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Weak turbulence of short equatorial waves

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Abstract

We derive a normal form of nonlinear equations for short equatorial waves considered in the framework of the rotating shallow water model. We show dynamical splitting of equatorial Rossby and inertia-gravity waves. We derive an effective Hamiltonian for the short inertia-gravity waves and consider their kinetics using the weak turbulence approach. Stationary power-law energy spectra are obtained. They have different slopes for eastward and westward propagating waves due to the fact that resonant triads of inertia-gravity waves exist only in specific regions of the phase-space.

1. Introduction

The equatorial region in the atmosphere and ocean is dynamically special because the vertical component of the Earth's angular velocity changes sign at the equator. This leads to the appearance of the whole family of special waves propagating in the so-called equatorial wave-guide, cf. [1]. The equatorial waves are known to play an important role in the dynamical processes both in the ocean and in the atmosphere which determine the Earth climate. In the ocean, they are crucial for El Niño phenomenon, [2,3], while in the atmosphere they are at the origin of such fundamental phenomena as tropospheric Madden–Julian oscillation, [4] and stratospheric quasi-biennial oscillation (QBO) [5].

In the present Letter we will concentrate on short equatorial waves. Due to the confined geometry of the equatorial wave-guide their dynamics is a good candidate for the application of the wave (or weak) turbulence approach which requires a large ensemble of weakly nonlinear random-phase waves. The form of energy and momentum spectra of short equatorial inertia-gravity waves is of utmost importance, e.g., in the QBO modeling [7], and the

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theory of wave turbulence provides the tools for obtaining such spectra. In what follows, in order to describe equatorial waves and their interactions we will use the simplest possible rotating shallow water model. It is worth noting, that in spite of its highly idealized character, this model is frequently used in interpreting data and fits the observations to a good extent [8,9].

2. Wave turbulence: a reminder

We remind here the basic notions of the wave turbulence approach which will be used in what follows. The material is standard, see [10,11] for details. A systematic (although not unique) way to realize the weak turbulence approach is through the Hamiltonian description. The full Hamiltonian H of the system expressed in terms of the Fourier-transforms $a_{\mathbf{k}}(t)$ of the physical fields contains the Hamiltonian of free waves:

$$H_2 = \int d\mathbf{k}\,\omega(\mathbf{k}) a_{\mathbf{k}} a^*_{\mathbf{k}}, \tag{1}$$

and the interaction term H_{int} which may be expanded in powers of $a_{\mathbf{k}}$ and its complex conjugate $a^*_{\mathbf{k}}$: $H_{\text{int}} = H_3 + H_4 + \cdots$ with

$$\begin{aligned} H_3 = {} & \frac{1}{2}\int d\mathbf{k}\,d\mathbf{k}_1\,d\mathbf{k}_2\,\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2) V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} a^*_{\mathbf{k}} a_{\mathbf{k}_1} a_{\mathbf{k}_2} + \text{c.c.} \\ & + \frac{1}{2}\int d\mathbf{k}\,d\mathbf{k}_1\,d\mathbf{k}_2\,\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2) U_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} a_{\mathbf{k}} a_{\mathbf{k}_1} a_{\mathbf{k}_2} + \text{c.c.}, \end{aligned} \tag{2}$$

$$H_4 = \frac{1}{2}\int d\mathbf{k}\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3\,\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} a^*_{\mathbf{k}} a^*_{\mathbf{k}_1} a_{\mathbf{k}_2} a_{\mathbf{k}_3}. \tag{3}$$

The frequency spectrum $\omega(\mathbf{k})$ is given by the dispersion relation. It is of *decay* type if the following equation:

$$\omega(\mathbf{k}_1+\mathbf{k}_2) = \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) \tag{4}$$

has (nonzero) solutions. Otherwise the spectrum is of *non-decay* type.

The equations of motion are, correspondingly,

$$\dot{a}_{\mathbf{k}} = -i\omega(\mathbf{k}) a_{\mathbf{k}} - i\frac{\delta H_{\text{int}}}{\delta a^*_{\mathbf{k}}}. \tag{5}$$

The wave-amplitudes are always supposed to be small and, hence, the interaction Hamiltonian gives only small corrections to the linear solutions.

The main hypothesis of the weak turbulence theory is that weak interactions of a large number of waves lead to the phase randomization (a central limit theorem-like argument) and result in the Gaussian statistics of the wave field. From the full dynamical description (5) one passes then to a statistical description, where the system is described by a set of correlation functions of complex amplitudes.

Gaussianity for spatially uniform medium means that all odd-order correlators vanish and that the even-order correlators are expressed in terms of the (real) quadratic one: $\langle a_{\mathbf{k}} a^*_{\mathbf{k}'}\rangle = N(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}')$. The main goal of the wave turbulence theory is, thus, to determine the evolution of this quantity which is given by the kinetic equation:

$$\dot{N}(\mathbf{k}) = \mathcal{I}\big[N(\mathbf{k})\big], \tag{6}$$

where expression in the r.h.s. is called collision integral. Obviously, the stationary solutions of the kinetic equation, i.e., such $N(\mathbf{k})$ that annihilate the collision integral, are of importance.

