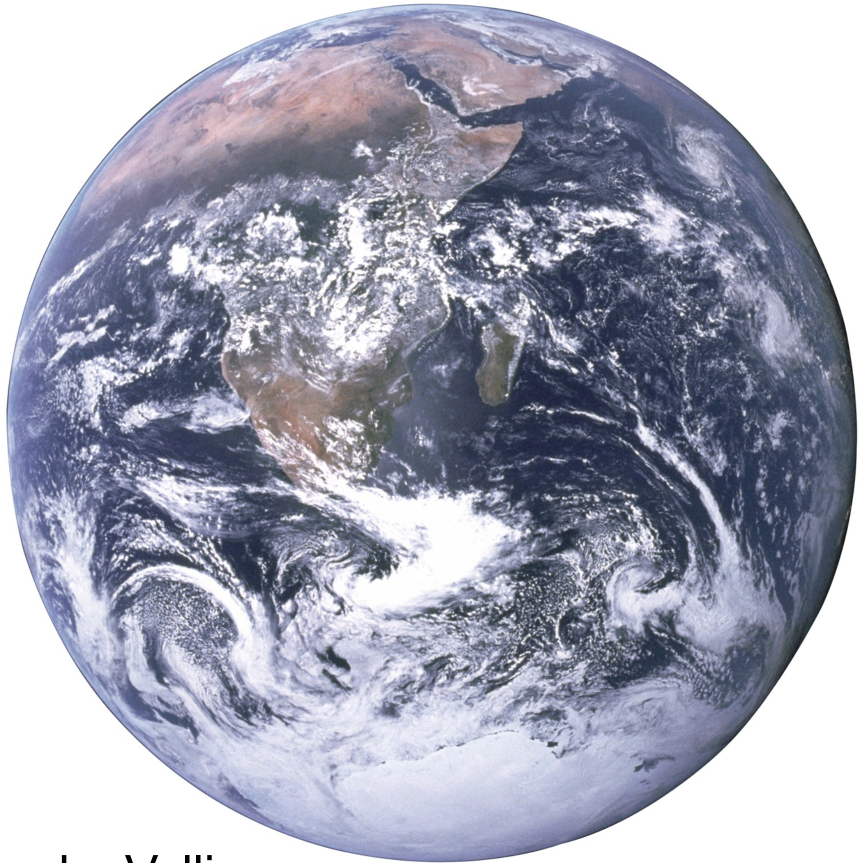
Geophysical Fluid Dynamics



Caroline Muller

References:

« Atmospheric and Oceanic Fluid Dynamics » by Vallis

« Physics of Climate » by Peixoto & Oort

« Introduction to Geophysical Fluid Dynamics » by Cushman-Roisin

Geophysical Fluid Dynamics

A- Introduction to GFD

Definition of GFD, scales
Effect of rotation
Effect of stratification

B- Coriolis force and equations in rotating frame

Rotating frame of reference
Unimportance of centrifugal force
Equations on a sphere
Traditional approximation, the f-plane, the beta-plane

C- Homogeneous flow

Equations of motion, adimensional numbers
Hydrostatic balance
Rapidly rotating flow, geostrophic balance
Vorticity dynamics, Rossby waves

D- Stratification effects

Equations for the ocean, Boussinesq approximation
Equations for the atmosphere, potential temperature
Stable and unstable to convection, Brunt Vaisala frequency

Recitation: Nov 26th (help on homework) + December 10th

A- intro to GFD

- Atmospheric circulation
- Ocean circulation
- Scales of interest

What sets the atmosphere and the oceans in motion?

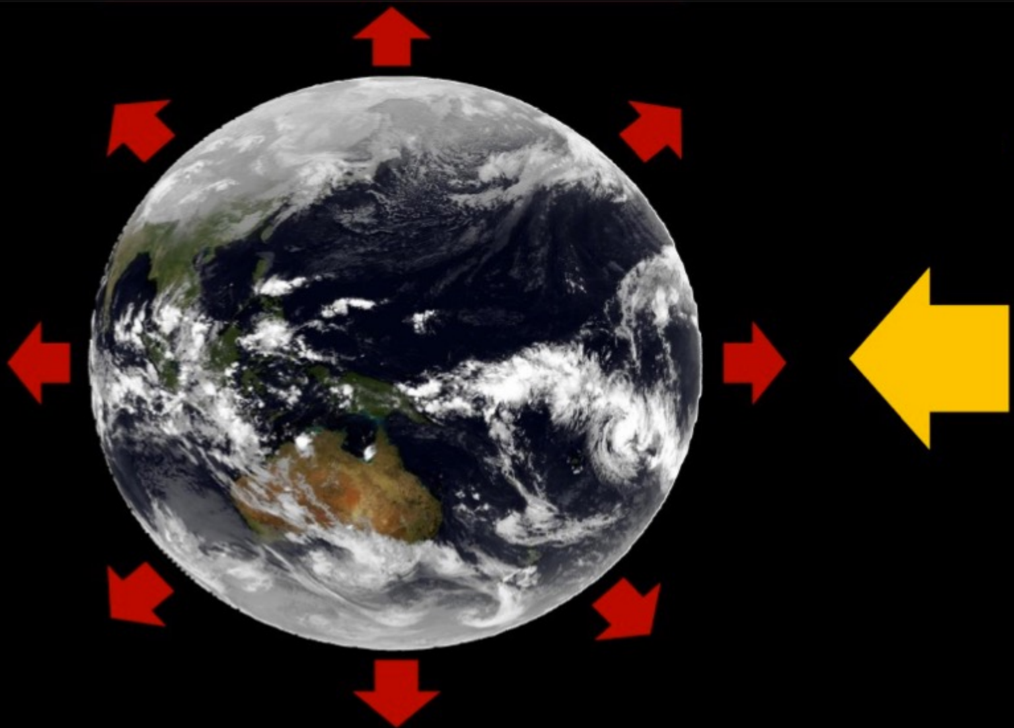
Solar forcing

Energy balance at the top of the atmosphere:

- Earth receives solar visible radiation from the sun
- Earth emits infrared radiation to space

Outgoing infrared radiation
Depends on temperature

Incoming visible radiation



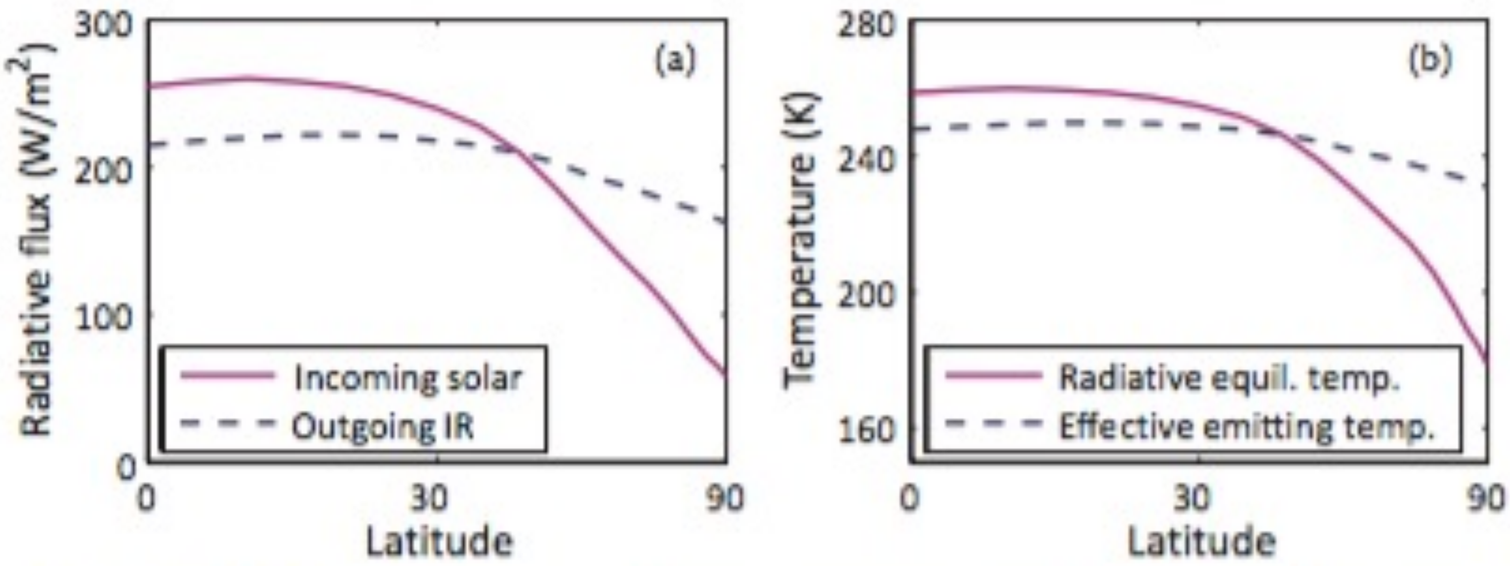


Fig. 14.1 (a) The (approximate) observed net average incoming solar radiation and outgoing infrared radiation at the top of the atmosphere, as a function of latitude (plotted on a sine scale). (b) The temperatures associated with these fluxes, calculated using $T = (R/\sigma)^{1/4}$, where R is the solar flux for the radiative equilibrium temperature and where R is the infrared flux for the effective emitting temperature. Thus, the solid line is an approximate radiative equilibrium temperature

Vallis

⇒ Radiative equilibrium temperature has steeper gradient than consistent with outgoing longwave radiation

⇒ Atmospheric and oceanic circulations transport heat poleward

In other words, radiative forcing maintains pole-equator T gradient; circulation smoothes it

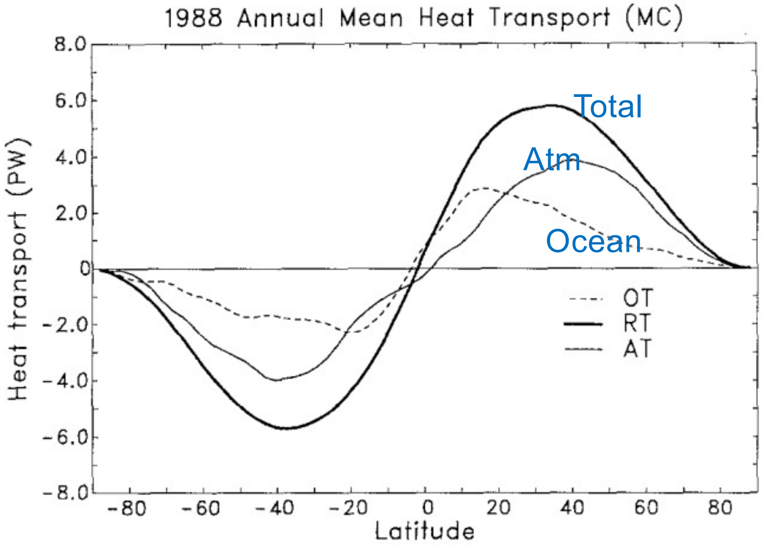
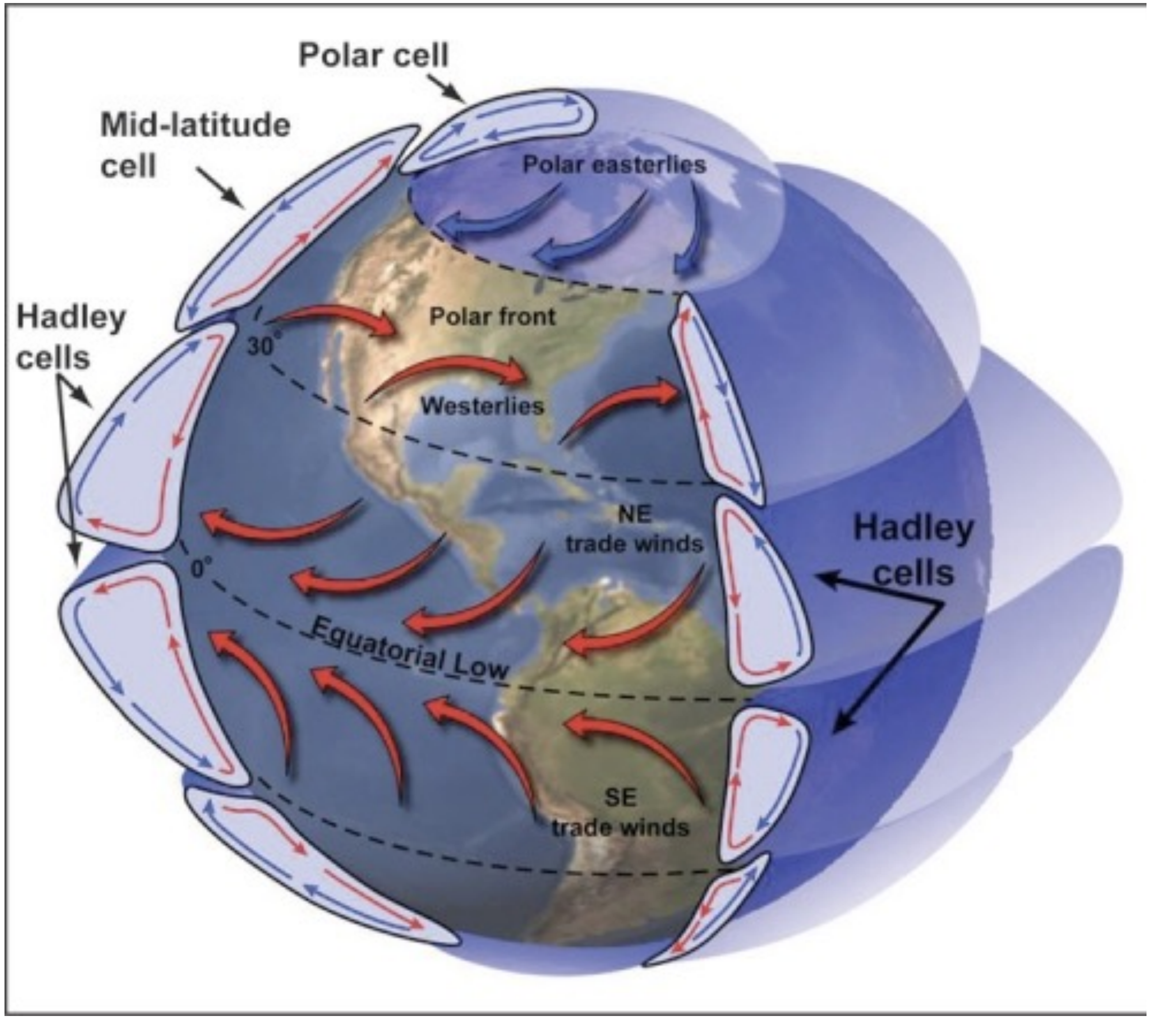


Fig. 16. The top-of-the-atmosphere required northward heat transport from satellite radiation measurements RT, the estimated atmospheric transports AT, and the ocean transports OT computed as a residual, for 1988 in PW

Atmospheric cells



Atmospheric circulation from satellite : visible / infrared

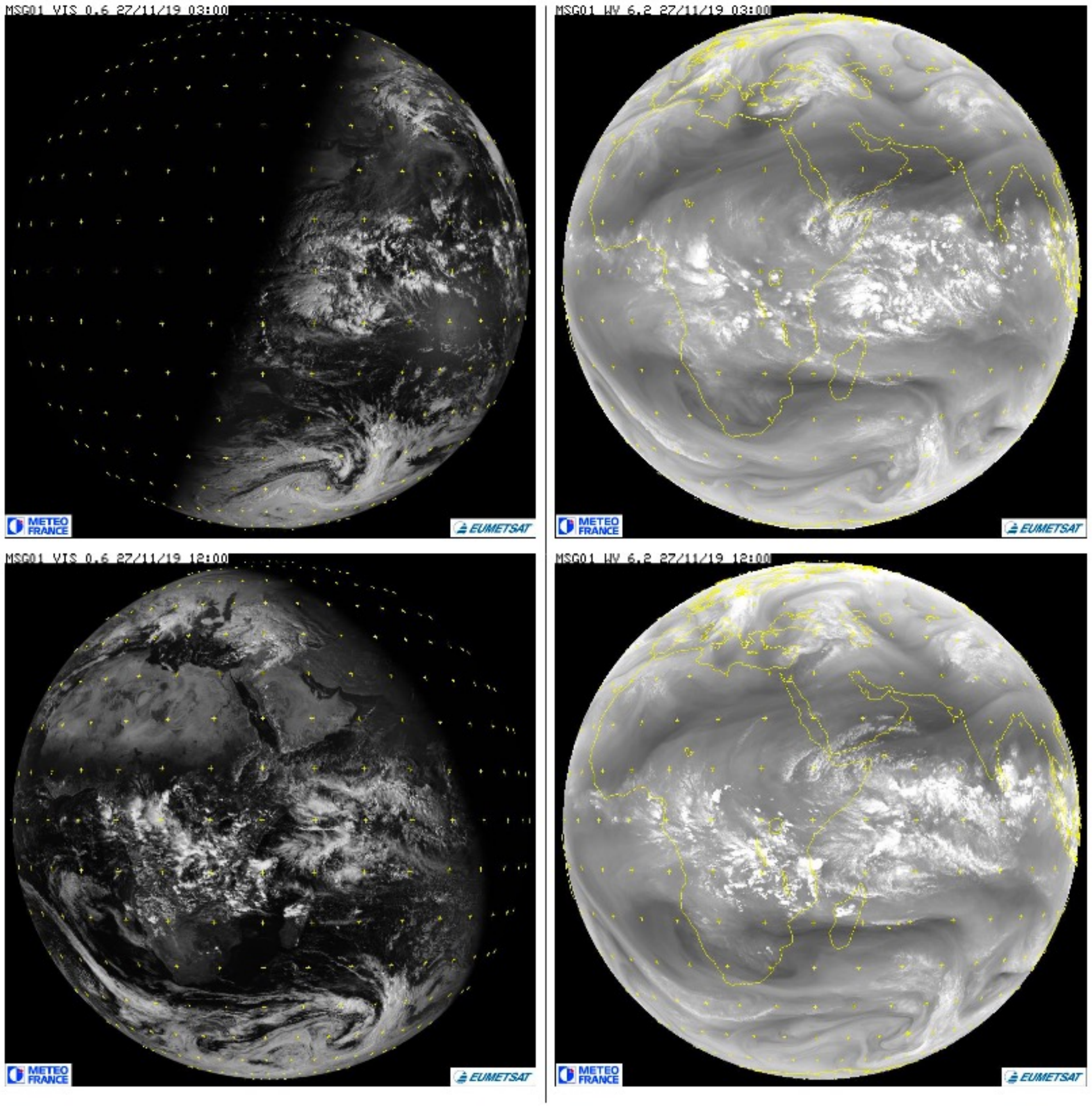
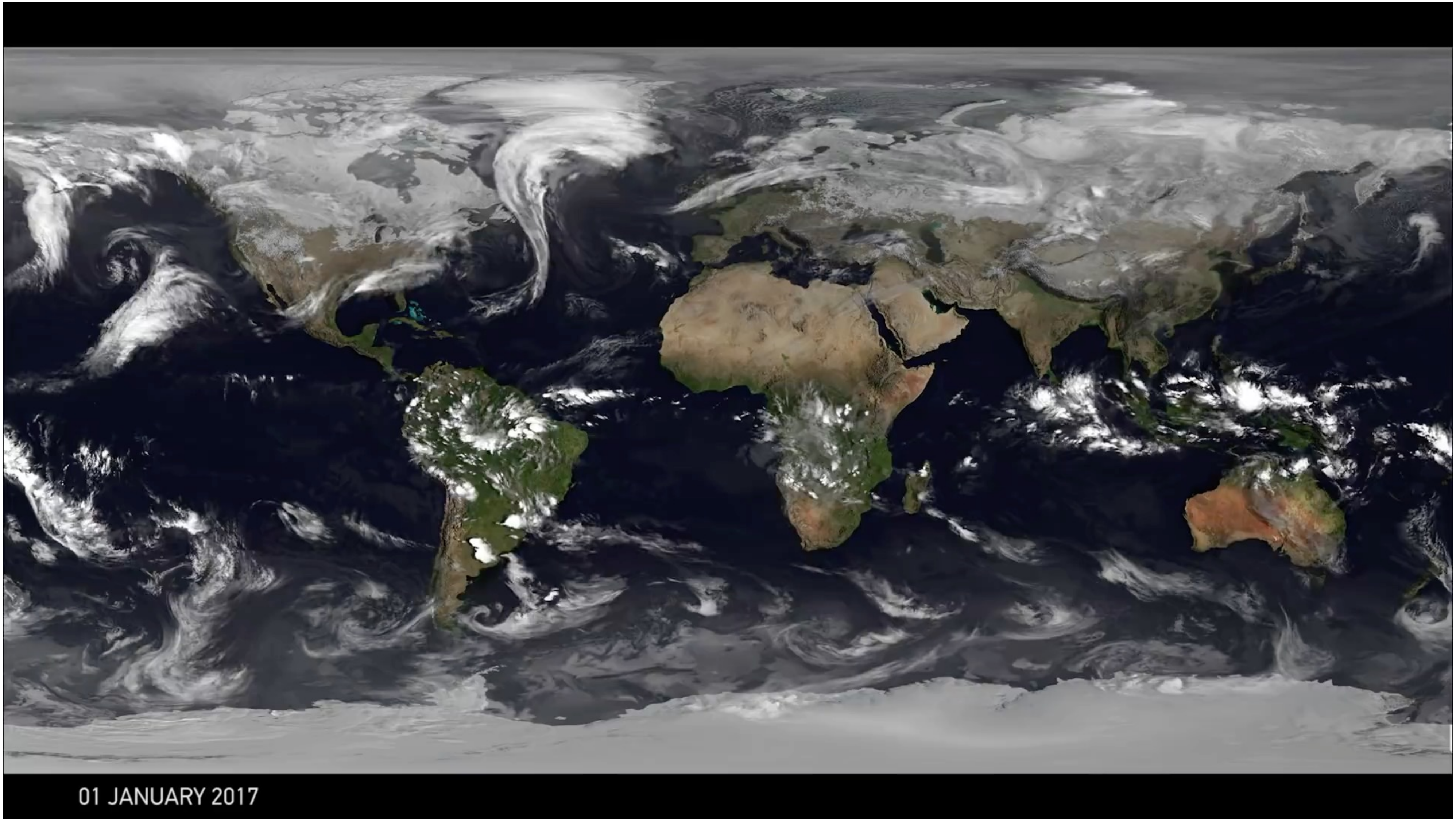


