,\left(2\,\frac{p^2}{p_H^2}\,\left(b_{\mathbf{p}}\,\overline{b}_{\mathbf{p}} + b_{-\mathbf{p}}\,\overline{b}_{-\mathbf{p}}\right)\right) \end{aligned} \tag{160}$$

We limit ourselves by a study of purely wave regimes, thus, excluding completely the vortical mode and mean flow/stratification corrections. A physical basis for this separation is the Lagrangian (pointwise) conservation of the potential vorticity $q = (\nabla \times \mathbf{v})(\nabla \xi + N^2 \hat{\mathbf{x}}_3)$. For a single harmonic wave potential vorticity is identically zero. This is not true anymore for a superposition of waves and, in fact, waves do produce vorticity and mean bouyancy already in the first order of perturbation theory. However, for a Gaussian ensemble of waves (see below) the statistically averaged potential vorticity is still zero which is not the case in the present of vortical modes or mean buoyancy corrections. We, thus, have a separation criterion and by limiting ourselves by zero mean potential vorticity configurations we discard all non-wave corrections.

The first order corrections, therefore, are

$$w^{(1)}(\mathbf{p},\,t) = -\int d\mathbf{k}\,d\mathbf{l}\;\delta(\mathbf{k}+\mathbf{l}-\mathbf{p})\sum_{\pm} B^{\pm}_{kl,p} \Big[\frac{b_{\mathbf{k}}b_{\mathbf{l}}}{\Omega_k+\Omega_l\mp\Omega_p-i\delta}+\frac{b_{\mathbf{k}}\bar{b}_{-\mathbf{l}}}{\Omega_k-\Omega_l\mp\Omega_p-i\delta} -\frac{\bar{b}_{-\mathbf{k}}b_{\mathbf{l}}}{\Omega_k-\Omega_l\mp\Omega_p+i\delta}-\frac{\bar{b}_{-\mathbf{k}}\bar{b}_{-\mathbf{l}}}{\Omega_k+\Omega_l\mp\Omega_p+i\delta}\Big] \tag{161}$$

$$\xi^{(1)}(\mathbf{p},\,t) = -iN\frac{p}{p_H}\int d\mathbf{k}\,d\mathbf{l}\;\delta(\mathbf{k}+\mathbf{l}-\mathbf{p})\sum_{\pm}(\pm)B^{\pm}_{kl,p} \Big[\frac{b_{\mathbf{k}}b_{\mathbf{l}}}{\Omega_k+\Omega_l\mp\Omega_p-i\delta}+\frac{b_{\mathbf{k}}\bar{b}_{-\mathbf{l}}}{\Omega_k-\Omega_l\mp\Omega_p-i\delta} +\frac{\bar{b}_{-\mathbf{k}}b_{\mathbf{l}}}{\Omega_k-\Omega_l\mp\Omega_p+i\delta}+\frac{\bar{b}_{-\mathbf{k}}\bar{b}_{-\mathbf{l}}}{\Omega_k+\Omega_l\mp\Omega_p+i\delta}\Big] \tag{162}$$

where

$$B^{\pm}_{kl,p} = (p_H^2 + p_3 l_3 \frac{\mathbf{p}\cdot\mathbf{l}}{l_H^2} \pm pl\frac{p_H}{l_H})\frac{H_{kl}}{2p^2} \ , H_{kl} = \frac{\mathbf{k}\cdot\mathbf{l}}{k_H^2}k_3 - l_3. \qquad (163)$$

We next make the random phase approximation and suppose that due to the presence of a large amount of weakly interacting waves the complex wave amplitudes $b_{\mathbf{k}}$ assume Gaussian statistics

$$
\begin{aligned}
< b_{\mathbf{k}}\, \bar{b}_{\mathbf{l}} > \quad &= \quad N_{\mathbf{k}}\; \delta(\mathbf{k}-\mathbf{l})\,, \\
< b_{\mathbf{k}}\, b_{\mathbf{l}}\, \bar{b}_{\mathbf{m}}\, \bar{b}_{\mathbf{n}} > \quad &= \quad N_{\mathbf{k}}\; N_{\mathbf{l}}(\delta(\mathbf{k}-\mathbf{m})\delta(\mathbf{l}-\mathbf{n}) + \delta(\mathbf{k}-\mathbf{n})\delta(\mathbf{l}-\mathbf{m})),
\end{aligned}
\tag{164}
$$

where the wavenumber density $N_{\mathbf{k}} = N_{-\mathbf{k}}$ is a real-valued function and we again denote abusively the quadratic mean by N_k.