In the case of the decay spectrum it is sufficient to retain the cubic interactions only and the following three-wave collision integral results:

$$\mathcal{I}^{(3)}\big[N(\mathbf{k})\big] = \int d\mathbf{k}_1\,d\mathbf{k}_2\,[W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} f_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} - W_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}} f_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}} - W_{\mathbf{k}_2\mathbf{k}\mathbf{k}_1} f_{\mathbf{k}_2\mathbf{k}\mathbf{k}_1}], \tag{7}$$

where

$$W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = 2\pi |V_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}|^2 \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\delta\big(\omega(\mathbf{k}) - \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2)\big), \tag{8}$$

and

$$f_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = N(\mathbf{k})N(\mathbf{k}_1)N(\mathbf{k}_2)\big(N^{-1}(\mathbf{k}) - N^{-1}(\mathbf{k}_1) - N^{-1}(\mathbf{k}_2)\big). \tag{9}$$

In the non-decay spectrum case, the interaction Hamiltonian, in general, contains both H_3 and H_4 contributions. The cubic terms in the Hamiltonian may be removed by a canonical transformation of the dynamical variables [10], and the four-wave collision integral results

$$\mathcal{I}^{(4)}\big[N(\mathbf{k})\big] = \pi \int d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3\, W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} f_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}, \tag{10}$$

where

$$W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} = |T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}|^2 \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\delta\big(\omega(\mathbf{k}) + \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2) - \omega(\mathbf{k}_3)\big), \tag{11}$$

with

$$\begin{aligned} T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} = {} & -2\frac{V_{\mathbf{k}+\mathbf{k}_1,\mathbf{k},\mathbf{k}_1} V^*_{\mathbf{k}_2+\mathbf{k}_3,\mathbf{k}_2,\mathbf{k}_3}}{\omega(\mathbf{k}+\mathbf{k}_1) - \omega(\mathbf{k}) - \omega(\mathbf{k}_1)} - 2\frac{V_{\mathbf{k},\mathbf{k}_2,\mathbf{k}-\mathbf{k}_2} V^*_{\mathbf{k}_3,\mathbf{k}_1,\mathbf{k}_3-\mathbf{k}_1}}{\omega(\mathbf{k}_3 - \mathbf{k}_1) + \omega(\mathbf{k}_1) - \omega(\mathbf{k}_3)} \\ & - 2\frac{V_{\mathbf{k},\mathbf{k}_3,\mathbf{k}-\mathbf{k}_3} V^*_{\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2-\mathbf{k}_1}}{\omega(\mathbf{k}_2 - \mathbf{k}_1) + \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2)} - 2\frac{V_{\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_1-\mathbf{k}_3} V^*_{\mathbf{k}_2,\mathbf{k},\mathbf{k}_2-\mathbf{k}}}{\omega(\mathbf{k}_3 - \mathbf{k}_1) + \omega(\mathbf{k}_1) - \omega(\mathbf{k}_3)} \\ & - 2\frac{V_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1-\mathbf{k}_2} V^*_{\mathbf{k}_3,\mathbf{k},\mathbf{k}_3-\mathbf{k}}}{\omega(\mathbf{k}_2 - \mathbf{k}_1) + \omega(\mathbf{k}_1) - \omega(\mathbf{k}_2)} - 2\frac{U_{-\mathbf{k}-\mathbf{k}_1,\mathbf{k},\mathbf{k}_1} U^*_{-\mathbf{k}_2-\mathbf{k}_3,\mathbf{k}_2,\mathbf{k}_3}}{\omega(\mathbf{k}+\mathbf{k}_1) + \omega(\mathbf{k}) + \omega(\mathbf{k}_1)} + W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}, \end{aligned} \tag{12}$$

and

$$f_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} = N(\mathbf{k})N(\mathbf{k}_1)N(\mathbf{k}_2)N(\mathbf{k}_3)\big(N^{-1}(\mathbf{k}) + N^{-1}(\mathbf{k}_1) - N^{-1}(\mathbf{k}_2) - N^{-1}(\mathbf{k}_3)\big). \tag{13}$$

A standard method to look for stationary solutions of the kinetic equation, both in decay and nondecay cases is factorization. The integrand of the collision integral, by transformation of the integration variables, either in terms of frequencies [10], or in terms of wavevectors [12], is cast to the form of a product of several factors, each of which may be annihilated by some choice of $N(\mathbf{k})$, taking into account the frequency- and wavenumber delta-functions which are always present in the integrand. In its classical form the method is applied to the class of homogeneous dispersion laws of the form $\omega \propto |\mathbf{k}|^\alpha$. Two types of stationary power-law spectra are usually found in this way: the equilibrium spectra with equipartition of energy, and the Kolmogorov-type spectra with energy or wave-action flux through the spectrum.

3. Weak turbulence of equatorial waves in the rotating shallow water model

3.1. *Rotating shallow water model on the equatorial β-plane, a reminder*

The physics of the model consists in horizontal momentum and mass conservation in a layer of shallow water of constant density (equal to unity in what follows) on the tangent plane to a rotating planet in the hydrostatic approximation. Rotation enters via the Coriolis force. The centrifugal force is absorbed into effective gravity g. Stratification is rudimentary and is reduced to a single isopycnal surface $z = h(x, y, t)$, the free surface. Here x and y are zonal and meridional directions, respectively. The fluid depth at rest is h_0. Equations of motion in the absence of dissipation are

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + f(y)\hat{\mathbf{z}} \wedge \mathbf{v} + g\nabla h = 0, \tag{14}$$

$$\partial_t h + \nabla \cdot (\mathbf{v}h) = 0. \tag{15}$$

The Coriolis parameter is $f(y)$, its meridional dependence is the only remnant of the planet's sphericity. On the equatorial tangent plane $f = \beta y$.

If boundary conditions of exponential decay far from the equator (equatorial wave-guide) are imposed, the linearization of (14), (15) using the decomposition of all the fields in the parabolic cylinder functions of the form

$$\phi_n(y) = \frac{H_n(y)e^{-y^2/2}}{\sqrt{2^n n!\sqrt{\pi}}}, \tag{16}$$

where H_n are Hermite polynomials, gives the following wave solutions (dispersion relations are given in non-dimensional units) [1,6]:

- Kelvin waves with linear dispersion $\omega = k$,
- Yanai waves with the dispersion law $\omega^2 - k\omega - 1 = 0$,
- Rossby and inertia-gravity waves with the dispersion law

  $$\omega^3 - \big(k^2 + (2n+1)\big)\omega - k = 0$$

  (the lower frequency: Rossby waves, the higher frequencies: inertia-gravity waves).

Therefore, the inertia-gravity waves (IGW) are strongly dispersive at long wave-lengths and weakly dispersive at short wave-lengths. The Rossby waves at short wave-lengths are strongly-dispersive and strongly spatially anisotropic. The Kelvin waves are rigorously nondispersive. The short Yanai waves rejoin the Rossby-waves family for negative k_x and the IGW family for positive k_x (see Fig. 1).