Figure 1.7. *Satellite images from SATMOS - November 2019.*

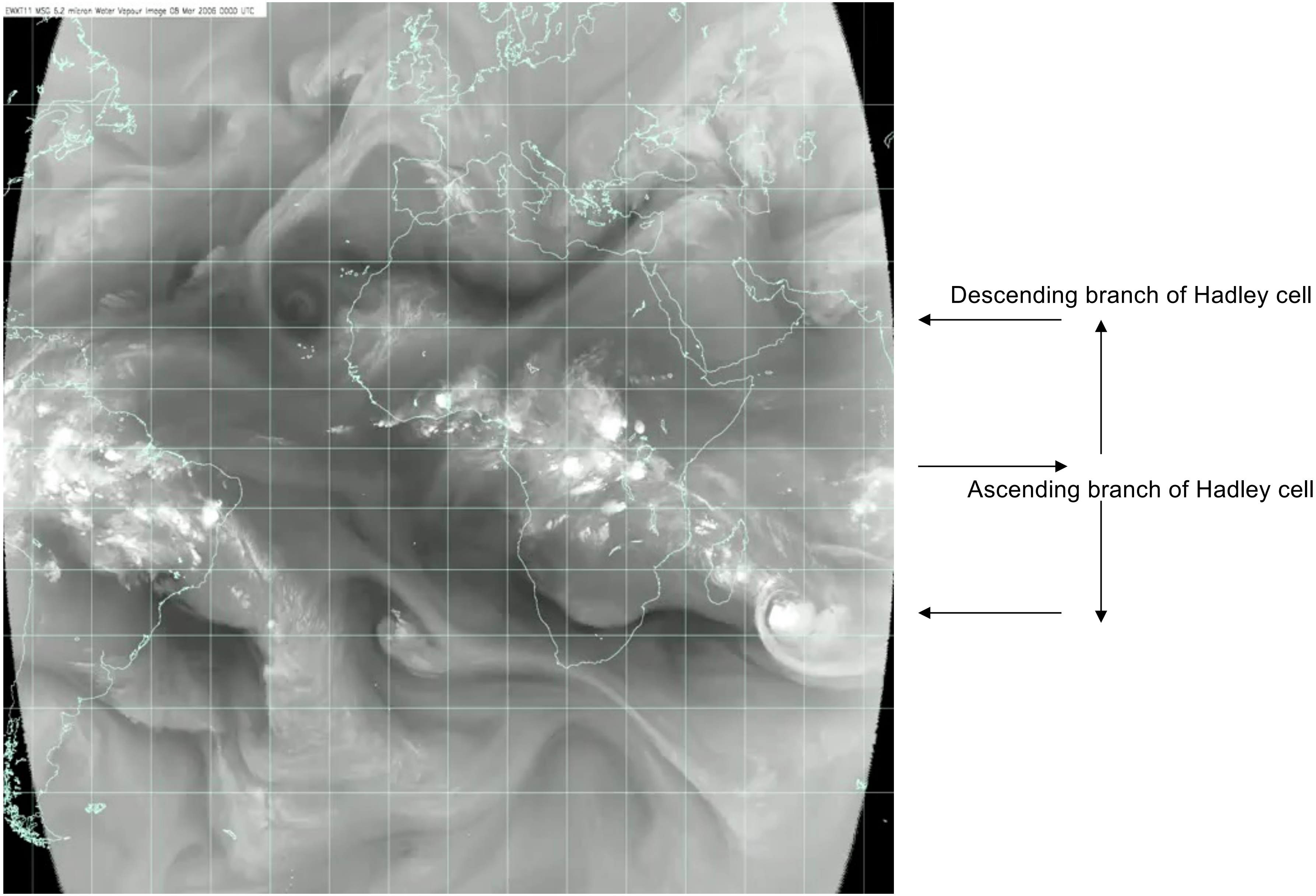
Atmospheric circulation redistributes energy



A Year of Weather

This visualisation, from geostationary satellites (infrared), shows an entire year of weather

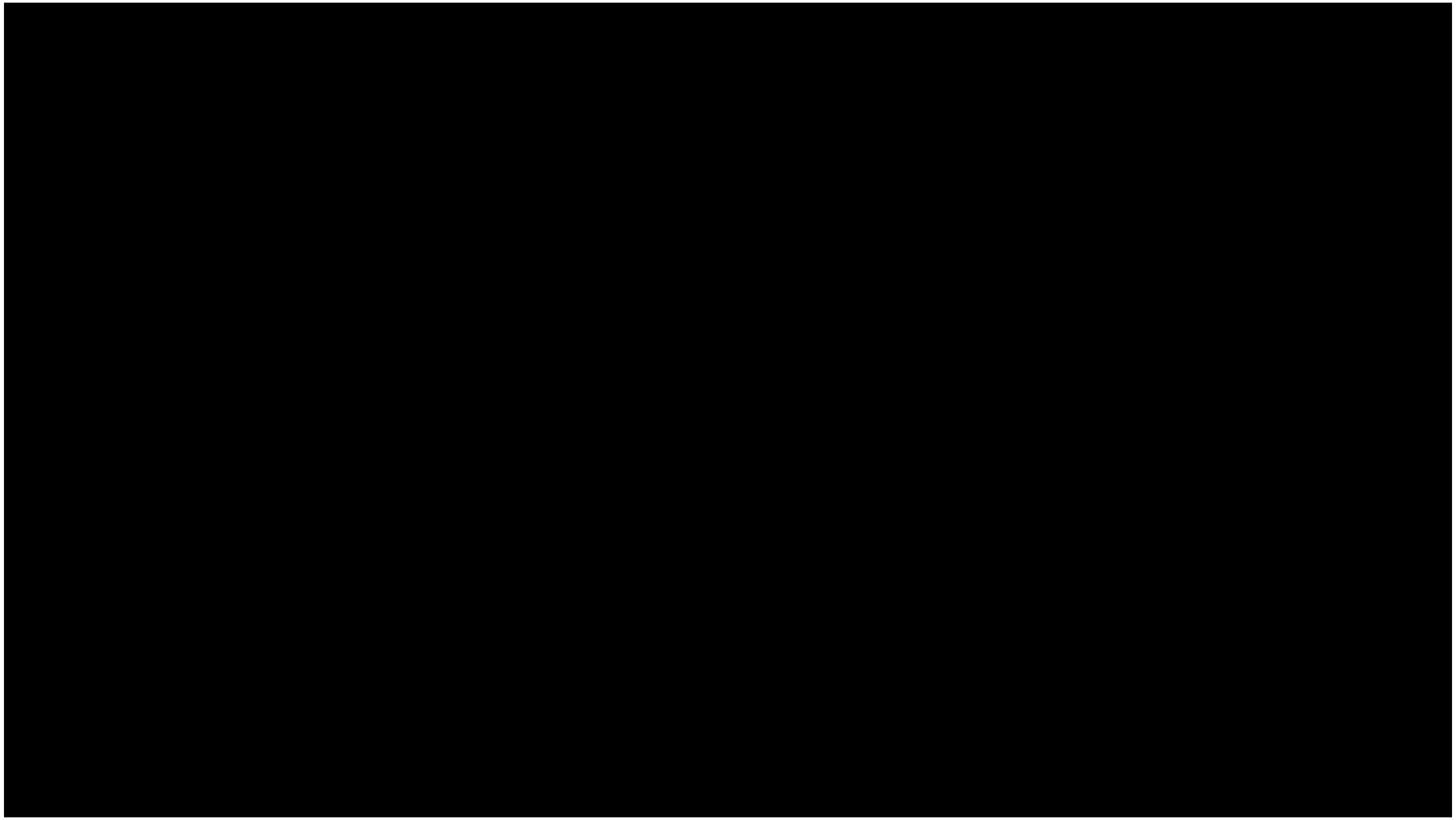
atmospheric water vapor (white=humid)



6.2_micron_wv_700-300-meteosat

Atmospheric circulation: Midlatitudes

Jet stream



NASA

Atmospheric circulation: Midlatitudes

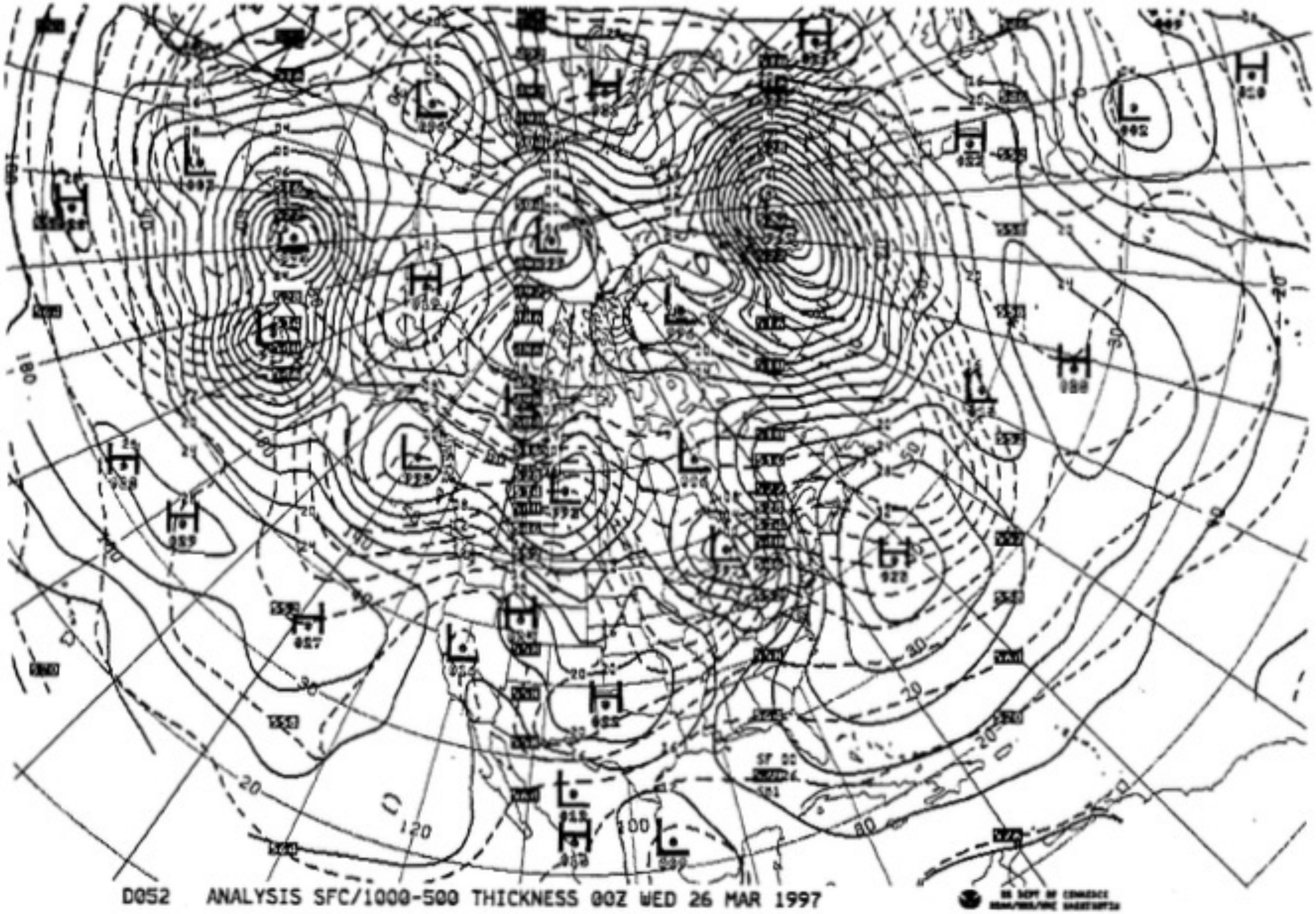


Figure 6.1: Surface pressure analysis (solid contours), 26 March 1997.

Atmospheric circulation: Midlatitudes

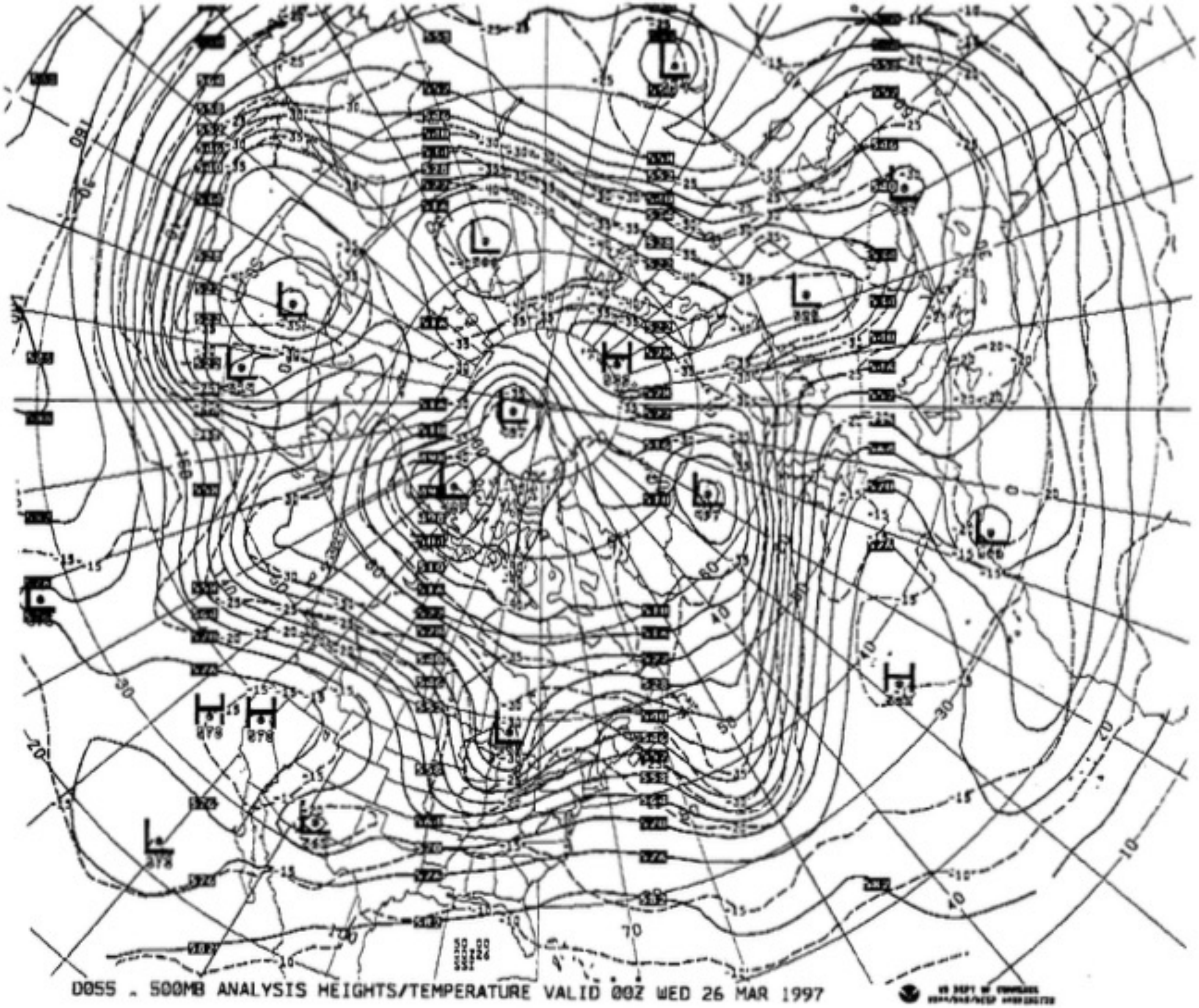


Figure 6.2: 500hPa analysis (solid contours), 26 Mar 1997.