>From (126) we obtain the following (exact) equations for averages of the Fourier components of the wave field:

$$
\begin{aligned}
< \dot{w}_{\mathbf{p}}\, w_{\mathbf{q}} > \quad &= \quad \frac{p_H^2}{p^2} < \xi_{\mathbf{p}}\, w_{\mathbf{q}} > \\
&- \quad \frac{i\, p_H^2}{p^2} \int d\mathbf{k}\, d\mathbf{l}\ \delta(\mathbf{k} + \mathbf{l} - \mathbf{p})\ I_{kl,p} < w_{\mathbf{k}}\, w_{\mathbf{l}}\, w_{\mathbf{q}} >, \\
< \dot{\xi}_{\mathbf{p}}\, \xi_{\mathbf{q}} > \quad &= \quad -N^2 < w_{\mathbf{p}}\, \xi_{\mathbf{q}} > \\
&- \quad i \int d\mathbf{k}\, d\mathbf{l}\ \delta(\mathbf{k} + \mathbf{l} - \mathbf{p})\ H_{kl} < w_{\mathbf{k}}\, \xi_{\mathbf{l}}\, \xi_{\mathbf{q}} > . \qquad (165)
\end{aligned}
$$

where $I_{kl,p} = (1 + \frac{p_3 l_3}{l_H^2} \frac{\mathbf{p} \cdot \mathbf{l}}{p_H^2})\ H_{kl}$.

By adding these two equations and their complex conjugates, we get the following kinetic equation for the energy spectral density $\epsilon_{\mathbf{p}} = \frac{p^2}{p_H^2}(b_{\mathbf{p}}\,\bar{b}_{\mathbf{p}} + b_{-\mathbf{p}}\,\bar{b}_{-\mathbf{p}})$

$$< \dot{\epsilon}_p >= \frac{i}{2} \int d\mathbf{k}\, d\mathbf{l}\, \delta(\mathbf{k}+\mathbf{l}-\mathbf{p})\, I_{kl,p}\, [< w_{-\mathbf{k}}\, w_{-\mathbf{l}}\, w_{\mathbf{p}} > - < w_{\mathbf{k}}\, w_{\mathbf{l}}\, w_{-\mathbf{p}} >]$$
$$+ \frac{i}{2N^2} \int d\mathbf{k}\, d\mathbf{l}\; \delta(\mathbf{k}+\mathbf{l}-\mathbf{p})\; H_{kl}\; [< w_{-\mathbf{k}}\, \xi_{-\mathbf{l}}\, \xi_{\mathbf{p}} > - < w_{\mathbf{k}}\, \xi_{\mathbf{l}}\, \xi_{-\mathbf{p}} >] \quad (166)$$

and calculate the r.h.s. of this equation perturbatively using (164) and Sokhotsky formulas.

We get

$$
\begin{aligned}
<\dot{\epsilon}_{\mathbf{p}}> \quad \sim \quad & 2\pi \frac{p^2}{p_H^2} \int d\mathbf{k}\, d\mathbf{l}\ \delta(\mathbf{k}+\mathbf{l}-\mathbf{p})\{ \\
B^{+}_{\{kl\},p} \quad & [B^{+}_{\{kl\},p} N_k N_l - D^{-}_{lp,k} N_p N_l - D^{-}_{kp,l} N_p N_k]\ \delta(\Omega_k + \Omega_l - \Omega_p) + \\
D^{+}_{lk,p} \quad & [D^{-}_{pk,l} N_p N_k + C^{+}_{pl,k} N_p N_l + D^{+}_{lk,p} N_k N_l]\ \delta(\Omega_l + \Omega_p - \Omega_k) + \\
D^{+}_{kl,p} \quad & [D^{-}_{pl,k} N_p N_l + C^{+}_{pk,l} N_p N_k + D^{+}_{kl,p} N_k N_l]\ \delta(\Omega_k + \Omega_p - \Omega_l)\}
\end{aligned}
\qquad (167)
$$

where $C^{\pm}_{kl,p} = B^{\pm}_{kl,p} - B^{\pm}_{lk,p}$, $D^{\pm}_{kl,p} = B^{+}_{kl,p} \pm B^{-}_{lk,p}$.