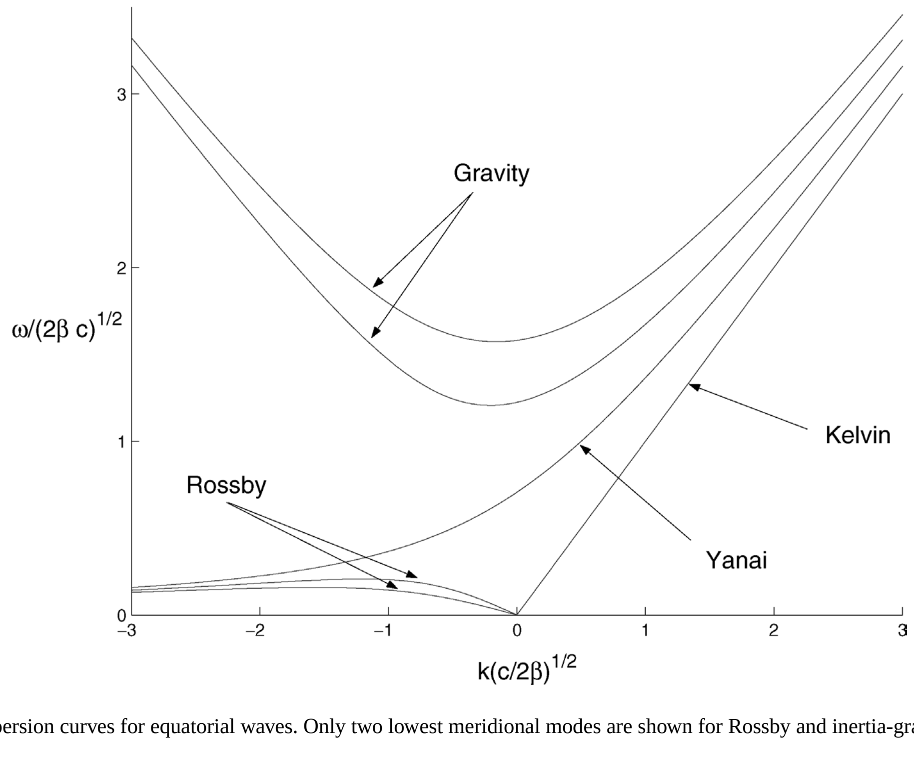

Fig. 1. Dispersion curves for equatorial waves. Only two lowest meridional modes are shown for Rossby and inertia-gravity waves.

Below, we will apply the wave turbulence approach to the short equatorial waves. They are still confined in the equatorial wave-guide but their spectrum may be considered as continuous. Being continuously generated in the atmosphere by deep tropical convection and topography, they are likely to form a "wave soup" which is a key element of the wave turbulence approach. As is clear from Fig. 1, the short IGW are almost nondispersive, in contradistinction with the short Rossby waves. This is not surprising, because the system (14) in the absence of rotation is just a 2d gas dynamics and, hence, in this case the IGW are analogous to the sound waves with the "sound speed" $c = \sqrt{gh_0}$.

3.2. *The normal form of the equations for short equatorial waves*

The Hamiltonian structure of hydrodynamics on the β-plane, the equatorial β-plane being a particular case of it, is known to be noncanonical (cf. [13] and references therein). Hence, the application of the above-exposed formalism is not straightforward and the dynamical system should be first processed. We, hence, try to bring the dynamical equations (14) to the standard normal form.

The nondimensional equations for the velocities u, v and the deviation of the height field from the rest value $z = \frac{h-h_0}{h_0}$ on the equatorial beta-plane are

$$u_t + uu_x + vu_y - \beta yv + z_x = 0, \tag{17}$$

$$v_t + uv_x + vv_y + \beta yu + z_y = 0, \tag{18}$$

$$z_t + u_x + v_y + (uz)_x + (vz)_y = 0. \tag{19}$$

For simplicity we suppose that there is a unique small parameter ϵ, and assume weak nonlinearity $u, v, z \sim \epsilon$ and weak inhomogeneity $\beta \sim \epsilon$. The leading-order part of the system (17)–(19) is a system with constant coefficients. The smallness of the β-term means, in fact, that we are considering motions with a characteristic scale small with respect to the equatorial deformation radius $R_e = \frac{(gh_0)^{1/4}}{\sqrt{\beta}}$.

Introducing the Fourier-transforms

$$\big(u(\mathbf{r}), v(\mathbf{r}), z(\mathbf{r})\big) = \int (u_\mathbf{k}, v_\mathbf{k}, z_\mathbf{k}) \exp(i\mathbf{kr})\, d\mathbf{k}, \tag{20}$$

where $\mathbf{r} = (x, y)$ and $\mathbf{k} = (k_1, k_2)$ and the integration is over the whole plane, we rewrite Eqs. (17)–(19) in a symmetric form

$$\begin{pmatrix} \partial u_\mathbf{k}/\partial t \\ \partial v_\mathbf{k}/\partial t \\ \partial z_\mathbf{k}/\partial t \end{pmatrix} + \begin{pmatrix} 0 & -i\beta\frac{\partial}{\partial k_2} & ik_1 \\ i\beta\frac{\partial}{\partial k_2} & 0 & ik_2 \\ ik_1 & ik_2 & 0 \end{pmatrix} \begin{pmatrix} u_\mathbf{k} \\ v_\mathbf{k} \\ z_\mathbf{k} \end{pmatrix} + \begin{pmatrix} ik_1/2\int(u_\mathbf{l}u_\mathbf{m} + v_\mathbf{l}v_\mathbf{m})\,d\lambda - \int \Omega_\mathbf{l}v_\mathbf{m}\,d\lambda \\ ik_2/2\int(u_\mathbf{l}u_\mathbf{m} + v_\mathbf{l}v_\mathbf{m})\,d\lambda + \int \Omega_\mathbf{l}u_\mathbf{m}\,d\lambda \\ ik_1\int z_\mathbf{l}u_\mathbf{m}\,d\lambda + ik_2\int z_\mathbf{l}v_\mathbf{m}\,d\lambda \end{pmatrix} = 0, \tag{21}$$

where $\Omega = v_x - u_y$, $\Omega_\mathbf{l} = il_1v_\mathbf{l} - il_2u_\mathbf{l}$, $d\lambda = \delta(\mathbf{k} - \mathbf{l} - \mathbf{m})\, d\mathbf{l}\, d\mathbf{m}$.