Atmospheric circulation: midlatitude frontal systems

Frontal systems:

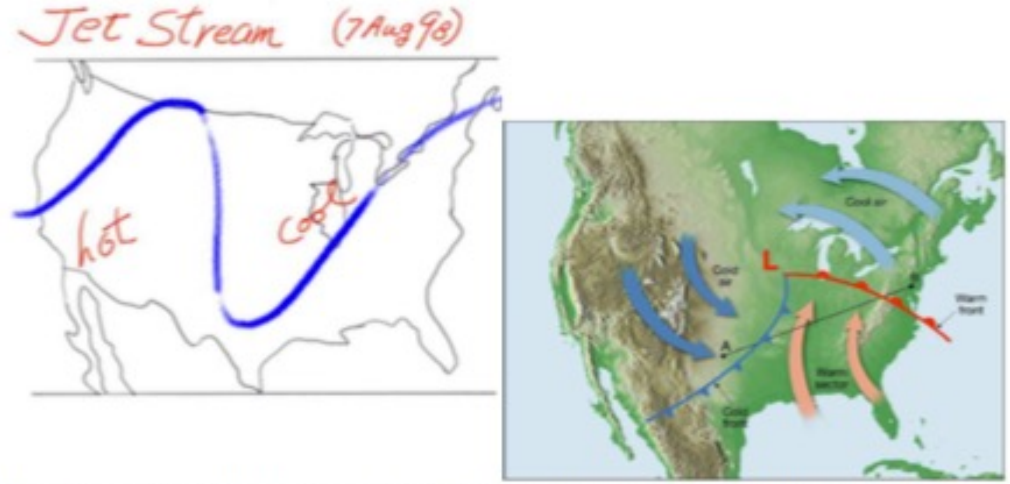
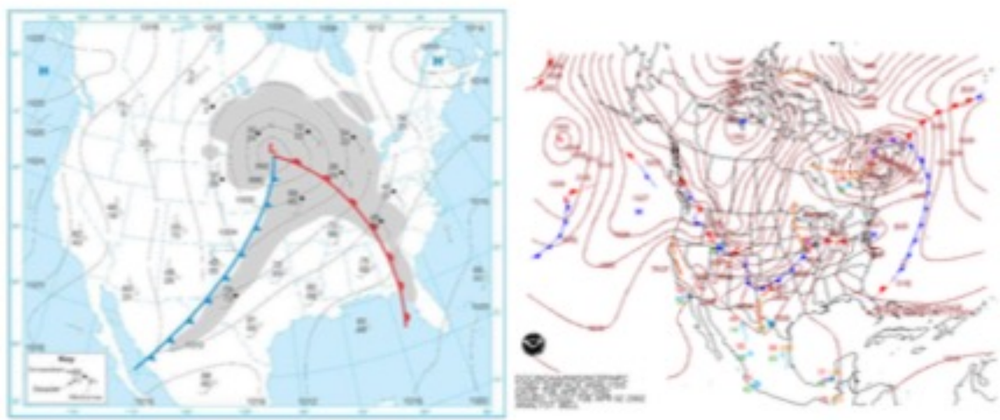


Figure 6 : représentation d'un courant-jet d'altitude. Figure 7 : représentation d'une dépression et ses fronts associés.



Figures 8 et 9 : cartes atmosphériques d'une situation météorologique (pression de surface et fronts).

High latitudes => Low/high pressure systems and associated fronts

Ocean circulation

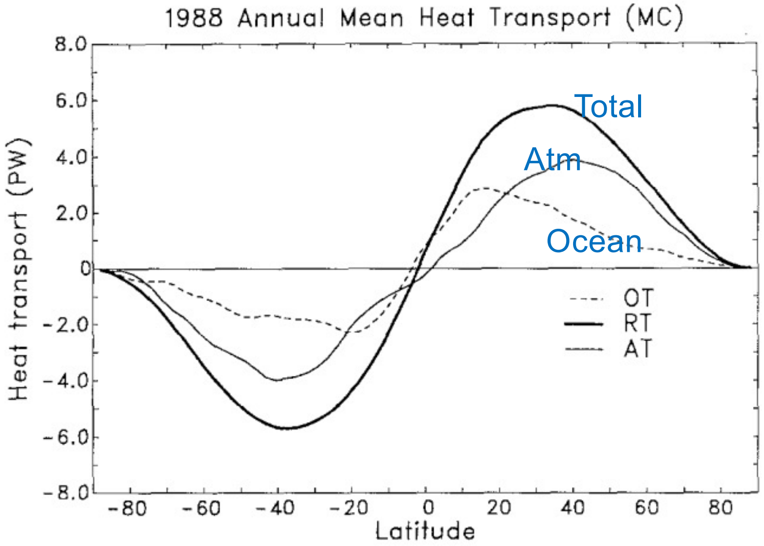


Fig. 16. The top-of-the-atmosphere required northward heat transport from satellite radiation measurements RT, the estimated atmospheric transports AT, and the ocean transports OT computed as a residual, for 1988 in PW

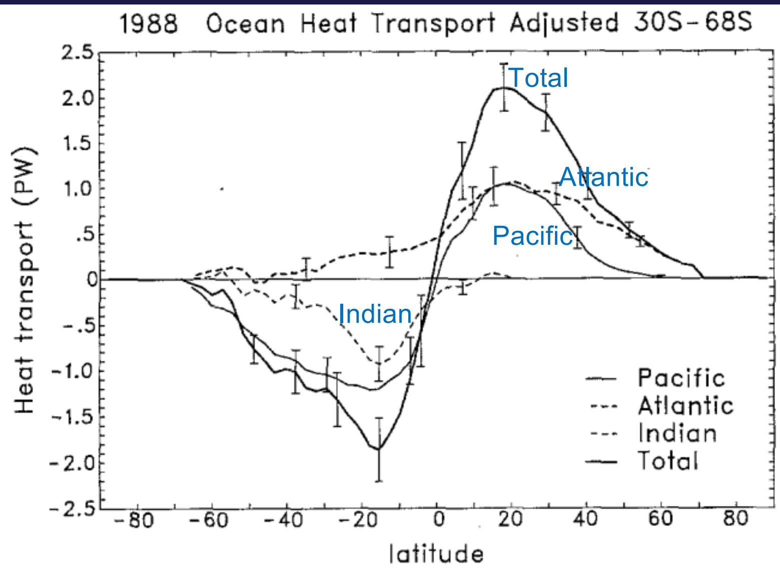
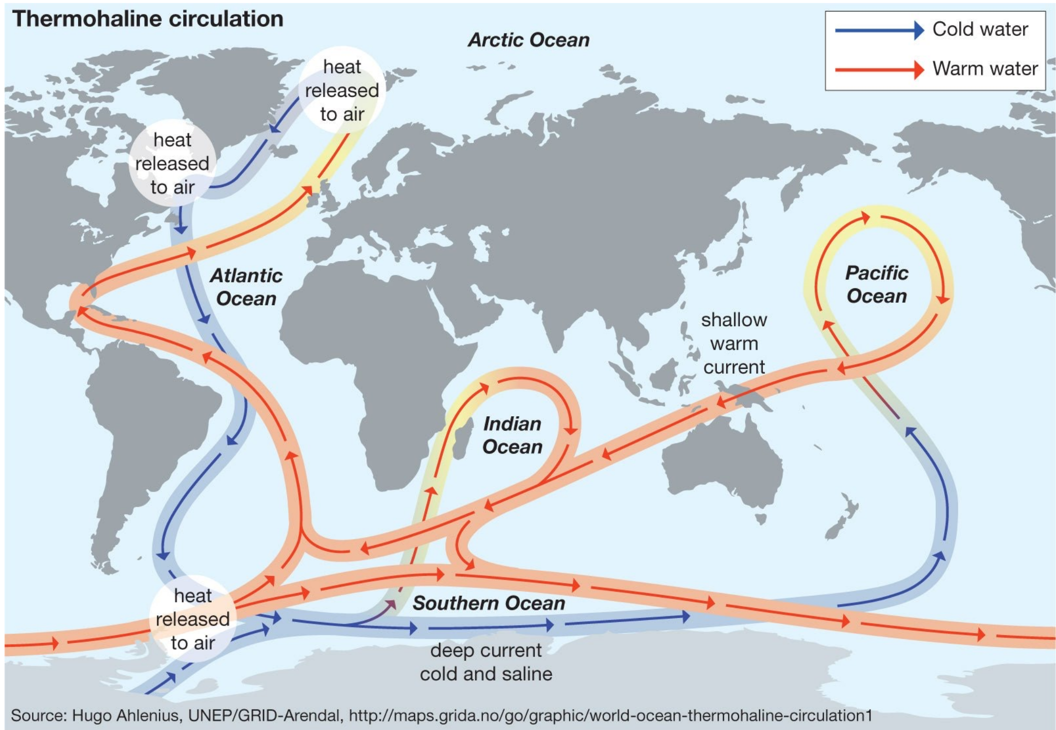


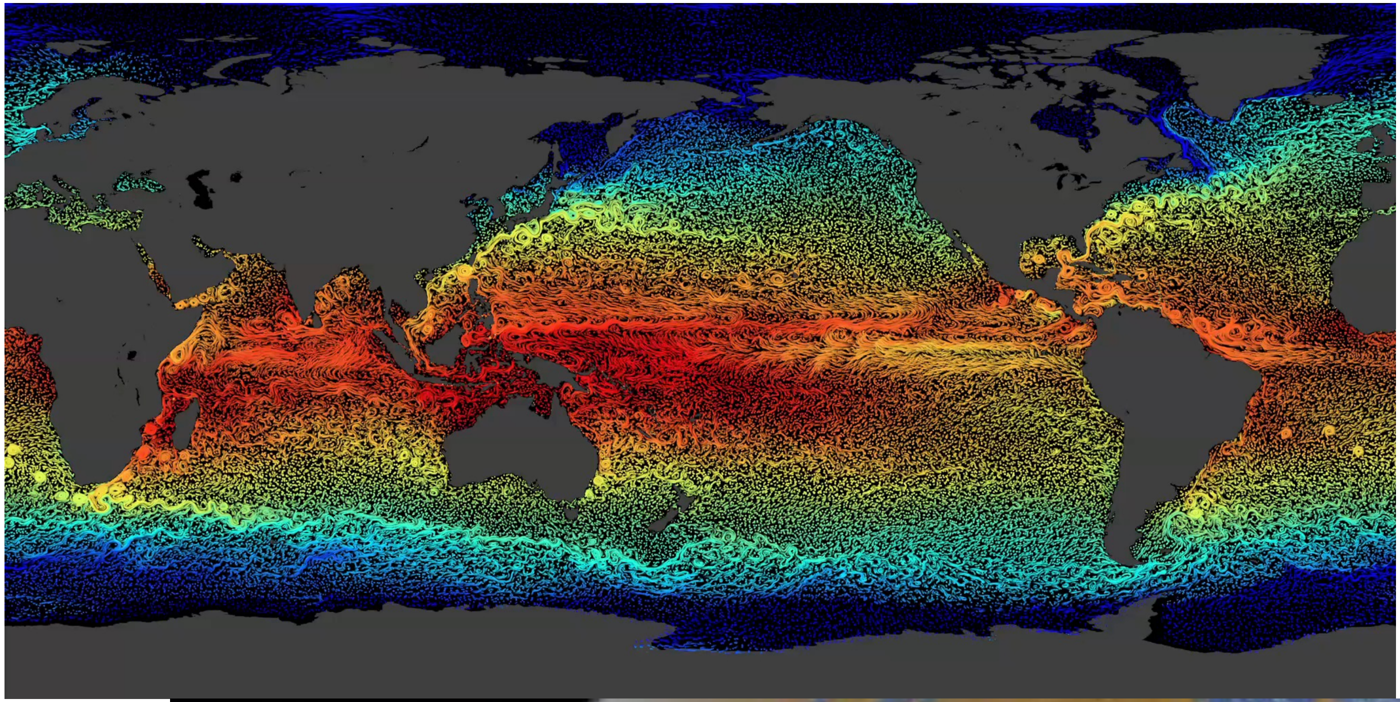
Fig. 17. The poleward ocean heat transports in each ocean basin and summed over all oceans (*total*), as computed from the net
urface, integrated from 65°N and ad-
988 in PW. As this calculation does not
n throughflow, the Pacific and Indian
d be combined

Ocean overturning circulation



Trenberth Solomon 94

Ocean circulation redistributes energy



Impact of land: boundaries for ocean currents, low heat capacity

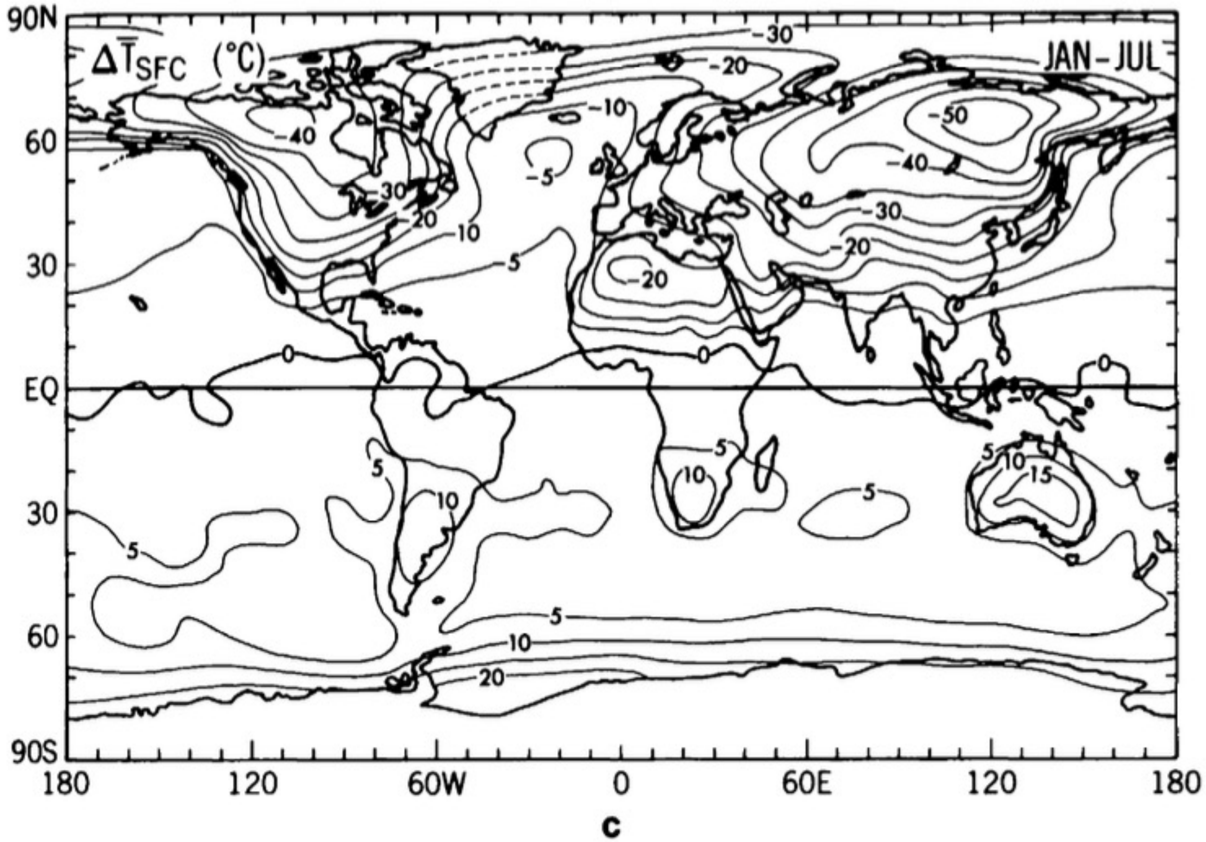


FIGURE 7.4. Horizontal distributions of the surface air temperature (in °C) for January (a) and July (b) after National Climatic Data Center (1987), and for the January–July difference (c) based on the 1963–73 analyses in Oort (1983).