This is a typical kinetic equation with two δ-functions assuring that the collision process conserves energy and momentum. However, some transformations are necessary in order to cast it to the standard form. A change of variables should be made in order to have a standard *frequency*$\times$*wavenumber density* form for $\epsilon_{\mathbf{p}}$:

$$N_p \to N \frac{p_H^3}{p^3} N_p$$

thus giving $\epsilon_p = 2\Omega_p N_p$ in terms of new N_p which is usually interpreted as "quasiparticle" density. We correspondingly modify the interaction coefficients

$$\mathcal{B}_{kl,p} = B^{+}_{\{kl\},p} (\frac{p}{kl})^{\frac{3}{2}}, \qquad \mathcal{C}_{kl,p} = C^{+}_{kl,p} (\frac{p}{kl})^{\frac{3}{2}} \qquad \mathcal{D}^{\pm}_{kl,p} = D^{\pm}_{kl,p} (\frac{p}{kl})^{\frac{3}{2}} \tag{168}$$

and finally get:

$$
\begin{aligned}
&< \dot{N}_{\mathbf{p}} > \sim \pi N \int d\mathbf{k}\, d\mathbf{l}\ \delta(\mathbf{k}+\mathbf{l}-\mathbf{p}) \{ \\
&\mathcal{B}_{\{kl\},p} \left[\frac{k_H^3 l_H^3}{p_H^3} \mathcal{B}_{\{kl\},p} N_k N_l - k_H^3 \mathcal{D}^-_{lp,k} N_p N_l - l_H^3 \mathcal{D}^-_{kp,l} N_p N_k \right] \delta(\Omega_k + \Omega_l - \Omega_p) + \\
&\mathcal{D}^+_{lk,p} \left[k_H^3 \mathcal{D}^-_{pk,l} N_p N_k + l_H^3 \mathcal{C}^+_{pl,k} N_p N_l + \frac{k_H^3 l_H^3}{p_H^3} \mathcal{D}^+_{lk,p} N_k N_l \right] \delta(\Omega_l + \Omega_p - \Omega_k) + \\
&\mathcal{D}^+_{kl,p} \left[l_H^3 \mathcal{D}^-_{pl,k} N_p N_l + k_H^3 \mathcal{C}^+_{pk,l} N_p N_k + \frac{k_H^3 l_H^3}{p_H^3} \mathcal{D}^+_{kl,p} N_k N_l \right] \delta(\Omega_k + \Omega_p - \Omega_l) \}
\end{aligned}
\tag{169}
$$

we limit ourselves by systems of waves (wave packets) propagating almost vertically, i.e. $k_3 \gg k_H$ and, thus, having low frequencies. In this case we get

$$\frac{k_H^3 l_H^3}{p_H^3}\mathcal{B}_{kl,p} = k_H^3 \mathcal{D}^-_{kp,l} = l_H^3 \mathcal{D}^-_{lp,k}, \;\; l_H^3 \mathcal{C}_{pl,k} = -k_H^3 \mathcal{D}^-_{pk,l} = -\frac{k_H^3 l_H^3}{p_H^3}\mathcal{D}^+_{lk,p} \tag{170}$$

and the collision integral in the r.h.s. of (169) greatly simplifies

$$\begin{aligned} < \dot{N}_{\mathbf{p}} > \;\; \sim \;\; \pi N \int d\mathbf{k}\, d\mathbf{l}\; \delta(\mathbf{k}+\mathbf{l}-\mathbf{p})\{ \\ X_{kl,p}[N_k N_l - N_p N_k - N_p N_l]\; \delta(\Omega_k + \Omega_l - \Omega_p) + \\ Y^+_{kl,p}[N_k N_l + N_p N_k - N_p N_l]\; \delta(\Omega_l + \Omega_p - \Omega_k) + \\ Y^+_{lk,p}[N_k N_l + N_p N_l - N_p N_k]\; \delta(\Omega_k + \Omega_p - \Omega_l)\} \end{aligned} \tag{171}$$

Here

$$X_{kl,p} = (\tilde{k}_H + \tilde{l}_H + \tilde{p}_H)^2 \frac{(p_3^2 - k_3 l_3)^2}{16\, k\, l\, p\, k_H\, l_H\, p_H} \Big(\frac{\tilde{p}_H^2 \; - \; \tilde{k}_H\, \tilde{l}_H}{p_3^2 - k_3 l_3} p_3 - \frac{\tilde{l}_H^2}{l_3} - \frac{\tilde{k}_H^2}{k_3} \Big)^2, \tag{172}$$

$$Y^{\pm}_{kl,p} = (\tilde{k}_H - \tilde{l}_H + \tilde{p}_H)^2 \frac{(k_3^2 + l_3 p_3)^2}{16\, k\, l\, p\, k_H\, l_H\, p_H} \Big(\frac{\tilde{k}_H^2 \; \pm \; \tilde{l}_H\, \tilde{p}_H}{k_3^2 + l_3 p_3} k_3 - \frac{\tilde{p}_H^2}{p_3} \pm \frac{\tilde{l}_H^2}{l_3} \Big)^2 \tag{173}$$

and $\tilde{k}_H = sign[k_3] k_H$.