We diagonalize the main part by a change of variables

$$\begin{pmatrix} u_\mathbf{k} \\ v_\mathbf{k} \\ z_\mathbf{k} \end{pmatrix} = \begin{pmatrix} \frac{-ik_2}{|\mathbf{k}|} & \frac{k_1}{\sqrt{2}|\mathbf{k}|} & \frac{-k_1}{\sqrt{2}|\mathbf{k}|} \\ \frac{ik_1}{|\mathbf{k}|} & \frac{k_2}{\sqrt{2}|\mathbf{k}|} & \frac{-k_2}{\sqrt{2}|\mathbf{k}|} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} a_\mathbf{k} \\ b_\mathbf{k} \\ c_\mathbf{k} \end{pmatrix}. \tag{22}$$

For real-valued u, v, z we have from (22)

$$a_\mathbf{k} = \bar{a}_{-\mathbf{k}}, \qquad c_\mathbf{k} = \bar{b}_{-\mathbf{k}}. \tag{23}$$

As a result of the diagonalization we get

$$\begin{pmatrix} \partial a_{\mathbf{k}}/\partial t \\ \partial b_{\mathbf{k}}/\partial t \\ \partial c_{\mathbf{k}}/\partial t \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & i|\mathbf{k}| & 0 \\ 0 & 0 & -i|\mathbf{k}| \end{pmatrix} \begin{pmatrix} a_{\mathbf{k}} \\ b_{\mathbf{k}} \\ c_{\mathbf{k}} \end{pmatrix} + \beta \begin{pmatrix} -\frac{ik_1}{|\mathbf{k}|^2} & \frac{1}{\sqrt{2}}\frac{\partial}{\partial k_2} & -\frac{1}{\sqrt{2}}\frac{\partial}{\partial k_2} \\ \frac{1}{\sqrt{2}}\frac{\partial}{\partial k_2} & -\frac{ik_1}{2|\mathbf{k}|^2} & \frac{ik_1}{2|\mathbf{k}|^2} \\ -\frac{1}{\sqrt{2}}\frac{\partial}{\partial k_2} & \frac{ik_1}{2|\mathbf{k}|^2} & -\frac{ik_1}{2|\mathbf{k}|^2} \end{pmatrix} \begin{pmatrix} a_{\mathbf{k}} \\ b_{\mathbf{k}} \\ c_{\mathbf{k}} \end{pmatrix} + NL = 0. \tag{24}$$

Here the nonlinear terms NL have the form:

$$NL = \begin{pmatrix} \int (U^{(0)}_{\mathbf{klm}} a_{\mathbf{l}} a_{\mathbf{m}} + U^{(1)}_{\mathbf{klm}} a_{\mathbf{l}} b_{\mathbf{m}} + U^{(2)}_{\mathbf{klm}} a_{\mathbf{l}} c_{\mathbf{m}})\, d\lambda \\ \int (V^{(0)}_{\mathbf{klm}} a_{\mathbf{l}} b_{\mathbf{m}} + V^{(1)}_{\mathbf{klm}} a_{\mathbf{l}} a_{\mathbf{m}} + V^{(2)}_{\mathbf{klm}} a_{\mathbf{l}} c_{\mathbf{m}})\, d\lambda \\ \int (W^{(0)}_{\mathbf{klm}} a_{\mathbf{l}} c_{\mathbf{m}} + W^{(1)}_{\mathbf{klm}} a_{\mathbf{l}} a_{\mathbf{m}} + W^{(2)}_{\mathbf{klm}} a_{\mathbf{l}} b_{\mathbf{m}})\, d\lambda \end{pmatrix} + \begin{pmatrix} 0 \\ \int (V^{(3)}_{\mathbf{klm}} b_{\mathbf{l}} b_{\mathbf{m}} + V^{(4)}_{\mathbf{klm}} c_{\mathbf{l}} c_{\mathbf{m}} + V^{(5)}_{\mathbf{klm}} b_{\mathbf{l}} c_{\mathbf{m}})\, d\lambda \\ \int (W^{(3)}_{\mathbf{klm}} c_{\mathbf{l}} c_{\mathbf{m}} + W^{(4)}_{\mathbf{klm}} b_{\mathbf{l}} b_{\mathbf{m}} + W^{(5)}_{\mathbf{klm}} c_{\mathbf{l}} b_{\mathbf{m}})\, d\lambda \end{pmatrix}. \tag{25}$$

with interaction coefficients which can be easily found from (21), (22).

Thus, the variable $a_{\mathbf{k}}$ describes the short equatorial Rossby waves with the dispersion law $\Omega_{\mathbf{k}} = -\frac{\beta k_1}{|\mathbf{k}|^2}$ and the variables $b_{\mathbf{k}}$, $c_{\mathbf{k}}$ describe the short inertia-gravity waves with the dispersion law $\omega_{\mathbf{k}} = |\mathbf{k}| - \frac{\beta k_1}{2|\mathbf{k}|^2}$.