Peixoto and Oort

Atmosphere: vertical temperature profile

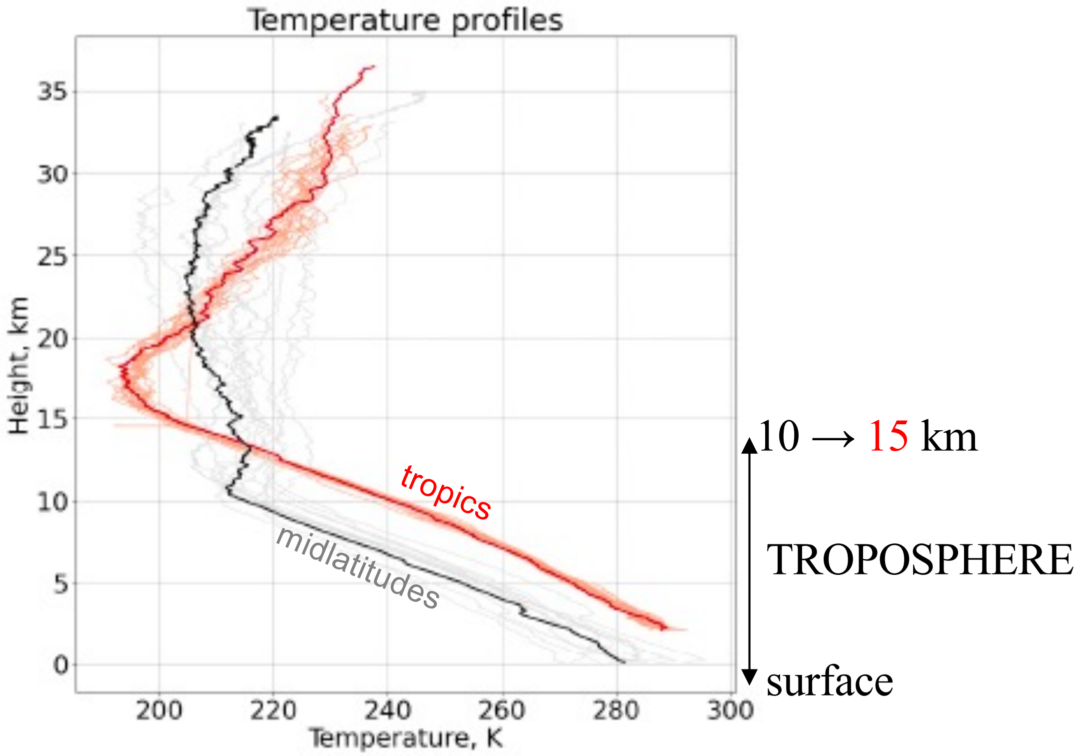
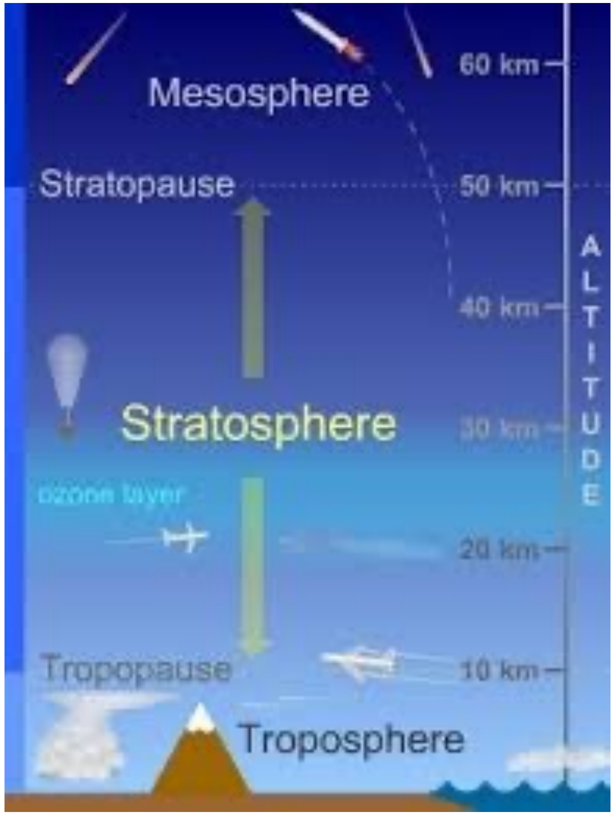
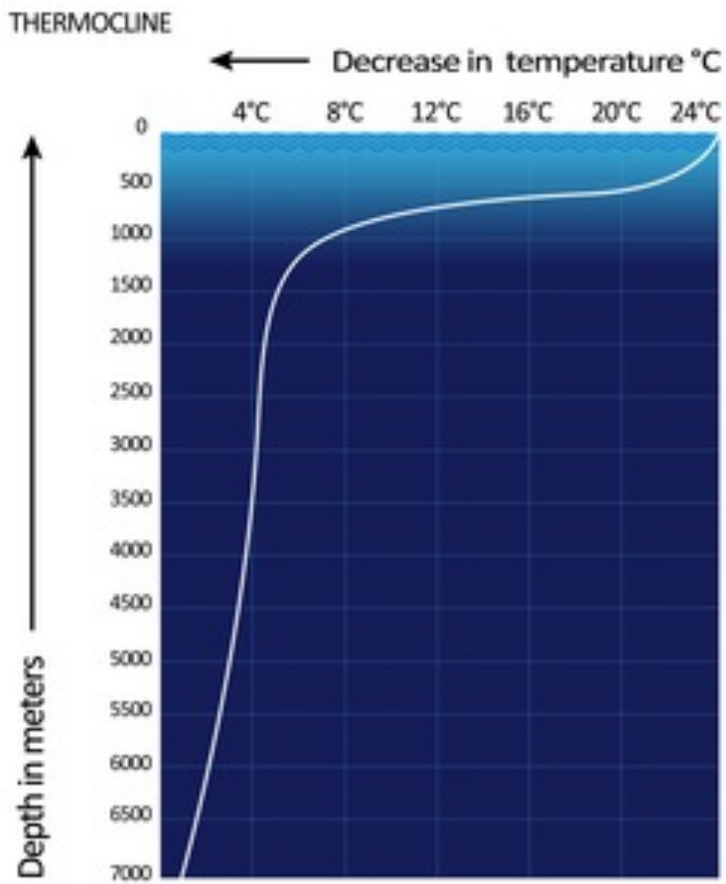


Figure 1.5. *Temperature profiles from radiosoundings from the Global Climate Observing System (GCOS) Reference Upper-Air Network (GRUAN). Sounding from 2019 for midlatitudes (Lindenberg, Germany, 52.2° N, grey lines) and tropics (Réunion Island, France, -20.9° S, red lines).*



=> Troposphere has vertical temperature gradient - stratified

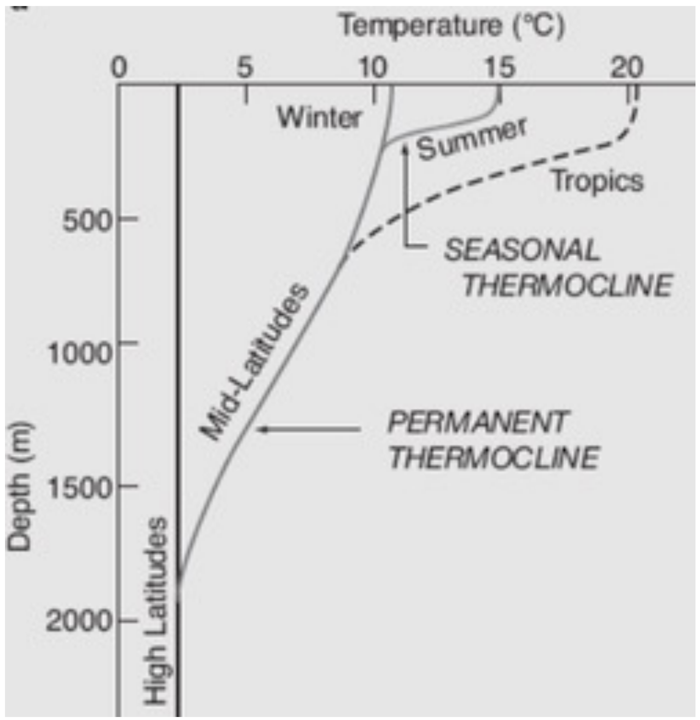
Ocean: vertical temperature profile



100s meters: Mixed layer

~1km : Thermocline (strong T gradient)

Deep ocean
Averaged ocean depth: 4km



⇒ Ocean density varies with depth

⇒ stratified

Effect of rotation

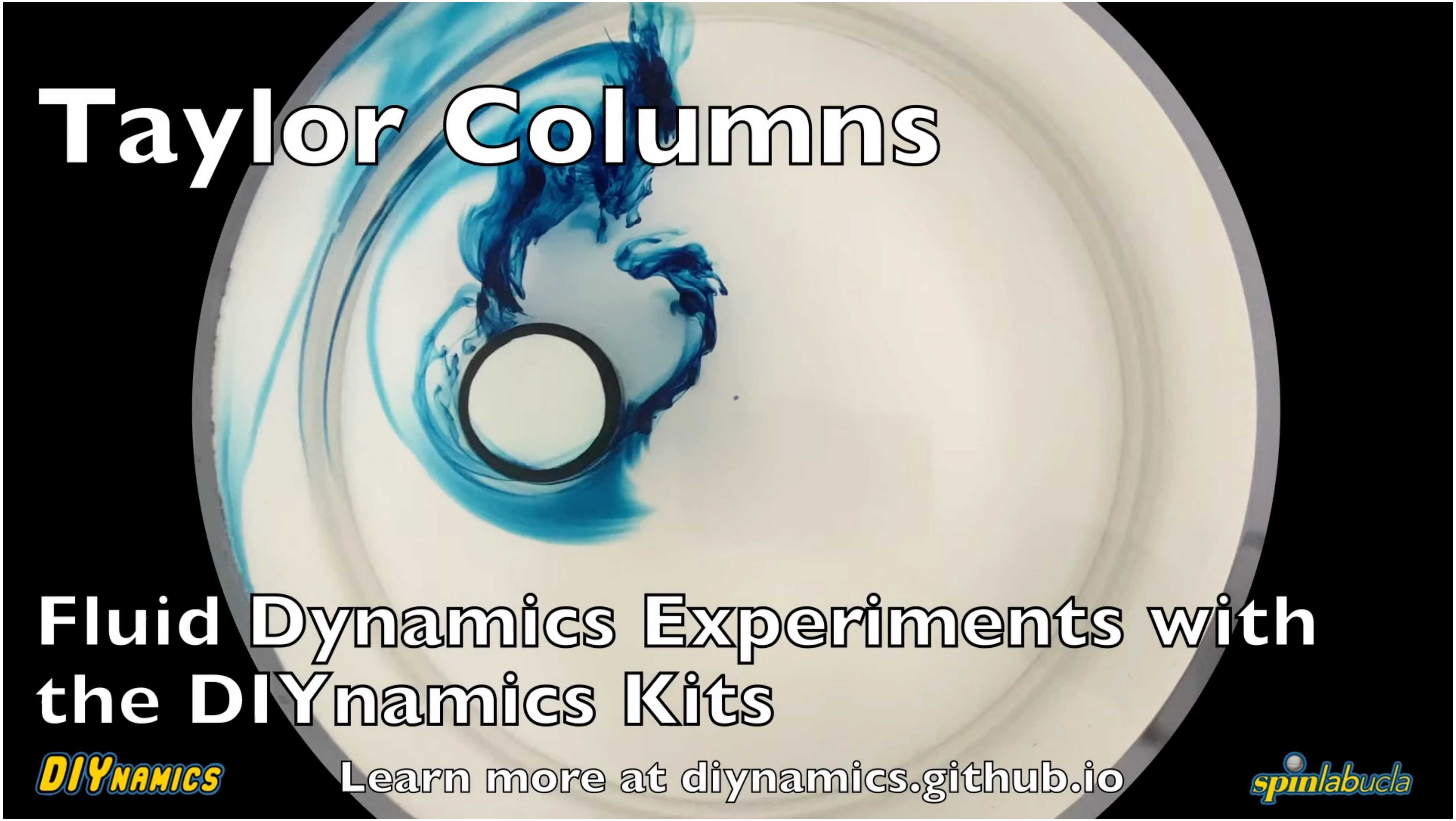
Coriolis

Rotation imparts vertical rigidity to the fluid

Rapidly rotating fluid: Taylor columns

Flow over topography/obstacle

Example GFD: ice melting around seamounts

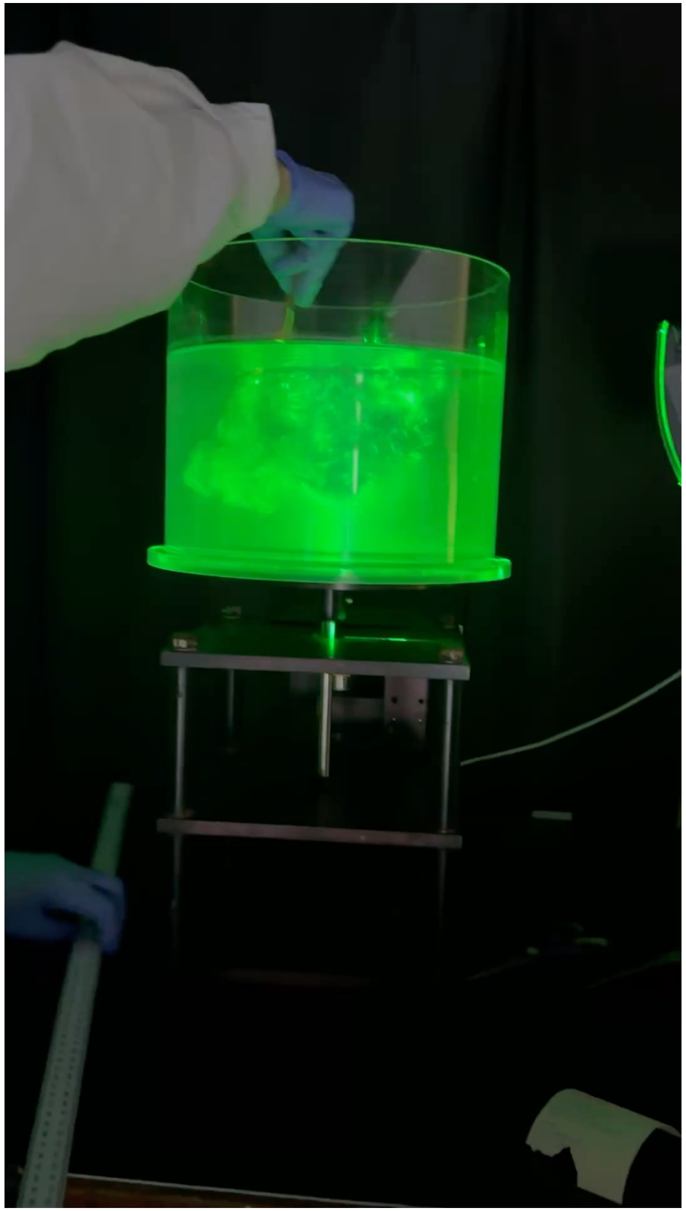

Taylor Columns
Fluid Dynamics Experiments with
the DIYnamics Kits
DIYNAMICS
Learn more at diynamics.github.io
spinlabucla

No rotation => 3d turbulence

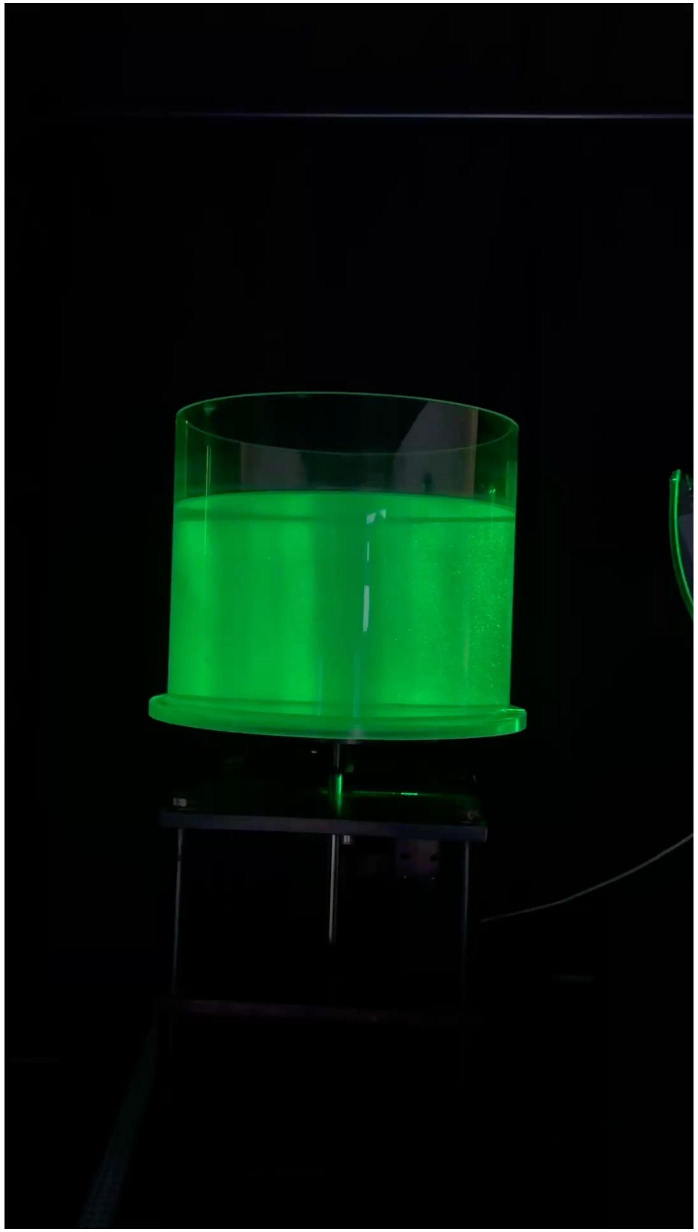
Start rotation

Courtesy: Sima Doga

With rotation => Taylor columns

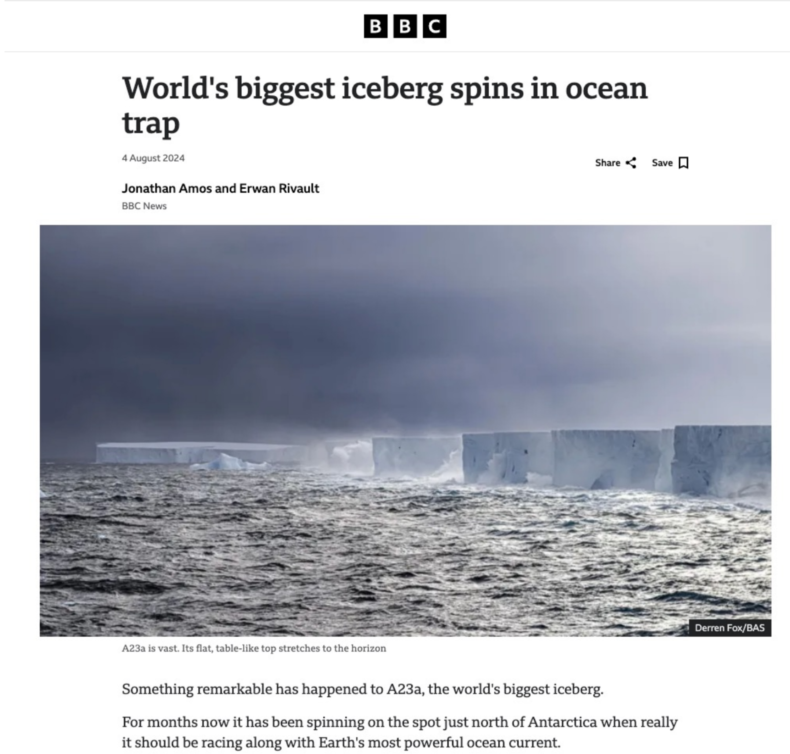


Rapid rotation



Courtesy: Sima Doga

Iceberg trapped in a Taylor column


BBC

World's biggest iceberg spins in ocean trap

4 August 2024

Share Save

Jonathan Amos and Erwan Rivault

BBC News

Derren Fox/BAS

A23a is vast. Its flat, table-like top stretches to the horizon

Something remarkable has happened to A23a, the world's biggest iceberg.

For months now it has been spinning on the spot just north of Antarctica when really it should be racing along with Earth's most powerful ocean current.

