Stability of turbulent Kolmogorov flow

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1 Introduction

The instability of the Kolmogorov flow $U = \sin y$ has received a large interest in the literature (see [8, 11, 2, 3, 7, 1, 12] and references therein). This flow exhibits a large-scale instability of the negative viscosity type for Reynolds $Re < \sqrt{2}$. For slightly supercritical conditions, the perturbation follows Cahn-Hilliard equation characterized by an inverse cascade of metastable states with scale growing in time. This cascade involves merging of jets until the gravest mode is reached (or without limit in an unbounded system). It can be halted by adding friction or dispersive effects like Rossby waves to generate a stable solution with multiple alternated jets [5, 7, 6]. Such stabilizing mechanisms have been advocated to explain features observed in the atmosphere of fast rotating Jovian planets and in numerical simulations of turbulence on a rotating sphere [9, 4].

The theory, however, is valid at very low Re while geophysical fluids are characterized by very large Re. Finding rigorously large-scale instabilities at large Re is a formidable task, probably out of reach at the moment. In this paper we address the problem of Kolmogorov flow instability when molecular viscosiy is replaced by one of the popular parameterization for small-scale turbulence representing the motion at scales smaller than the Kolmogorov flow. This is clearly a non rigorous approach, but it provides hints on large-scale instabilities at large Re and, hopefully, on the character of such instabilities.

2 The stability of a turbulent Kolmogorov flow in a Clark-Smagorinsky model

As in the standard problem [11], we use the framework of the two-dimensional incompressible Navier-Stokes equation. Introducing the streamfunction ψ such that $(u = \partial_y \psi, v = -\partial_x \psi)$, our basic equation is

$$\partial_t \nabla^2 \psi - \frac{\partial(\psi, \nabla^2 \psi)}{\partial(x,y)} = -r \nabla^2 \psi + D + F \qquad (1)$$

where r is a friction and the dissipation D follows the Clark-Smagorinsky model [10] often used in LES or atmospheric simulations:

$$\begin{aligned} D = \frac{\Delta^2}{12} & \left((\partial_{xy^3}\psi + \partial_{x^3y}\psi)(\partial_{xx}\psi - \partial_{yy}\psi) + \partial_{xy}\psi(\partial_{y^4}\psi - \partial_{x^4}\psi) \right) \\ & + C_s \Delta^2 \partial_{xy} \left(4(\bar{S}\partial_{xy}\psi) + (\partial_{yy} - \partial_{xx})\bar{S}(\partial_{yy}\psi - \partial_{xx}\psi) \right) \end{aligned}$$

with $\bar{S} = \sqrt{4(\partial_{xy}\psi)^2 + (\partial_{xx}\psi - \partial_{yy}\psi)^2} + \nu$. In this model Δ depends on the filtering of the velocity and can be considered as a cutoff scale. The forcing F is chosen in order to maintain $\Psi(y) = A \cos y$ with $A > 0$ as a stationary solution of (1). In order to distinguish our case from the standard case where this flow is maintained against molecular diffusion, we call it the *turbulent* Kolmogorov flow. A small diffusion ν is added to regularize the solutions near $y = \pm\pi/2$

We assume scale separation between the Kolmogorov flow and the large-scale flow, and introduce slow variables $X = \epsilon x$ and $T = \epsilon^2 t$. Then it turns out that the flow depends only on (X, y, T) and we expand ψ as

$$\psi(X,y,t) = \Psi(y) + \phi_0(X,T) + \epsilon\psi_1(X,y,t) + \epsilon^2\psi_2(X,y,t) + \dots . \qquad (2)$$

Equation (1) is then expanded in ϵ and the perturbation problem is solved at successive orders. At each order n in the expansion we have to solve

$$C_s \Delta^2 \partial_{yy}((2A|\cos y| + \nu)\partial_{yy}\psi_n - r\psi_n) = \mathcal{H}\,, \qquad (3)$$

where $\mathcal{H}$ holds for a complicated expression involving solutions to lower order equations in the perturbation expansion. In order to satisfy (3), the integral $\int_0^{2\pi} \mathcal{H} dy$ must vanish for all (X, T), thus providing solvability conditions for each n.