The linear part of (24) may be diagonalized by an additional change of variables

$$\begin{pmatrix} a_{\mathbf{k}} \\ b_{\mathbf{k}} \\ c_{\mathbf{k}} \end{pmatrix} \to \begin{pmatrix} a_{\mathbf{k}} \\ b_{\mathbf{k}} \\ c_{\mathbf{k}} \end{pmatrix} + \beta \begin{pmatrix} 0 & -\frac{\partial}{\partial k_2}\frac{1}{\sqrt{2}|\mathbf{k}|} & -\frac{\partial}{\partial k_2}\frac{1}{\sqrt{2}|\mathbf{k}|} \\ \frac{i}{\sqrt{2}|\mathbf{k}|}\frac{\partial}{\partial k_2} & 0 & -\frac{k_1}{4|\mathbf{k}|^3} \\ \frac{i}{\sqrt{2}|\mathbf{k}|}\frac{\partial}{\partial k_2} & \frac{k_1}{4|\mathbf{k}|^3} & 0 \end{pmatrix} \begin{pmatrix} a_{\mathbf{k}} \\ b_{\mathbf{k}} \\ c_{\mathbf{k}} \end{pmatrix}. \tag{26}$$

The system at the leading order takes the form

$$\begin{aligned} &\begin{pmatrix} \partial a_{\mathbf{k}}/\partial t \\ \partial b_{\mathbf{k}}/\partial t \\ \partial c_{\mathbf{k}}/\partial t \end{pmatrix} + \begin{pmatrix} i\Omega_{\mathbf{k}} & 0 & 0 \\ 0 & i\omega_{\mathbf{k}} & 0 \\ 0 & 0 & -i\omega_{-\mathbf{k}} \end{pmatrix} \begin{pmatrix} a_{\mathbf{k}} \\ b_{\mathbf{k}} \\ c_{\mathbf{k}} \end{pmatrix} + \begin{pmatrix} \int (U^{(0)}_{\mathbf{klm}} a_{\mathbf{l}} a_{\mathbf{m}} + U^{(1)}_{\mathbf{klm}} a_{\mathbf{l}} b_{\mathbf{m}} + U^{(2)}_{\mathbf{klm}} a_{\mathbf{l}} c_{\mathbf{m}})\, d\lambda \\ \int (V^{(0)}_{\mathbf{klm}} a_{\mathbf{l}} b_{\mathbf{m}} + V^{(1)}_{\mathbf{klm}} a_{\mathbf{l}} a_{\mathbf{m}} + V^{(2)}_{\mathbf{klm}} a_{\mathbf{l}} c_{\mathbf{m}})\, d\lambda \\ \int (W^{(0)}_{\mathbf{klm}} a_{\mathbf{l}} c_{\mathbf{m}} + W^{(1)}_{\mathbf{klm}} a_{\mathbf{l}} a_{\mathbf{m}} + W^{(2)}_{\mathbf{klm}} a_{\mathbf{l}} b_{\mathbf{m}})\, d\lambda \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 \\ \int (V^{(3)}_{\mathbf{klm}} b_{\mathbf{l}} b_{\mathbf{m}} + V^{(4)}_{\mathbf{klm}} c_{\mathbf{l}} c_{\mathbf{m}} + V^{(5)}_{\mathbf{klm}} b_{\mathbf{l}} c_{\mathbf{m}})\, d\lambda \\ \int (W^{(3)}_{\mathbf{klm}} c_{\mathbf{l}} c_{\mathbf{m}} + W^{(4)}_{\mathbf{klm}} b_{\mathbf{l}} b_{\mathbf{m}} + W^{(5)}_{\mathbf{klm}} c_{\mathbf{l}} b_{\mathbf{m}})\, d\lambda \end{pmatrix} = 0. \end{aligned} \tag{27}$$

Hence, the Rossby waves split out, i.e., if $a_{\mathbf{k}} = 0$ at the initial moment, then $a_{\mathbf{k}} = 0$ for all times while these equations are applicable. As a result we obtain the closed system of equations for the short inertia-gravity waves

$$\begin{pmatrix} \partial b_{\mathbf{k}}/\partial t \\ \partial c_{\mathbf{k}}/\partial t \end{pmatrix} + \begin{pmatrix} i\omega_{\mathbf{k}} & 0 \\ 0 & -i\omega_{-\mathbf{k}} \end{pmatrix} \begin{pmatrix} b_{\mathbf{k}} \\ c_{\mathbf{k}} \end{pmatrix} + \begin{pmatrix} \int (V^{(3)}_{\mathbf{klm}} b_{\mathbf{l}} b_{\mathbf{m}} + V^{(4)}_{\mathbf{klm}} c_{\mathbf{l}} c_{\mathbf{m}} + V^{(5)}_{\mathbf{klm}} b_{\mathbf{l}} c_{\mathbf{m}})\, d\lambda \\ \int (W^{(3)}_{\mathbf{klm}} c_{\mathbf{l}} c_{\mathbf{m}} + W^{(4)}_{\mathbf{klm}} b_{\mathbf{l}} b_{\mathbf{m}} + W^{(5)}_{\mathbf{klm}} c_{\mathbf{l}} b_{\mathbf{m}})\, d\lambda \end{pmatrix} = 0. \tag{28}$$

These equations are equivalent to the following Hamiltonian system in terms of two space–time variables $\varphi(\mathbf{r}, t)$, $z(\mathbf{r}, t)$:

$$\begin{pmatrix} \dot{\varphi} \\ \dot{z} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta H/\delta\varphi \\ \delta H/\delta z \end{pmatrix} = 0, \tag{29}$$

with the Hamiltonian

$$H = \int \left(\frac{1}{2}(1+z)\left(\varphi_x^2 + \varphi_y^2\right) + \frac{1}{2} z^2 + \beta\varphi \Delta^{-1} z_x \right) dx\, dy. \tag{30}$$

In terms of the Fourier-transforms

$$\begin{pmatrix} \varphi_{\mathbf{k}} \\ z_{\mathbf{k}} \end{pmatrix} = \begin{pmatrix} \frac{i}{\sqrt{2k}} & -\frac{i}{\sqrt{2k}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} b_{\mathbf{k}} \\ c_{\mathbf{k}} \end{pmatrix} \tag{31}$$

these equations have the standard form (5). For real initial variables we have $c_{\mathbf{k}} = b^*_{-\mathbf{k}}$ therefore $b_{\mathbf{k}}$ can be considered as a single independent variable. The Hamiltonian is $H = H_2 + H_3$, with

$$H_2 = \int \omega_{\mathbf{k}} |b_{\mathbf{k}}|^2 \, d\mathbf{k}, \quad \omega_{\mathbf{k}} = k - \frac{\beta}{2} \frac{k_x}{k^2}, \; k = \sqrt{k_x^2 + k_y^2}, \tag{32}$$

where the frequency $\omega_{\mathbf{k}}$ is positive for the short waves. H_3 is of the form (2), where in the leading order in β

$$2U_{123} = V_{123} = \sqrt{18} \frac{k_1(\mathbf{k}_2, \mathbf{k}_3) + k_2(\mathbf{k}_3, \mathbf{k}_1) + k_3(\mathbf{k}_1, \mathbf{k}_2)}{\sqrt{k_1 k_2 k_3}}, \tag{33}$$

and an obvious short-hand notation for the indices is used.