Adimensional number that gives importance of rotation: Rossby number

Effect of stratification

Stratification

Stratification decouples stacked horizontal layers

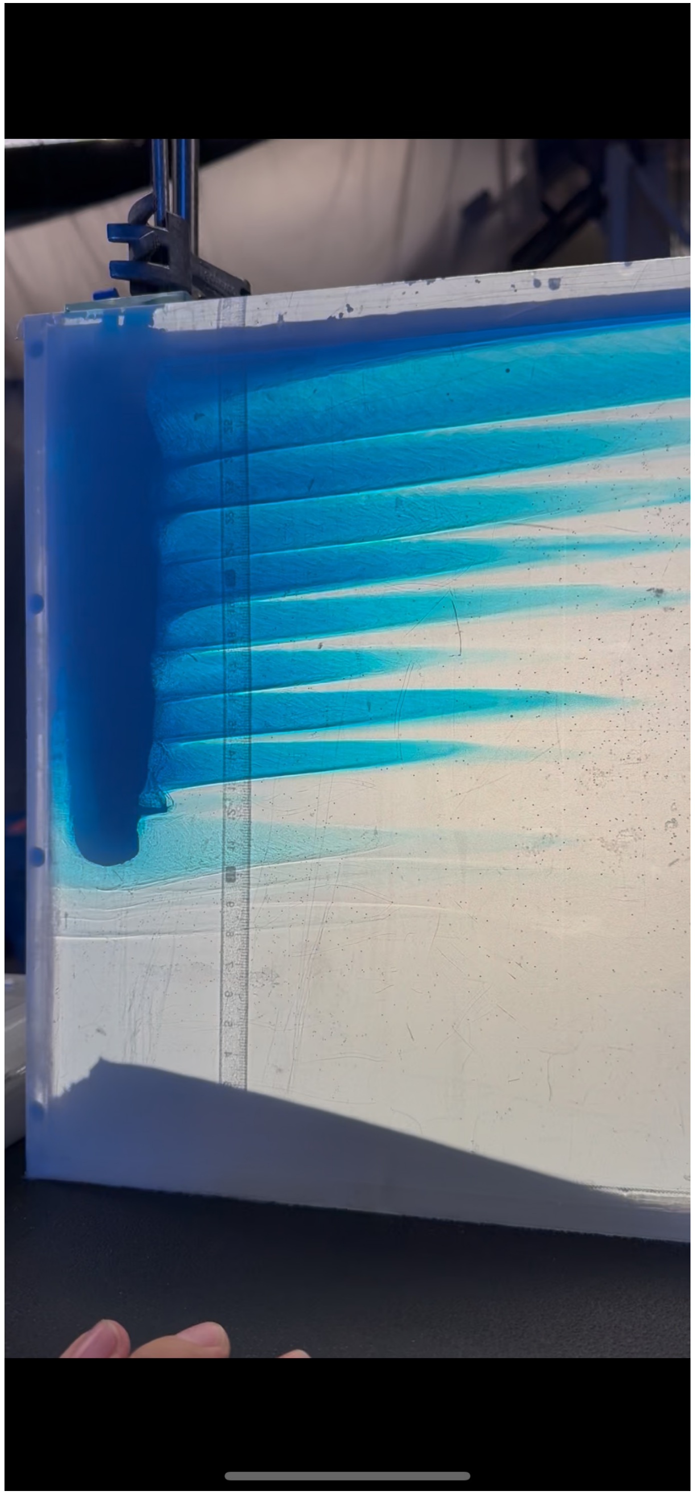
Layered motion with stratification

Pollution trapped

Flow over mountain – over or around

waves

Stratified fluid => horizontal spreading



Courtesy: Sima Doga

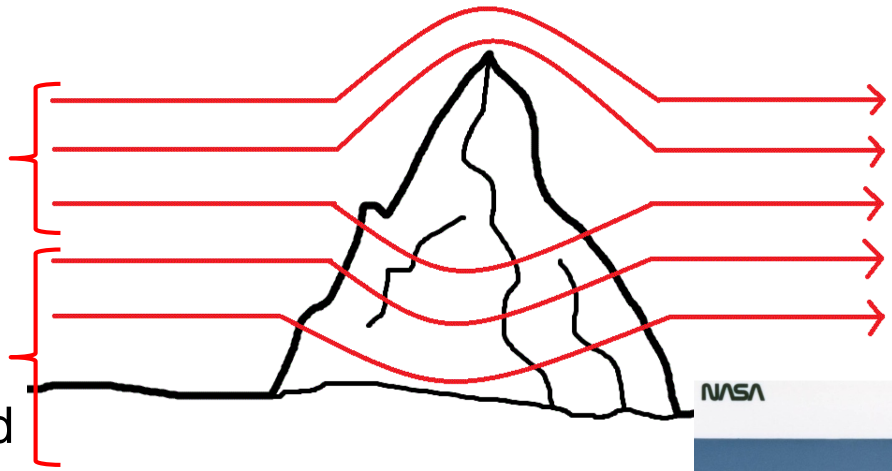
fire

pollution

Adimensional number that gives importance of stratification: Froude number

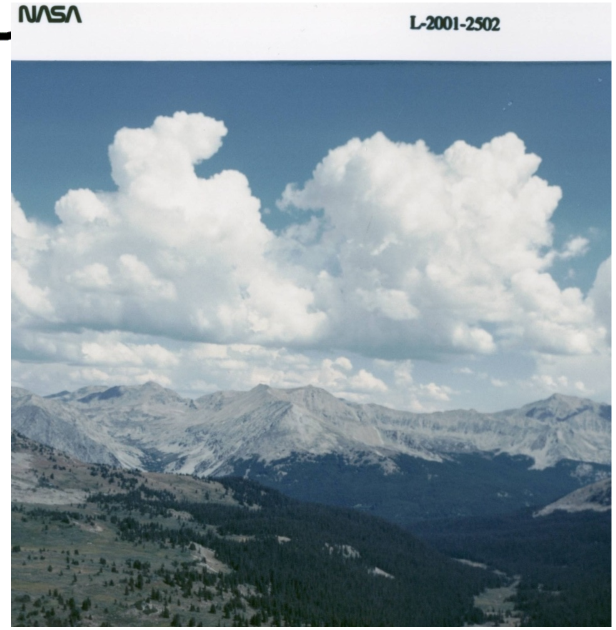
Fr > 1 => unblocked flow, passes over topography

Fr <1 => blocked flow, can not go above, flows around

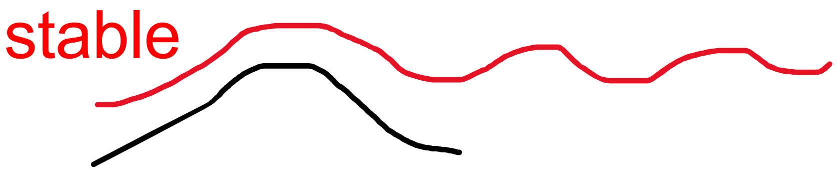


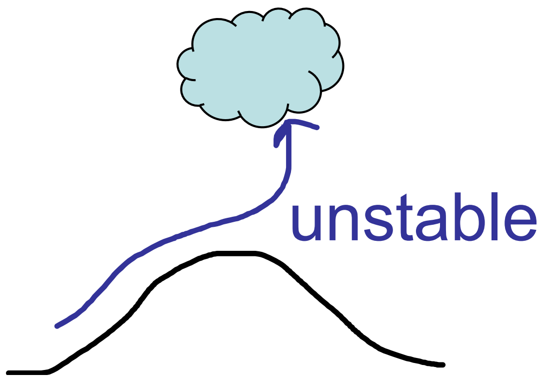
Orographic lift can lead to waves downstream

⇒ can be visible with condensation/cloud formation



Downwind waves behind Amsterdam island (Indian Ocean)

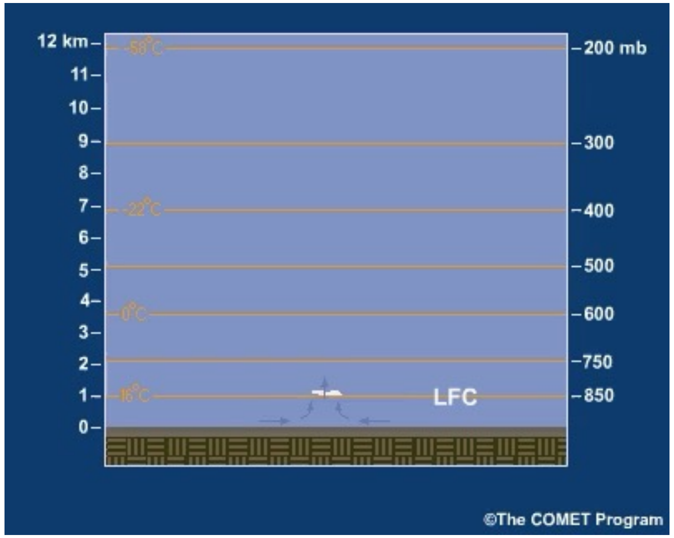




Ocean vs atmosphere

Differences:

Atm: water vapor, phase change, latent heat



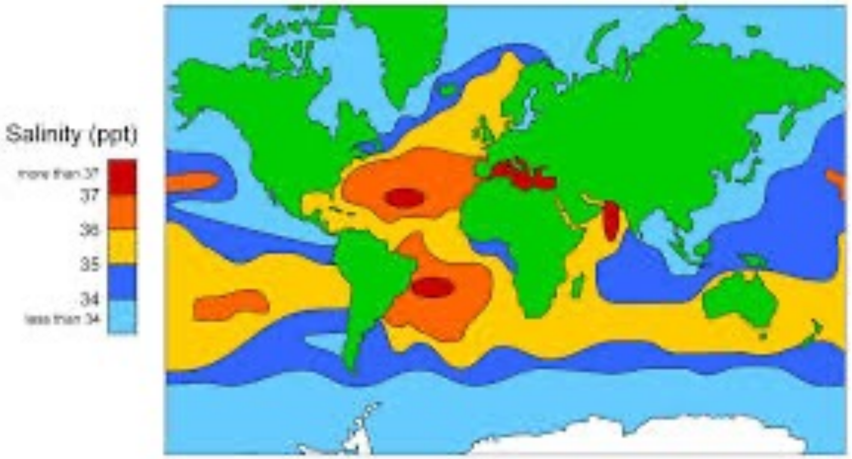
Ocean vs atmosphere

Differences:

Ocean:

obstacles / continents
=> gyres

salinity



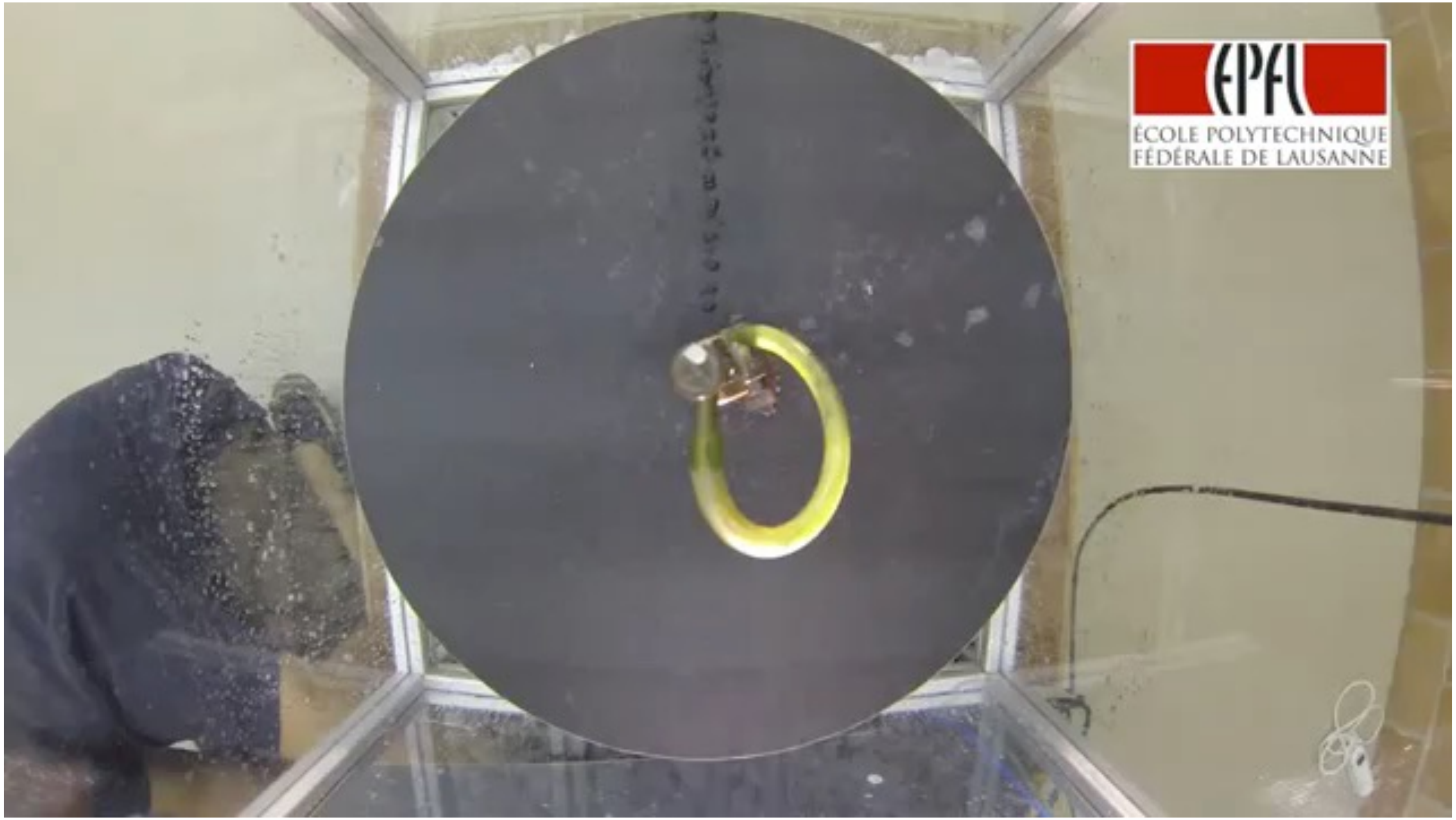
B-CORIOLIS

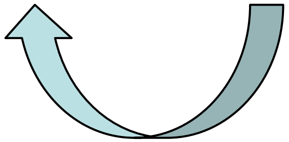
Coriolis

Centrifugal (outward) + Coriolis (deviation to the left here)

Table turns clockwise.
Yellow person will throw the ball to the green person in front.
Who receives it?

Coriolis





The table will turn clockwise.
How will the water jet behave ?

Coriolis

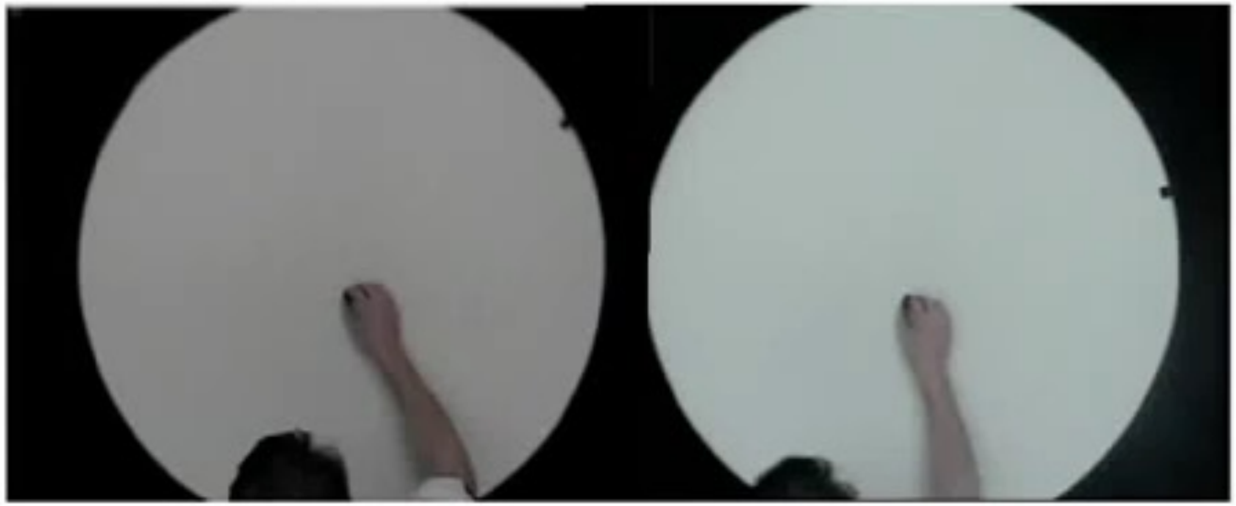


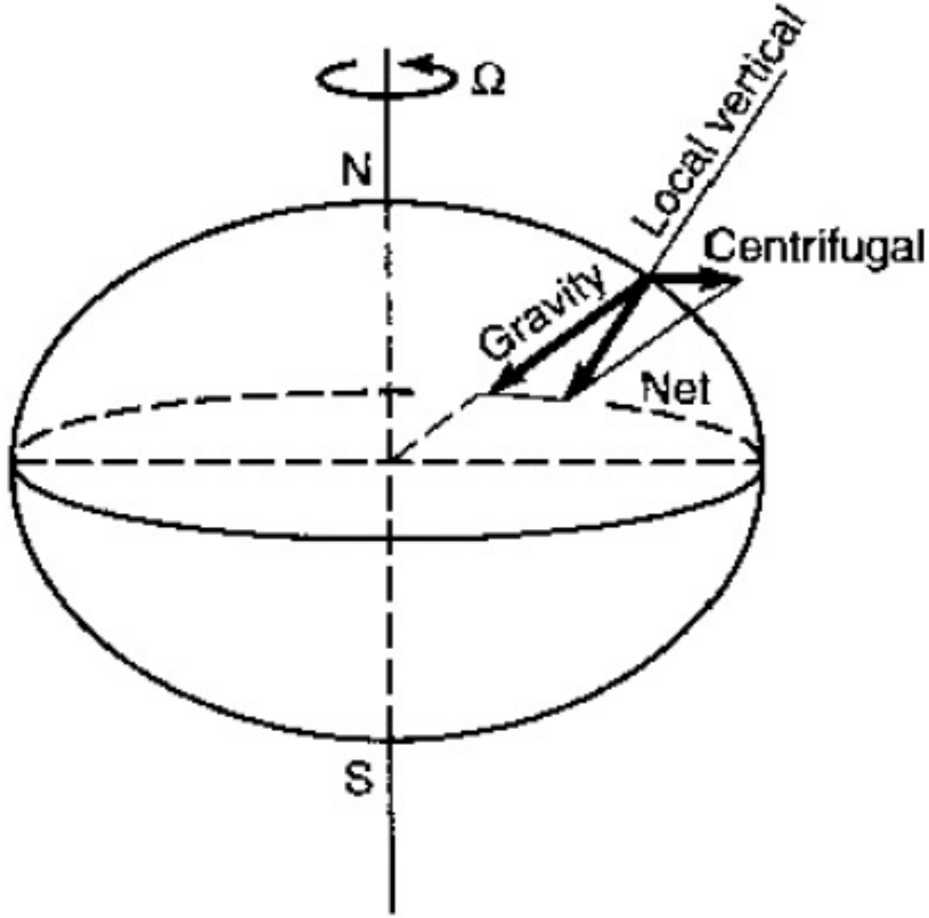
Table rotation counterclockwise

Fixed frame

Rotating frame

Experience with parabolic bottom to compensate the centrifugal force.

Unimportance of centrifugal force



How the flattening of the Earth (very exaggerated in this schematic) has flattened the Earth to reach a balance between gravity and centrifugal forces, leading to a net gravity force perpendicular to the surface.