Since the dissipation is a nonlinear function of the flow, the algebra of the perturbative expansion is considerably more intricate than in the standard problem. In practice it must be solved by symbolic calculations using Mathematica. The output of these calculations would fill several printed pages. The integrals appearing in the solvability conditions do not need to be calculated except for those involved in the amplitude equation below. The numerous other terms are found to vanish by the application of simple symmetry rules.

3 Results and discussion

It turns out that the solvability conditions at order 0 and 1 are automatically satisfied by (2). At order 2, we get a solvability condition

$$r = \frac{4AC_s\mathit{\Delta}^2}{\pi}\,,$$

that is imposed to get rid of a spurious instability entirely due to the Clark-Smagorinsky parameterization (i.e. without any coupling with the Jacobian in (1)). This effect should be taken into account in numerical simulations of the Kolmogorov flow.

The solvability condition is again satisfied at order 3 and, like in the standard problem, the solvability condition at order 4 provides the instability condition for ϕ_0. This equation is

$$\partial_{X^2T}\phi_0 = H\partial_{X^4}\phi_0 + G(\partial_X\phi_0)^2\partial_{X^2}\phi_0\,, \tag{4}$$

where H and G are two constants depending on the parameters $(C_s, \mathit{\Delta}, \nu)$ and on the solutions of the perturbation problem at orders 1 and 2.

It turns out that in the limit $\nu \to 0$

$$H = \frac{2\mathit{\Delta}^2 AC_s}{\pi} - \frac{\alpha}{2\mathit{\Delta}^2 C_s}\left(1 + \frac{\mathit{\Delta}^2}{12}\right)A\,,$$

where the coefficient α is numerically calculated as $\alpha = 0.18935\ldots$. If one further take the usual value 0.23 for C_s, it is found that $H < 0$ for $0 < \mathit{\Delta} < \mathit{\Delta}_c = 1.341\ldots$. Hence (4) exhibits negative viscosity for small enough cutoff scale, a result that is very similar to the standard Kolmogorov instability.

The new feature here is that the amplitude equation provides an additional nonlinear term that is absent in the standard problem where Cahn-Hilliard equation appears at sixth order. In the limit $\nu \to 0$, the coefficient in front of this term is

$$G = \frac{1}{AC_s\mathit{\Delta}^2}\left(\alpha_1 + \left(\frac{\alpha_2}{C_s} + \frac{\alpha_3}{C_s^2}\right)\frac{1}{\mathit{\Delta}^2} + \left(\frac{\alpha_4}{C_s^2} + \frac{\alpha_5}{C_s^3}\right)\frac{1}{\mathit{\Delta}^4} + \frac{\alpha_6}{C_s^3\mathit{\Delta}^6}\right)$$

with numerical coefficients $\alpha_1 = 0.012\ldots$, $\alpha_2 = -0.0052\ldots$, $\alpha_3 = -0.0029\ldots$, $\alpha_4 = -0.0026\ldots$, $\alpha_5 = 0.00073\ldots$ and $\alpha_6 = 0.0084\ldots$. It turns out that for $\mathit{\Delta} = \mathit{\Delta}_c$, we have $G \approx 0.219/A$, that is $G > 0$.

In the instability range, (4) admits stationary solutions that can be obtained in terms of elliptical functions. The dynamical study of (4) will be presented elsewhere.

We have shown that a large-scale instability of the Kolmogorov flow is obtained when a parameterized turbulent viscosity replaces the standard molecular viscosity. This result is reached to the price of two scale separation hypothesis, the first one between the Kolmogorov flow and the large-scale flow, and the second one between the small-scale turbulence represented by the Clark-Smagorinsky parameterization and the Kolmogorov flow. Nevertheless, the result is encouraging for the generic existence of large-scale instabilities of the negative viscosity type in fully turbulent flow. The robustness would need to be tested with a variety of parameterizations and numerical simulation. Our results also shows the necessity of damping spurious instabilities that are purely generated by the parameterization and may have polluted previous numerical investigations. We also find a new nonlinear term in the amplitude equation which is obtained without any assumption of slight supercriticality. The term is cubic but differs from the nonlinearity in the Cahn-Hilliard equation. Again, robustness and significance of this result requires further studies. This work has been supported by Project N.11397 CNR/CNRS 2002-2003

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