Therefore, the standard methods of the weak turbulence theory may be applied to the equatorial IGW. The results will, however, depend on the decay or nondecay character of the dispersion law (32). The following analysis shows that the dispersion law changes its type depending on orientation of the wave-vectors triad and that, in spite of the almost acoustic character of the short equatorial IGW, they can form nontrivial resonant triads.

3.3. Existence of resonant triads for short equatorial IGW

Consider the resonance conditions

$$\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2, \qquad \omega_{\mathbf{k}} = \omega_1 + \omega_2, \tag{34}$$

and write them in the new basis formed by $\mathbf{k} = (k_x, k_y)$ and $\mathbf{k}_\perp = (-k_y, k_x)$. Then

$$k = k_1 \cos\theta_1 + k_2 \cos\theta_2, \qquad 0 = k_1 \sin\theta_1 + k_2 \sin\theta_2, \tag{35}$$

$$k_1 + k_2 - k - \frac{\beta}{2} \left(\frac{\cos(\alpha + \theta_1)}{k_1} + \frac{\cos(\alpha + \theta_2)}{k_2} - \frac{\cos\alpha}{k} \right) = 0, \tag{36}$$

where α is the angle between $\mathbf{k}$ and the direction $\mathbf{e}_x$ and $\cos\theta_j = (\frac{\mathbf{k}}{k}, \frac{\mathbf{k}_j}{k_j})$, $\sin\theta_j = (\frac{\mathbf{k}_\perp}{k}, \frac{\mathbf{k}_j}{k_j})$, $j = 1, 2$. We rewrite the frequency equation as

$$\left(2k_1 \sin^2\frac{\theta_1}{2} + 2k_2 \sin^2\frac{\theta_2}{2} \right) - \frac{\beta\cos\alpha}{2} \left(\frac{\cos\theta_1}{k_1} + \frac{\cos\theta_2}{k_2} - \frac{1}{k} \right) - \frac{\beta\sin\alpha}{2} \left(\frac{\sin\theta_1}{k_1} + \frac{\sin\theta_2}{k_2} \right) = 0. \tag{37}$$

The characteristic scales are $k_1 \sim k_2 \sim 1 \gg \beta > 0$, therefore angles are small: $\theta_1, \theta_2 \ll 1$. For small θ_j the resonance equations are simplified:

$$k = k_1 + k_2, \qquad 0 = k_1\theta_1 + k_2\theta_2, \tag{38}$$

$$k_1\theta_1^2/2 + k_2\theta_2^2/2 - \frac{\beta\cos\alpha}{2} \left(\frac{1}{k_1} + \frac{1}{k_2} - \frac{1}{k} \right) - \frac{\beta\sin\alpha}{2} \left(\frac{\theta_1}{k_1} + \frac{\theta_2}{k_2} \right) = 0. \tag{39}$$

Thus two situations are possible. First, assume that the third bracket in (37) is small: $\cos\alpha \gg \theta_j \sin\alpha$. Then the first and the second ones are in balance: $\theta_j^2 \sim \beta\cos\alpha$, and

$$\frac{\cos\alpha}{\sin^2\alpha} \gg \beta, \tag{40}$$

i.e., $\mathbf{k}$ in this case is situated in a rather large sector around the x-axis.

Second, if the second bracket is small: $\cos\alpha \ll \theta_j \sin\alpha$, then $\theta_j \sim \beta\sin\alpha$ and therefore

$$\frac{\cos\alpha}{\sin^2\alpha} \ll \beta, \tag{41}$$

i.e., $\mathbf{k}$ is situated in a narrow sector around the y-axis in this case.

In the first case we have the following balance at the main order

$$2k_1 \sin^2\frac{\theta_1}{2} + 2k_2 \sin^2\frac{\theta_2}{2} = \frac{\beta\cos\alpha}{2}\left(\frac{1}{k_1} + \frac{1}{k_2} - \frac{1}{k}\right) \tag{42}$$

and because of

$$\frac{1}{k_1} + \frac{1}{k_2} - \frac{1}{k_1+k_2} = \frac{(k_1+k_2/2)^2 + 3k_2^2/4}{k_1k_2(k_1+k_2)} > 0$$

solution exists only for $\cos\alpha > 0$.

In the second case we have

$$2k_1 \sin^2\frac{\theta_1}{2} + 2k_2 \sin^2\frac{\theta_2}{2} = \frac{\beta\sin\alpha}{2}\left(\frac{\sin\theta_1}{k_1} + \frac{\sin\theta_2}{k_2}\right) \tag{43}$$

and $\theta_1, \theta_2 \sim \beta \ll 1$. Existence of solution is independent of the sign of $\sin\alpha$ because there always exist a trivial solution $\theta_1 = \theta_2 = 0$.

Hence, the resonant triads exist (1) in a relatively wide sector around the x-axis in the right half-plane in the **k**-space; (2) in two narrow segments around the y-axis. Apart from the narrow regions around the y-axis, there are no resonant triads in the left half-plane in the **k**-space. The resonant triads are formed by almost parallel wave-vectors.