Cartesian frame of reference

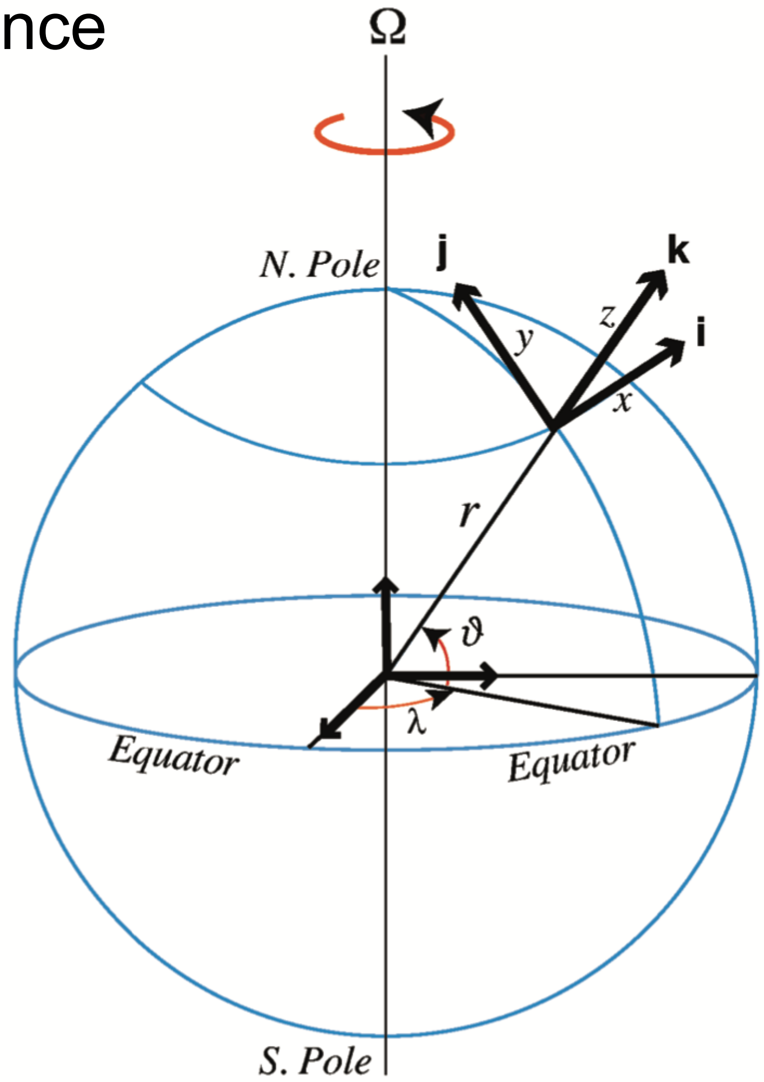


Fig. 2.3 The spherical coordinate system. The orthogonal unit vectors **i**, **j** and **k** point in the direction of increasing longitude λ, latitude ϑ, and altitude z. Locally, one may apply a Cartesian system with variables x, y and z measuring distances along **i**, **j** and **k**.

Traditional approximation: only consider the component of Earth rotation along local vertical $f = 2\,\Omega\,\sin(y)$ *(due to small vertical/horizontal aspect ratio of geophysical flows)*

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\,\Delta\mathrm{u}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\,\Delta\mathrm{v}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + \nu\,\Delta\mathrm{w}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$

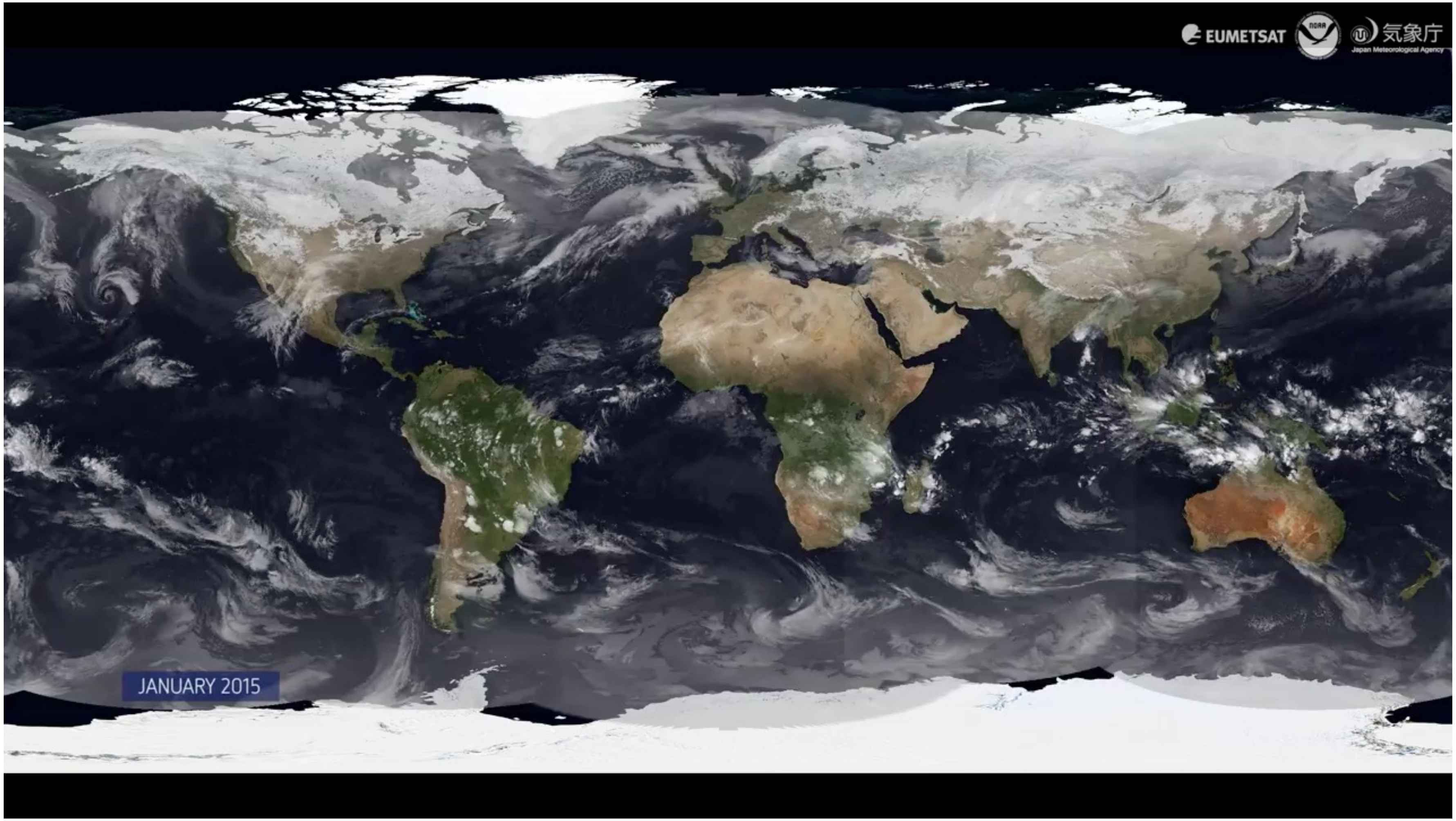
On the f-plane : $f = f_0$

On the β-plane : $f = f_0 + \beta y$

C-HOMOGENEOUS ROTATING FLOW

Rossby number

Coriolis matters for large-scale motion (U/(fL)<<1)
e.g. mid latitude low/high pressure ; tropical cyclones…

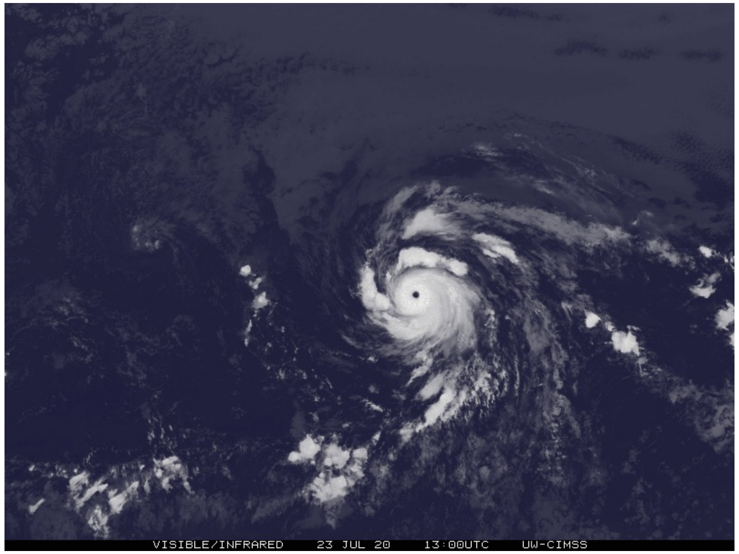


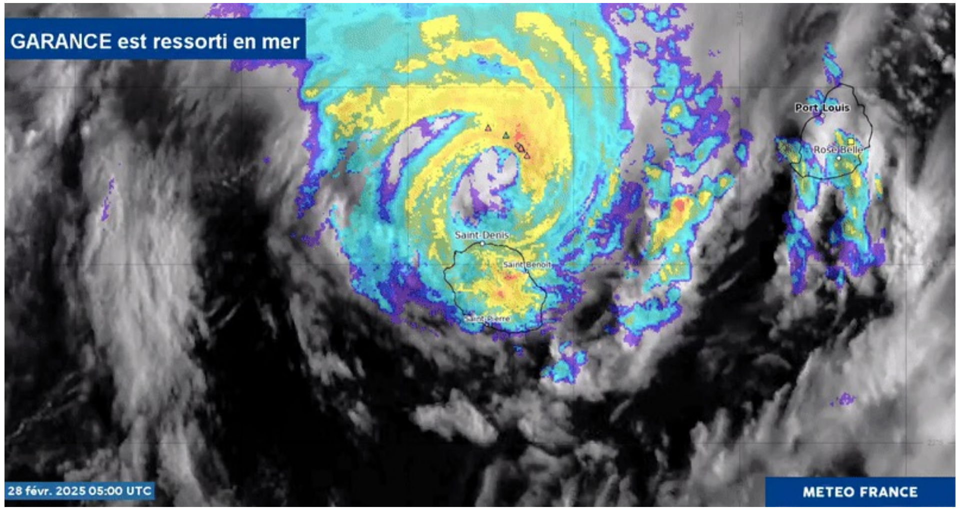
A Year of Weather 2015

Rossby number

TROPICAL CYCLONE: northern hemisphere

Southern hemisphere



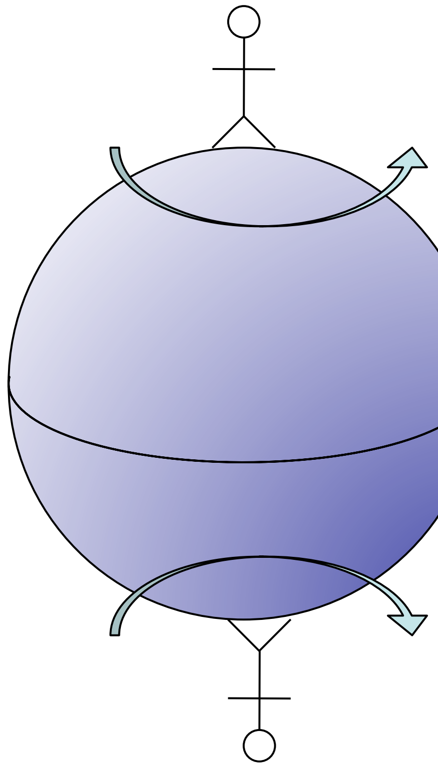


Cyclonic rotation around low pressure
Anticyclonic rotation around high pressure

Cyclonic : which rotates in the same direction as the Earth

$\Rightarrow$ counter clockwise in the northern hemisphere
$\Rightarrow$ clockwise in the southern hemisphere

Cyclonic rotation around low pressure
Anticyclonic rotation around high pressure

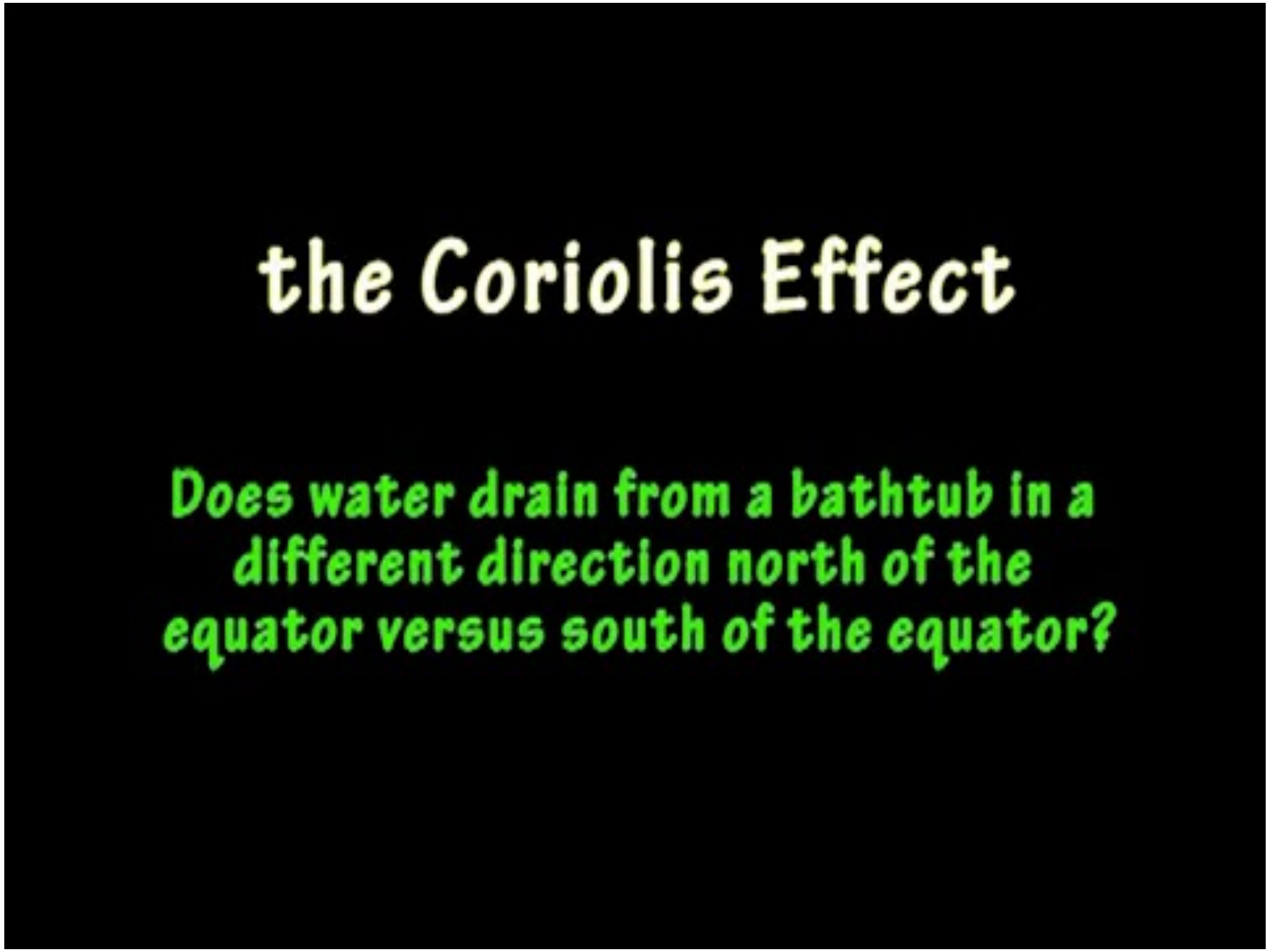


Cyclonic : which rotates in the same direction as the Earth

$\Rightarrow$ counter clockwise in the northern hemisphere
$\Rightarrow$ clockwise in the southern hemisphere

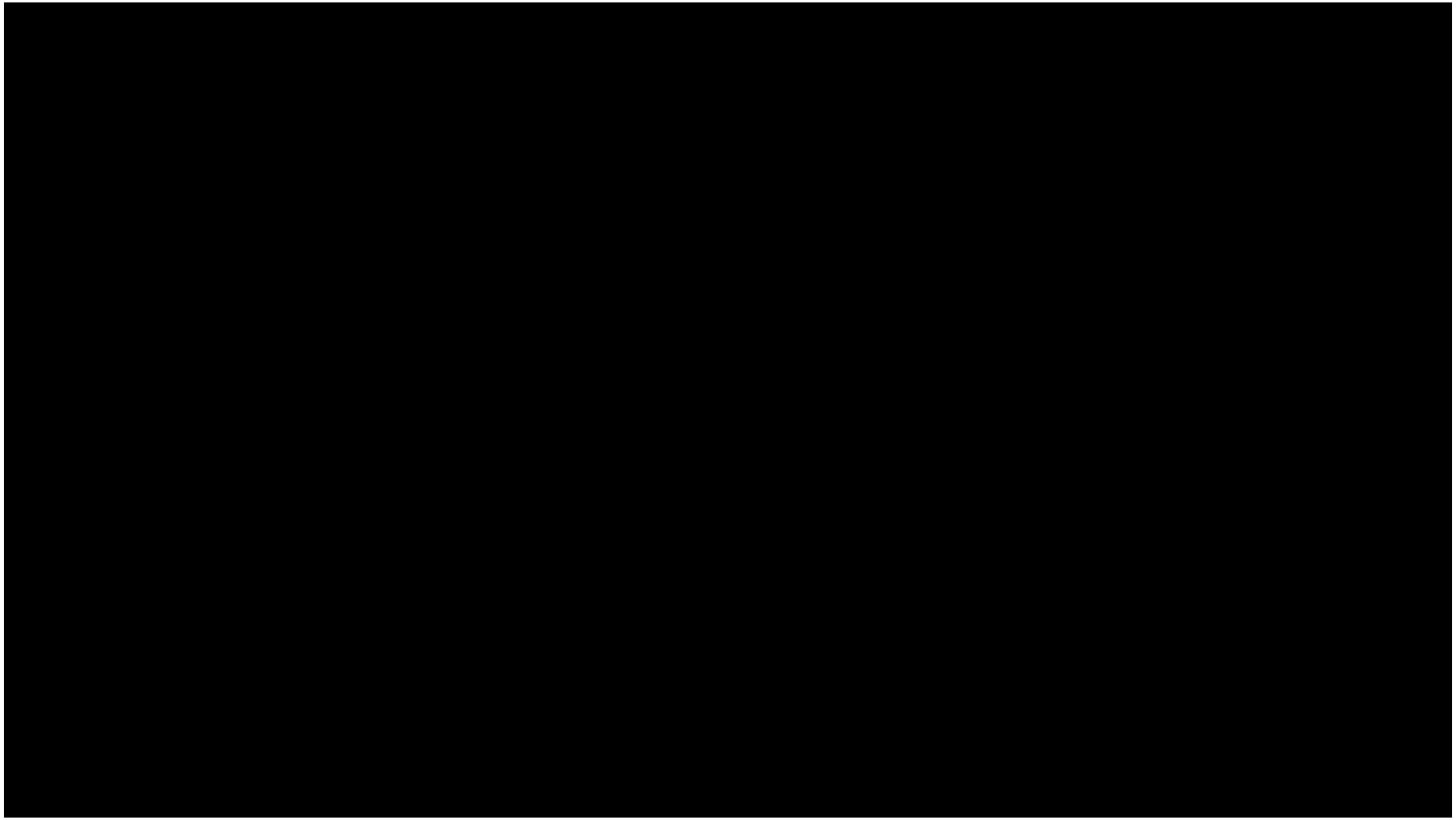
Rossby number: Urban legends

Sink/bathtub vortex



Rossby number: Urban legends

Sink/bathtub vortex



Hydrostatic balance:

Excellent approximation for large-scale motion $(H/L)^2 << 1$

Not for aspect ration ~ 1 (e.g. clouds where vertical acceleration important)

Remark on cloud formation: upward motion and cooling

$\Rightarrow$ condensation of water vapor into liquid/ice

Courtesy : Octave Tessiot

Full equations for frictionless homogeneous fluid

with dynamic pressure p

(= departure from hydrostatic pressure: $\mathrm{p_{tot}} = \mathrm{p_0} + \mathrm{p}$, with $\frac{\partial p_0}{\partial z} = -\,\rho_0\, g$ *)*

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

(*Will see in recitation*): **Rapidly rotating fliud**

⇒ Geostrophic wind (⇔ balance between Coriolis and pressure gradient)

⇒ Cyclonic around low pressure

August

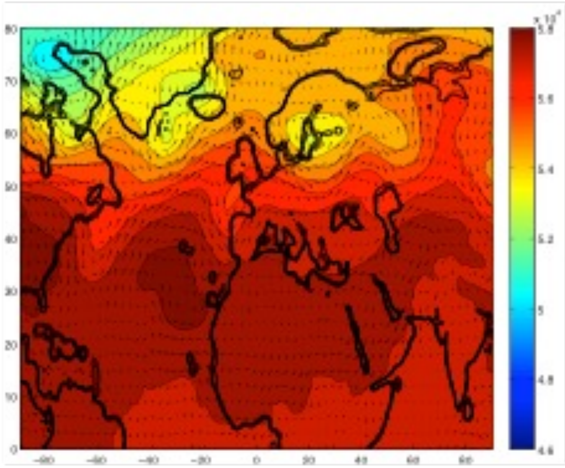


Figure 19 : vitesse et géopotentiel à 500hPa le 15 août 2003

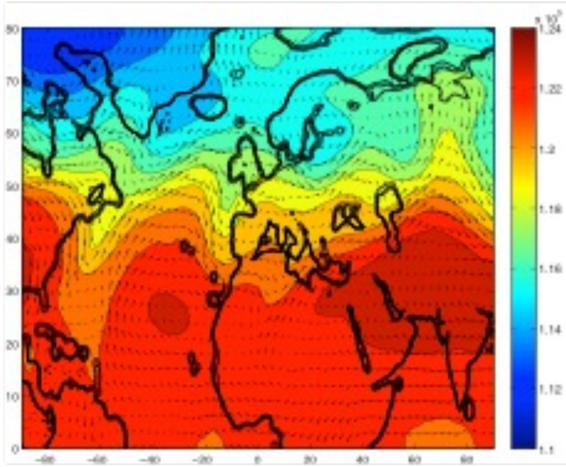
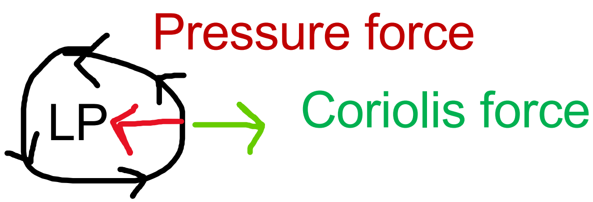


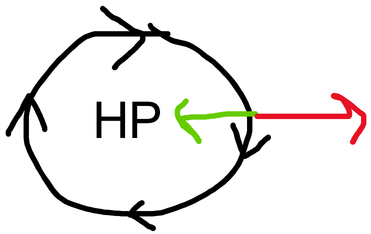
Figure 20: vitesse et géopotentiel à 200hPa.

Wind speed and geopotential height
(high geopotential height anomaly ⇔ high pressure)

Northern hemisphere=>Coriolis to the right of velocity



Low pressure LP => cyclonic



High pressure HP=>anticyclonic

(*Will see in recitation*): **Rapidly rotating fliud**

⇒ Geostrophic wind (⇔ balance between Coriolis and pressure gradient)

August

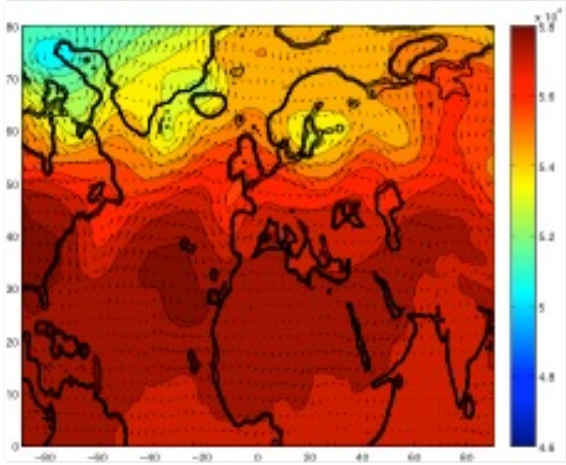


Figure 19 : vitesse et géopotentiel à 500hPa le 15 août 2003

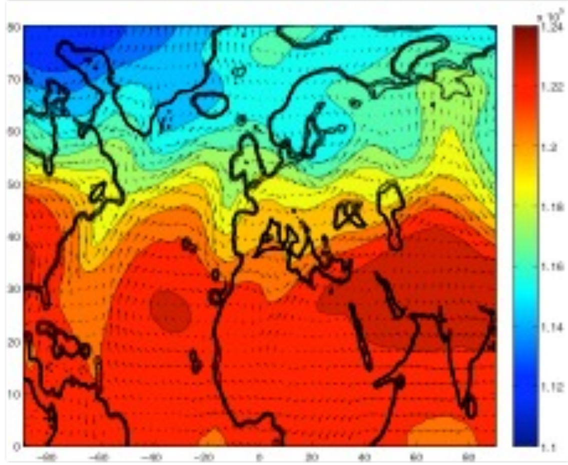


Figure 20: vitesse et géopotentiel à 200hPa.

December

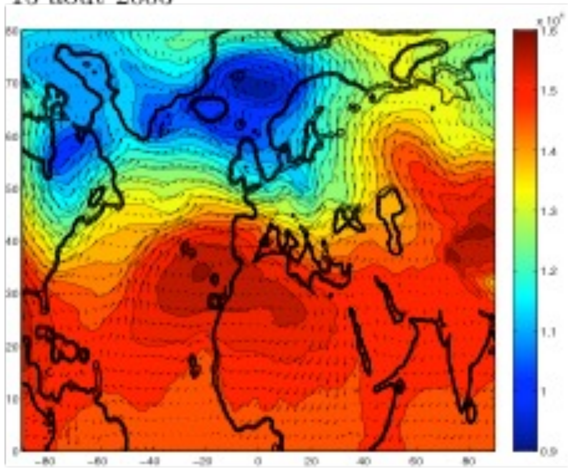


Figure 21 : vitesse et géopotentiel à 850hPa le 26 décembre 1999 .

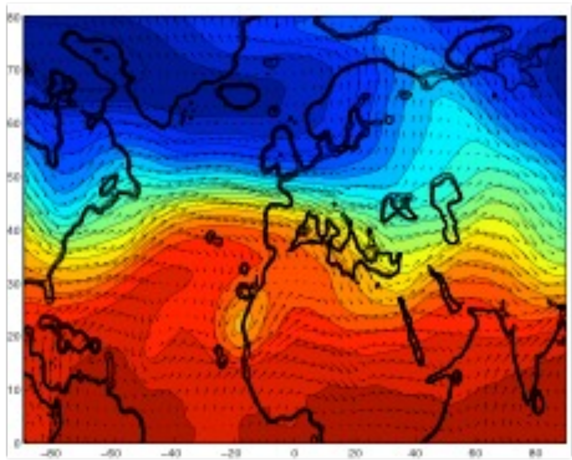
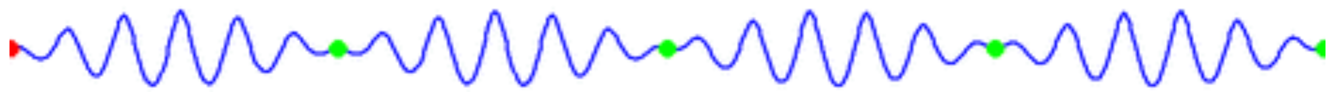


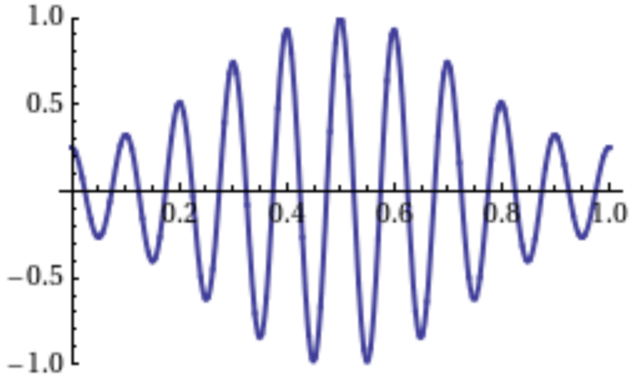
Figure 22: vitesse et géopotentiel à 200hPa.

Some properties:
- geostrophic wind flows parallel to iso-lines of geopotential height
- stronger when contours closer ($\| \overrightarrow{u_g} \| \propto \| \nabla p \|$)
- non divergent on the f-plane

Vorticity dynamics – Rossby waves

Phase vs group velocity





Recall : circulation: Midlatitudes

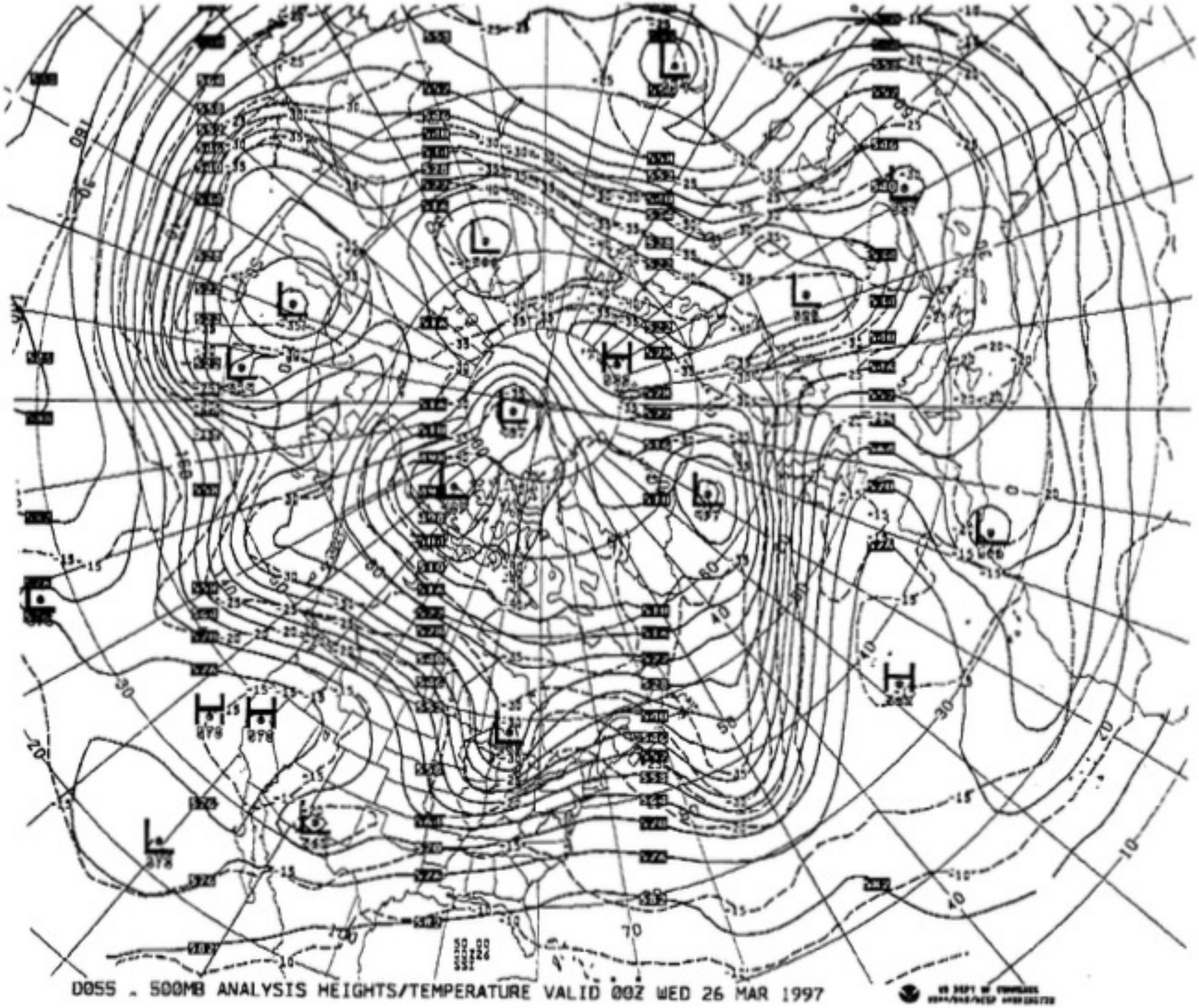


Figure 6.2: 500hPa analysis (solid contours), 26 Mar 1997.

Time height average => mode 3

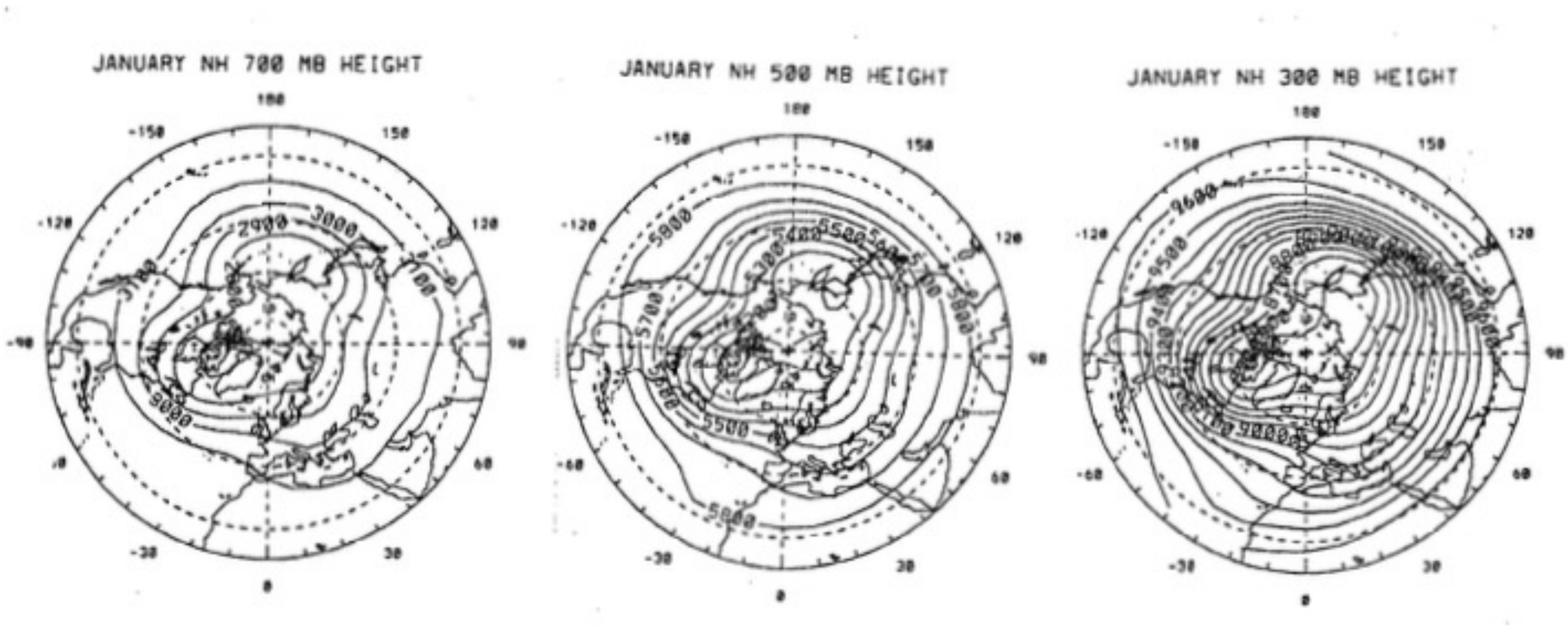
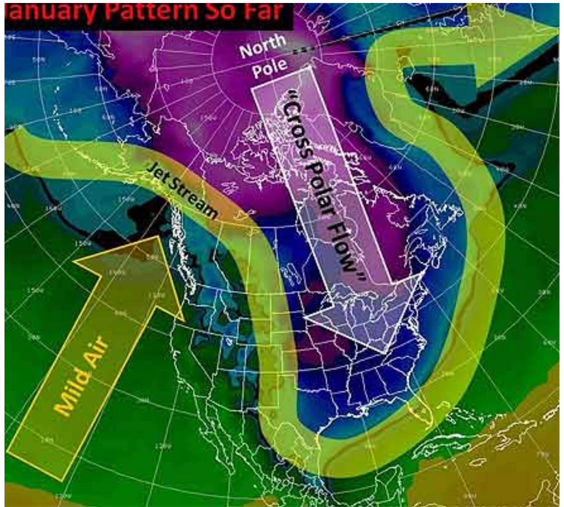


Figure 6.3: Long-term January mean heights, 300 - 700hPa.

$\Rightarrow$ Stationary wave pattern barotropic

Vorticity dynamics – Rossby waves



Monthly average temperatures (Northeast North Dakota)

(Based on climate data from Cavalier and Towner)

Temperature	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Mean Max (F)	16	20	34	54	69	77	84	83	72	59	37	23
Mean Min (F)	−4	0	14	30	42	52	58	55	45	39	18	4
Mean Max (C)	−9	−7	1	12	21	25	29	28	22	15	3	−5
Mean Min (C)	−20	−18	−10	−1	6	11	14	13	7	4	−8	−16

D-STRATIFICATION

OCEAN

D.1 Boussinesq approximation

D.2 Energy equation and equation of state for the ocean

D.3 Static stability

ATMOSPHERE

D.4 Ideal gas law and energy equation for atmosphere

D.5 Static stability

D.1 Boussinesq approximation

Recall the overall equations

$$
\begin{aligned}
\frac{Du}{Dt} - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\
\frac{Dv}{Dt} + fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \\
\frac{Dw}{Dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \\
\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{u} &= 0.
\end{aligned}
$$

=> 4 equations for 5 unknowns. Completed with equation of state and energy equation

A common further approximation is the Boussinesq approximation $\delta\rho \ll \rho_0$.
OK for ocean: $\delta\rho \sim$ few ≤ 3 kg m^{-3}; $\rho_0 \sim 1000$ kg m^{-3}
OK for shallow atmospheric layer $\delta\rho \sim$ few g m^{-3}; $\rho_0 \sim 1$ kg m^{-3}
BUT NOT OK for atmospheric motion in general, $\rho(z)$ decreases strongly with altitude from ρ_0 to ≈ 0.