We will look for horizontally isotropic and vertically averaged stationary solutions of (171) by assuming that wavenumber densities N_p depend neither on orientation of the horizontal wavenumber in the horizontal plane nor on the sign of the vertical wavenumber. In this case we may perform explicitely all angular integrations in the r.h.s. of the kinetic equation and, by assuming a scaling form for solutions

$$N_p \sim p_H^{\alpha} \; |p_3|^{\beta} \tag{174}$$

we get:

$$
\begin{aligned}
< \dot{N}(p_H, p_3) > \sim \int dk_H\, dl_H\, dk_3\, dl_3\;\; \delta(k_3 + l_3 - p_3)\frac{k_H l_H}{\triangle}\{ \\
X_{kl,p}\Big[k_H^{\alpha}\, l_H^{\alpha}\, k_3^{\beta}\, l_3^{\beta} - k_H^{\alpha}\, p_H^{\alpha}\, k_3^{\beta}\, p_3^{\beta} - l_H^{\alpha}\, p_H^{\alpha}\, l_3^{\beta}\, p_3^{\beta}\Big] \\
\times\Big[1-(\frac{p_H}{k_H})^{2\alpha+6}(\frac{p_3}{k_3})^{2\beta+2}-(\frac{p_H}{l_H})^{2\alpha+6}(\frac{p_3}{k_3})^{2\beta+2}\Big]\;\delta\Big(\frac{k_H}{k3}+\frac{l_H}{l_3}-\frac{p_H}{p_3}\Big)\}+ \\
2\int dk_H\, dl_H\, dk_3\, dl_3\;\; \delta(k_3 + l_3 - p_3)\frac{k_H l_H}{\triangle}\{ \\
Y^{+}_{kl,p}\;\Big[k_H^{\alpha}\, l_H^{\alpha}\, k_3^{\beta}\, l_3^{\beta} + k_H^{\alpha}\, p_H^{\alpha}\, k_3^{\beta}\, p_3^{\beta} - l_H^{\alpha}\, p_H^{\alpha}\, l_3^{\beta}\, p_3^{\beta}\Big] \\
\times\Big[1-(\frac{p_H}{k_H})^{2\alpha+6}(\frac{p_3}{k_3})^{2\beta+2}+(\frac{p_H}{l_H})^{2\alpha+6}(\frac{p_3}{l_3})^{2\beta+2}\Big]\;\delta\Big(\frac{l_H}{l_3}+\frac{p_H}{p_3}-\frac{k_H}{k_3}\Big)\}
\end{aligned}
\tag{175}
$$

with $\triangle = \frac{1}{4}[2(p_H^2 k_H^2 + p_H^2 l_H^2 + l_H^2 k_H^2) - p_H^4 - k_H^4 - l_H^4]^{\frac{1}{2}}$

A direct substitution shows that there exist two sets of scaling exponents providing solutions. One annihilating the first bracket is a Rayleigh- Jeans type solution $\alpha = -1$, $\beta = 1$ leading to the equipartition of energy:

$$N_k \sim \frac{k_3}{k_H}, \qquad \epsilon_k = const \tag{176}$$

One annihilating the second bracket is a Kolmogorov-type solution $\alpha = -\frac{7}{2}$, $\beta = -\frac{1}{2}$ giving

$$N_k \sim k_H^{-\frac{7}{2}} k_3^{-\frac{1}{2}}, \qquad \epsilon_k \sim k_H^{-\frac{5}{2}} k_3^{-\frac{3}{2}}. \tag{177}$$

A simple check of dimensions shows that it corresponds to a constant energy flux through the wave spectrum

3.3 Historical remarks and bibliography

The presentation in this section follows the paper of Caillol and Zeitlin (2000) where the kinetic equations in Eulerian variables were obtained and power-law spectra for internal gravity waves were first derived. Power-law spectra for *unidirectional* internal gravity waves were derived by Daubner and zeitlin (1996). The weak turbulence ideas were first applied to the internal gravity waves in the stratified fluid by Pelinovsky and Raevsky (1977) by using Clebsch variables. There is a huge literature on observed atmospheric and oceanic spectra, cf. e.g. Bacmeister, J.T. *et al*, 1996, for the atmosphere. The resonant interactions and kinetics of the internal waves in the ocean were intensively discussed in the context of theoretical explanation of the celebrated empirical Garrett - Munk spectrum, cf. e.g. the review Müller, P. *et al*, 1986, and references therein.

There are, however, some pecularities of applying Hamiltonian formalism to non-canonical Hamiltonian system, such as (126) which were not adressed in the earlier studies. Lagrangian variables used, e.g. in Olbers (1976) and other works, are subject to incompressibility constraint and are not canonical, which makes their use rather tricky. The search of exact power-law solutions of the kinetic equations was never undertaken either.

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