3.4. *The three-wave kinetic equation for equatorial IGW*

The kinetic equations are, therefore, different in the regions of the phase-space where the resonant triads are allowed and forbidden, respectively. In the first case the kinetic equation with the three-wave collision integral (7) applies. Dispersion is weak, and techniques similar to those used to factorize the *four-wave* collision integral in [14] may be applied for $N(\mathbf{k}) = k^s$ to the *three-wave* one in the case $\cos\alpha \gg \beta\sin^2\alpha$.

To construct a transformation from one resonance surface to another one in (7) we rewrite the delta-functions $\delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)$ and $\delta(\omega_k - \omega_1 - \omega_2)$ in the basis $(\mathbf{k}, \mathbf{k}_\perp)$. The projections of the argument of $\delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)$ on $\mathbf{k}$ and $\mathbf{k}_\perp$ are given in (35). The argument of the frequency delta-function in the new basis is given by (36).

Thus, in order to transform, for example, the third integral in (7) we take the vector $\mathbf{k}$ and consider the resonance surface $\mathbf{k} + \mathbf{q}_1 = \mathbf{q}_2$ (to avoid confusion we use here a different notation $\mathbf{q}_{1,2}$ for dummy integration variables). We redirect the axes along $\mathbf{q}_2$ and its transverse $\mathbf{q}_2^\perp$. In this way we get for the wavenumber delta-function

$$q_2 = k\cos\xi_1 + q_1\cos\xi_2, \qquad 0 = k\sin\xi_1 + q_1\sin\xi_2, \tag{44}$$

where ξ_1 is the angle between $\mathbf{k}$ and $\mathbf{q}_2$, ξ_2 is the angle between $\mathbf{q}_2$ and $\mathbf{q}_1$. For the frequency delta-function we get

$$0 = k + q_1 - q_2 - \frac{\beta}{2}\left(\frac{\cos\alpha}{k} + \frac{\cos(\alpha+\xi_1+\xi_2)}{q_1} - \frac{\cos(\alpha+\xi_1)}{q_2}\right). \tag{45}$$

We take into account that the angles ξ_1, and ξ_2 are small, as θ_1, θ_2 before. Using this fact we get the approximate resonance conditions

$$q_2 = k + q_1, \qquad 0 = k\xi_1 + q_1\xi_2, \tag{46}$$

$$\left(k\xi_1^2/2 + q_1\xi_2^2/2\right) - \frac{\beta\cos\alpha}{2}\left(\frac{1}{k} + \frac{1}{q_1} - \frac{1}{q_2}\right) - \frac{\beta\sin\alpha}{2}\left(\frac{\xi_1+\xi_2}{q_1} - \frac{\xi_2}{q_2}\right). \tag{47}$$

We assume that the third bracket is small and get the final form of (47)

$$\left(k\xi_1^2/2 + q_1\xi_2^2/2\right) - \frac{\beta\cos\alpha}{2}\left(\frac{1}{k} + \frac{1}{q_1} - \frac{1}{q_2}\right) = 0. \tag{48}$$

Using this approximation we will match the three resonance surfaces in (7) by the following transformation. It consists of a dilatation

$$\lambda_1 k_1 = k, \qquad \lambda_1 k_2 = q_1, \qquad \lambda_1 k = q_2, \tag{49}$$

and a transformation of angles

$$\theta_1/\lambda_1 = \xi_1, \qquad \theta_2/\lambda_1 = \xi_2. \tag{50}$$

After the transformation the arguments of three respective delta-functions in the collision integral take the form

$$q_2 - k + q_1 = \lambda_1(k - k_1 - k_2), \tag{51}$$

$$k\xi_1 + q_1\xi_2 = \lambda_1^0(k_1\theta_1 + k_2\theta_2), \tag{52}$$

$$k\xi_1^2/2 + q_1\xi_2^2/2 - \frac{\beta\cos\alpha}{2}\left(\frac{1}{k} + \frac{1}{q_1} - \frac{1}{q_2}\right) = \lambda_1^{-1}\left[k_1\theta_1^2/2 + k_2\theta_2^2/2 - \frac{\beta\cos\alpha}{2}\left(\frac{1}{k_1} + \frac{1}{k_2} - \frac{1}{k}\right)\right]. \tag{53}$$

The second integral in (7) is treated in a similar way.

Hence, up to some factors, which are powers of λ_1, the three resonant surfaces in (7) are the same, cf. (35), (37). The next step is a transformation of the interaction coefficient $|V_{2k1}|^2$ and computation of respective Jacobians in the integration measure. For small angles the interaction coefficient can be approximated as $|V_{2k1}|^2 = q_2 k q_1$, cf. (33). After the transformation, thus, it acquires the factor λ_1^3. It is easy to see that Jacobians give a factor λ_1. Hence, assuming an isotropic distribution $N(\mathbf{k}) = k^s$ we write $\mathcal{I}^{(3)}(\mathbf{k})$ in a factorized form

$$\mathcal{I}^{(3)}(\mathbf{k}) = \pi k^r \int |V_{k12}|^2 \delta(k - k_1 - k_2)\delta(k_1\theta_1 + k_2\theta_2)\delta\left(k_1\theta_1^2/2 + k_2\theta_2^2/2 - \frac{\beta\cos\alpha}{2}\left(\frac{1}{k_1} + \frac{1}{k_2} - \frac{1}{k}\right)\right)$$
$$\times N_k N_1 N_2 \left(N_k^{-1} - N_1^{-1} - N_2^{-1}\right)\left(k^{-r} - k_1^{-r} - k_2^{-r}\right) d\mathbf{k}_1\, d\mathbf{k}_2. \tag{54}$$

Here $r = 2s + 5$ is a sum of powers coming from (1) f, which give $2s$, (2) interaction coefficient, which gives 3, (3) delta-functions, which give $-1 + 0 + 1 = 0$, and (4) two Jacobians which give 2.

We should note that the case of almost vertical resonant triads $\cos\alpha \ll \beta\sin^2\alpha$ does not allow such treatment.

3.5. The four-wave kinetic equation

The four-wave collision integral may be factorized as well in the nondecay regions of the phase-space following the method of [14]. The situation here is close to the short IGW on the mid-latitude f-plane where a kinetic equation and stationary spectra were obtained by the same method in [15]. We apply the following transformation of the wave-vectors $\mathbf{k}, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ which form a resonant quadruple

$$q_2 = \lambda k, \qquad q_1 = \lambda k_3, \qquad q_3 = \lambda k_1, \qquad k = \lambda k_2, \qquad \xi_j = \theta_j/\lambda.$$

This transformation, under hypothesis of a scaling solution $N(\mathbf{k}) \propto k^s$, engenders the following rescalings in the integrand of the collision integral: (1) the interaction coefficients: $|T|^2 \sim |\frac{V^2}{\Delta\omega}| \sim (\frac{\lambda^3}{\lambda^{-1}})^2 = \lambda^8$, (2) the f-factors: $f \sim \lambda^{3s}$, (3) the delta-functions: $\delta(k) \sim \lambda^{-1}$, $\delta(\omega) \sim \lambda^1$, (4) the integration measure (Jacobian): $J \sim \lambda^3$.

We thus get the following factorized form of the collision integral

$$\mathcal{I}^{(4)}(\mathbf{k}) = k^r \int |T_{k123}|^2 \delta(k + k_1 - k_2 - k_3)$$
$$\times \delta\left(k_1\theta_1^2/2 - k_2\theta_2^2/2 - k_3\theta_3^2/2 - \frac{\beta\cos\alpha}{2}\left(k^{-1} + k_1^{-1} - k_2^{-1} - k_3^{-1}\right)\right)$$
$$\times N_k N_1 N_2 N_3 \left(N_k^{-1} + N_1^{-1} - N_2^{-1} - N_3^{-1}\right)\left(k^{-r} + k_1^{-r} - k_2^{-r} - k_3^{-r}\right) d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3$$

with $r = 3s + 11$.

3.6. Stationary solutions of the three- and four-wave kinetic equations

For the three-wave kinetic equation the following stationary solutions result: for $r = -1$ a Kolmogorov-type spectrum arises with $s = -3$, i.e., $N = k^{-3}$. The spectral energy density per unit k is $\epsilon_k = \omega_k k N_k = k^{-1}$. This solution is thus valid for ensembles of short equatorial IGW propagating eastward with the propagation angle with respect to the equator lying in the domain (40), which can be roughly estimated for $\beta = 1/10$ as the sector $-\pi/4, +\pi/4$. The second, equilibrium solution with $s = -1$ corresponds to the equipartition of energy. The standard dimensional arguments (cf. [11]) show that this solution correspond to a constant energy flux through the spectrum.

For the four-wave kinetic equation there exist two stationary Kolmogorov-type solutions: $r = -1$ and $N = k^{-4}$, and $r = 0$ and $N = k^{-11/3}$. They would work, e.g., for westward-propagating equatorial IGW with the propagation angle sufficiently far from the meridional direction (in the similar sector). The equilibrium solution again corresponding to $s = -1$. Again, the standard dimensional arguments show that the first solution corresponds to a constant energy flux, while the second one corresponds to a constant wave-action flux.

An important question is that of locality of the obtained spectra, i.e., convergence of the collision integral on the obtained solutions (cf. [11,17]), which is necessary to realize a Kolmogorov-type energy or wave-number cascade (local flux) through the spectrum. By construction, only locality with respect to interactions with small scales is meaningful within our model. Analysis of the collision integrals shows that the above-obtained solution of the three-wave kinetic equation is local, while both solutions of the four-wave kinetic equation are not which means that the three-wave spectrum is formed by local interactions of neighboring in the phase-space wave triads, while the four-wave spectra are formed by interactions of the distant triads. It should be noted that due to related divergencies in the collision integrals the nonlocal spectra are, formally speaking, not the solutions of the kinetic equations (cf. [11]). Their relevance is to be verified by direct simulations of the four-wave kinetic equation.

4. Conclusions and discussion

We established a normal form of equations describing dynamics of short equatorial waves and showed that short equatorial IGW and short equatorial Rossby waves are dynamically split and noninteracting in the leading order. This fact is not surprising because of the wide spectral gap between this two kinds of equatorial waves, see Fig. 1. Kinetics of the short equatorial Rossby waves may be considered along the same lines as that of the Rossby waves on the mid-latitude beta-plane, cf. [16,17].

The kinetics of the short equatorial IGW propagating *eastward* is described by a three-wave kinetic equation which admits Kolmogorov-type power-law stationary solutions of the form $N(\mathbf{k}) \propto k^{-3}$. This results in energy density (per unit volume in the phase-space) $\epsilon(k) \propto k^{-2}$ (which is equivalent to k^{-1} per unit k).

Short equatorial IGW propagating *westward* are described by a four-wave kinetic equation which admits two stationary Kolmogorov-type solutions: $N = k^{-4}$, and $N = k^{-11/3}$ with corresponding energy densities $\epsilon(k) \propto k^{-3}$ and $\epsilon(k) \propto k^{-8/3}$, respectively. Due to the ultraviolet nonlocality of these spectra direct simulations of the four-wave kinetic equations are needed to check their relevance.

The east–west asymmetry of the kinetic equations is rather remarkable and is due to the fact that short eastward propagating IGW, in spite of their small dispersion can form resonant triads, which is impossible for their westward propagating counterparts. Note that an asymmetry between eastward and westward IGW spectra seem to be necessary for correct modeling of the QBO (cf. [7], although only heuristic spectra of a specific shape were considered in this paper).

Narrow (in k_x) almost vertically propagating packets of short IGW also obey a three-wave kinetic equation. However we were not able to factorize the collision integral and find stationary spectra in this case.

We should emphasize that anisotropy is weak for the short equatorial IGW and may be considered as a correction to the acoustic dispersion law, which explains the fact that isotropic stationary spectra arise in the leading approximation. The situation is different as compared to strongly anisotropic situation, e.g., mid-latitude Rossby waves [17]. It should be noted that, to our knowledge, the example of short equatorial IGW is the first one where dispersion law close to the acoustic one allows for resonant triads due to its specific anisotropy.

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