*If $\delta\rho \ll \rho_0$, we can write $\rho(x,y,z,t) = \rho_0 + \rho'$, $\rho' \ll \rho_0$. The the continuity equation becomes:

$$\frac{D\rho'}{Dt} + (\rho_0+\rho')\nabla\cdot\vec{u} = \underbrace{\frac{\partial \rho'}{\partial t} + u\frac{\partial \rho'}{\partial x} + v\frac{\partial \rho'}{\partial y} + w\frac{\partial \rho'}{\partial z}}_{\mathcal{O}(\epsilon)} + (\underbrace{\rho_0}_{\mathcal{O}(1)} + \underbrace{\rho'}_{\mathcal{O}(\epsilon)})\nabla\cdot\vec{u} = 0$$

$$\boxed{\Rightarrow \nabla\cdot\vec{u} \approx 0}$$

Says that conservation of mass becomes conservation of volume.

---

*Vertical momentum: Again, introduce dynamic pressure p' such that

$$\begin{aligned} p_{tot} &= p_0(z) + p'(x,y,z,t), \text{ with } p_0(z) = P_0 - \rho_0 g z \\ \Rightarrow (\rho_0+\rho')\frac{Dw}{Dt} &= -\frac{\partial p_0}{\partial z} - \frac{\partial p'}{\partial z} - (\rho_0+\rho')g \end{aligned}$$

$$\boxed{\Rightarrow \frac{Dw}{Dt} \approx -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho'}{\rho_0}g}$$

---

*Horizontal momentum:

$$\frac{1}{\rho_0+\rho'} \approx \frac{1}{\rho_0}\left(1-\frac{\rho'}{\rho_0}\right) \approx \frac{1}{\rho_0}$$

and

$$\frac{\partial p}{\partial x,y} = \frac{\partial p'}{\partial x,y}$$

$$\boxed{\Longrightarrow \begin{aligned} \frac{Du}{Dt} - fv &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \\ \frac{Dv}{Dt} + fu &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} \end{aligned}}$$

=> Boussinesq: ρ replaced by ρ_0 except in front of gravity

Still 4 (simplified) equations for 5 unknown

D.2 Energy equation and equation of state for the ocean

*Ocean equation of state

$$\rho = \rho_0[1 - \alpha_T(T - T_0) + \alpha_S(S - S_0)]$$

linear to first approximation, with S salinity. The salt equation is simply a conservation one

$$\boxed{\frac{DS}{Dt} = \kappa_S \Delta S,}\ \kappa_S \text{ coefficient of salt diffusion}$$

---

*Energy equation

$$dU = \delta Q - p\,dV,$$

where V denotes volume, which we write per unit mass $\alpha = 1/\rho$. δQ denote the rate of heating per unit mass. Writing c_v the specific (i.e. per unit mass) heat capacity at constant volume:

$$\Rightarrow c_v dT = \delta Q - p d\alpha \Leftrightarrow c_v dT - \frac{p}{\rho^2} d\rho = \delta Q$$

Thus, using the continuity equation, $\frac{D\rho}{Dt} = -\rho \nabla \cdot \vec{u}$, we obtain

$$c_v \frac{DT}{Dt} + \frac{p}{\rho} \underbrace{\nabla \cdot \vec{u}}_{=0} = \delta Q$$

Assuming all heating comes from diffusion $\propto \kappa_T \Delta T$, we obtain:

$$\boxed{\frac{DT}{Dt} = \kappa_T \Delta T}$$

We further assume $\kappa_T \approx \kappa_S \equiv \kappa$ ($\sim 10^{-2}$ m^2 s^{-1}), thus from the S, T equations and the equation of state:

$$\boxed{\frac{D\rho}{Dt} = \kappa \Delta \rho}$$

=> Our Boussinesq system of equations is complete

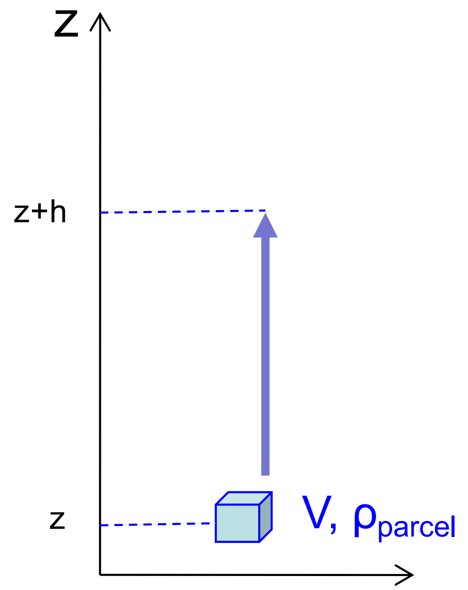
$$\begin{aligned}
\frac{Du}{Dt} - fv &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \\
\frac{Dv}{Dt} + fu &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} \\
\frac{Dw}{Dt} &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho'}{\rho_0}g \\
\nabla \cdot \vec{u} &= 0 \\
\frac{D\rho'}{Dt} &= \kappa \Delta \rho'
\end{aligned}$$

This system is sometimes written for buoyancy $b = -g\rho'/\rho_0$ (and omitting primes):

$$\begin{aligned}
\frac{Du}{Dt} - fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\
\frac{Dv}{Dt} + fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\
\frac{Dw}{Dt} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + b \\
\nabla \cdot \vec{u} &= 0 \\
\frac{Db}{Dt} &= 0 \text{ (neglecting diffusion)}
\end{aligned}$$

D.3 Static stability

Take volume V at height z with density $\rho(z)$ and displace it to height $z+h$. Assuming the fluid incompressible, $\rho_{parcel} = \rho(z)$ is conserved in the displacement.



The buoyancy force at $z+h$ is (LHS=mass $\times$ acceleration)

$$\rho(z)V\frac{d^2h}{dt^2} = g[\rho(z+h) - \rho_{parcel}(z)]V$$

Under the Boussinesq approximation

$$\rho_0 V\frac{d^2h}{dt^2} = g\frac{d\rho}{dz}hV$$

$$\Leftrightarrow \frac{d^2h}{dt^2} - \frac{g}{\rho_0}\frac{d\rho}{dz}h = 0$$

$$\Leftrightarrow \frac{d^2h}{dt^2} + N^2 h = 0,$$

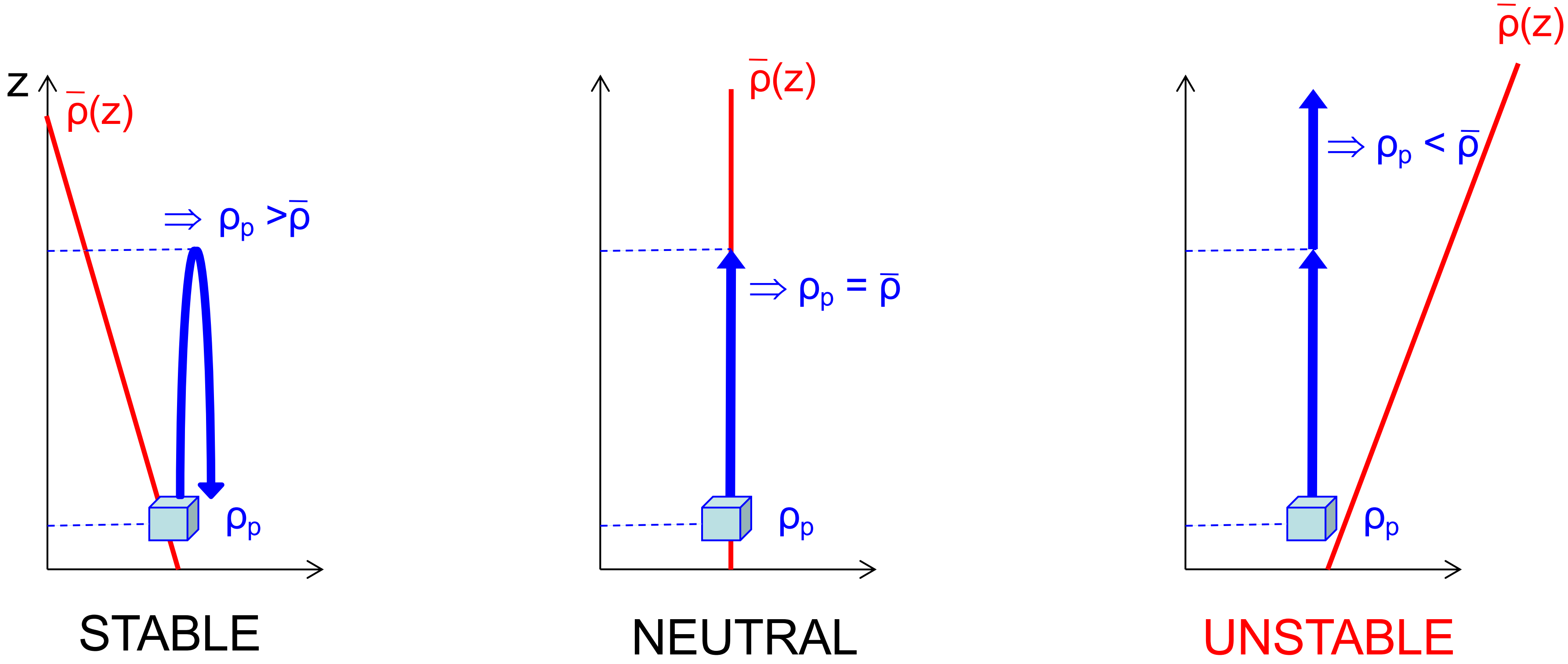
with $N^2 = -\frac{g}{\rho_0}\frac{d\rho}{dz} = \frac{db}{dz}$

-$N^2 > 0 \Rightarrow$ oscillatory solution $h = A\cos(Nt) + B\sin(Nt) \Rightarrow$ stable

-$N^2 < 0 \Rightarrow$ exponentially increasing solution $h \propto \exp(\pm|N|t) \Rightarrow$ unstable

Ocean: Boussinesq case

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho}{dz} = \frac{db}{dz}$$



N^2 called stratification frequency, or Brunt-Väisälä frequency

D-STRATIFICATION

OCEAN

D.1 Boussinesq approximation

D.2 Energy equation and equation of state for the ocean

D.3 Static stability

ATMOSPHERE

D.4 Ideal gas law and energy equation for atmosphere

D.5 Static stability

RELAX BOUSSINESQ APPROXIMATION

Recall the overall equations

$$\begin{aligned}
\frac{Du}{Dt} - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\
\frac{Dv}{Dt} + fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \\
\frac{Dw}{Dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \\
\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{u} &= 0.
\end{aligned}$$

=> Need additional equations as before (4 equations, 5 unknowns)

D.4 Ideal gas law and energy equation for atmosphere

*Ideal gas law $\boxed{p = \rho RT}$ (Ideal gas law in specific form, see recitation), R specific gas constant (but note that it depends on the gas).

Energy equation:

$$c_v dT + p\, d\alpha = \delta Q \text{ where } \alpha = 1/\rho \text{ specific volume.} \tag{49}$$

$$\Leftrightarrow c_p dT - \alpha dp = \delta Q \tag{50}$$

where c_v specific heat capacity at constant volume, c_p specific heat capacity at constant pressure, and where we've used $p\alpha = RT$ and $pd\alpha = d(p\alpha) - \alpha dp = RdT - \alpha dp$ and $c_v + R = c_p$.

Remark $c_v < c_p$ makes sense?

D.4 Ideal gas law and energy equation for atmosphere

*Ideal gas law $\boxed{p = \rho RT}$ (Ideal gas law in specific form, see recitation), R specific gas constant (but note that it depends on the gas).

Energy equation:

$$
\begin{aligned}
c_v dT + p\, d\alpha &= \delta Q \text{ where } \alpha = 1/\rho \text{ specific volume.} &(49)\\
\Leftrightarrow c_p dT - \alpha dp &= \delta Q &(50)
\end{aligned}
$$

where c_v specific heat capacity at constant volume, c_p specific heat capacity at constant pressure, and where we've used $p\alpha = RT$ and $pd\alpha = d(p\alpha) - \alpha dp = RdT - \alpha dp$ and $c_v + R = c_p$.

Remark $c_v < c_p$ makes sense?

$\Delta T_{cstt\, volume} > \Delta T_{cstt\, pressure}$ (figure 3), as some of the heating goes into increasing volume. Thus $c_v < c_p$. $c_{v,dry\, air} \approx 719$ J kg^{-1} K^{-1}, $c_{p,dry\, air} \approx 1005$ J kg^{-1} K^{-1}

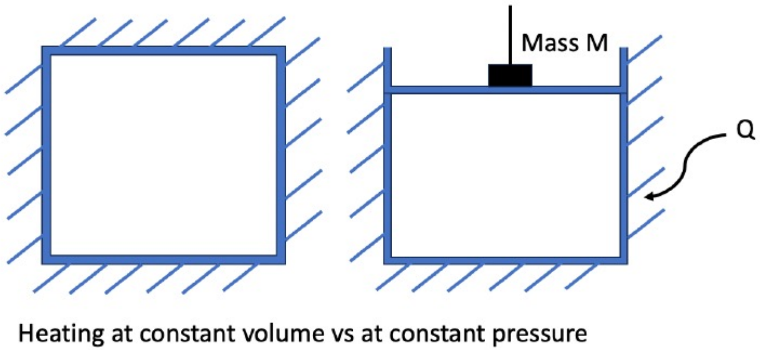


Figure 3: Heat added at constant volume vs constant pressure.

Energy equation becomes:

$$
\begin{aligned}
c_p dT - \frac{1}{\rho} dp &= c_p dT - \frac{RT}{p} dp = \delta Q \\
c_p T \left(\frac{dT}{T} - \frac{R}{c_p} \frac{dp}{p} \right) &= c_p T \frac{d\theta}{\theta} = \delta Q
\end{aligned}
$$

where $\theta = T \left(\frac{p}{p_0} \right)^{-R/c_p}$, p_0 reference pressure typically 1000 hPa.

Remark θ is an adiabatic invariant: $D\theta/Dt = 0$ under adiabatic displacement. It is called potential temperature, and corresponds to the temperature of a parcel of air brought adiabatically at p_0 (i.e. removes compressibility). θ allows to compare ρ at different p.

$\Longrightarrow$ the equation becomes

$$\frac{T}{\theta} \frac{D\theta}{Dt} = Q, \text{ where } Q \text{ is the rate of heating in K s}^{-1}$$

or

$$\boxed{\frac{D\theta}{Dt} = Q_\theta,} \text{ (where } Q_\theta = \dot{\theta} = \frac{\theta}{T} Q_T\text{)}$$

Thus the equations become:

$$
\begin{aligned}
\frac{Du}{Dt} - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\
\frac{Dv}{Dt} + fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \\
\frac{Dw}{Dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \\
\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{u} &= 0 \\
\frac{D\theta}{Dt} &= Q_\theta \\
p = \rho RT \Leftrightarrow \rho = f(p,\theta)
\end{aligned}
$$

This system is complete.

(the last equation p=ρ R T can be written between p ρ and θ using the definition of potential temperature θ:

$$\theta = T\left(\frac{p}{p_0}\right)^{-R/c_p}$$)

D.5 Static stability

Static stability can be investigated using a similar computation as in the ocean case above, but noting that buoyancy $b = -g\dfrac{\rho'}{\rho_0}$ is replaced with $b = g\dfrac{\theta'}{\theta_0}$. This comes from the observation that pressure perturbations in atmospheric motion are much smaller than temperature perturbations. In other words, $-\rho'/\rho \approx \theta'/\theta$. (Indeed, $p'/p \ll T'/T$ for all displacements slower than the speed of molecules $\sim$ speed of sound. So pressure adjusts much faster than temperature, and to a good approximation $\rho'/\rho \approx \theta'/\theta$ from $p = \rho RT \Rightarrow \rho'/\rho \approx -T'/T \approx -\theta'/\theta$ neglecting p'/p contributions.)

Thus, similarly in that case, stability is determined by the sign of $N^2 = \dfrac{db}{dz} = \dfrac{g}{\theta_0}\dfrac{d\theta}{dz}$ (to see this, your can redo the earlier static stability analysis in this now compressible case, where buoyancy $b = -g\dfrac{\rho'}{\rho_0}$ is replaced with $b = g\dfrac{\theta'}{\theta_0}$):

-$N^2 > 0 \Rightarrow$ oscillatory solution $h = A\cos(Nt) + B\sin(Nt) \Rightarrow$ stable

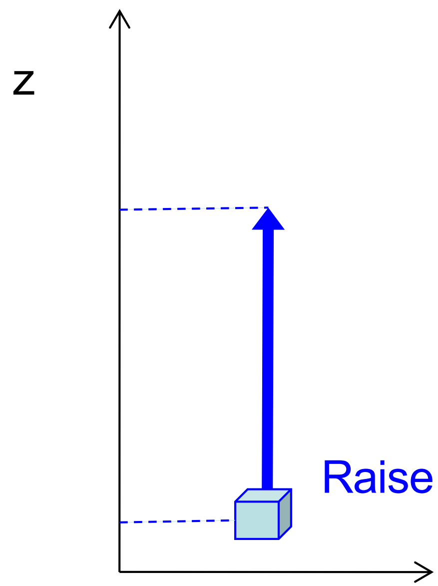
-$N^2 < 0 \Rightarrow$ exponentially increasing solution $h \propto \exp(\pm|N|t) \Rightarrow$ unstable

Atmospheric thermodynamics: instability

Dry convection

T decreases with height.
But p as well.

Density = ρ(T,p).
How determine stability? The parcel method



Raise parcel adiabatically. Comes back to initial position?

Atmospheric thermodynamics: instability

Dry convection

Potential temperature $\theta = T\,(p_0 / p)^{R/cp}$ conserved under adiabatic displacements :

Adiabatic displacement

1st law thermodynamics: d(internal energy) = Q (heat added) – W (work done by parcel)

$$c_v\, dT = -\, p\, d(1/\rho)$$

Since $p = \rho R T$, $\quad c_v\, dT = -\,p\, d(R T / p) = -\,R\, dT + R T\, dp / p$

Since $c_v + R = c_p$, $\quad c_p\, dT / T = R\, dp / p$

$$\Rightarrow d \ln T - R / c_p\, d \ln p = d \ln (T / p^{R/cp}) = 0$$

$$\Rightarrow T / p^{R/cp} = \text{constant}$$

Hence $\theta = T\,(p_0 / p)^{R/cp}$ potential temperature is conserved under adiabatic displacement

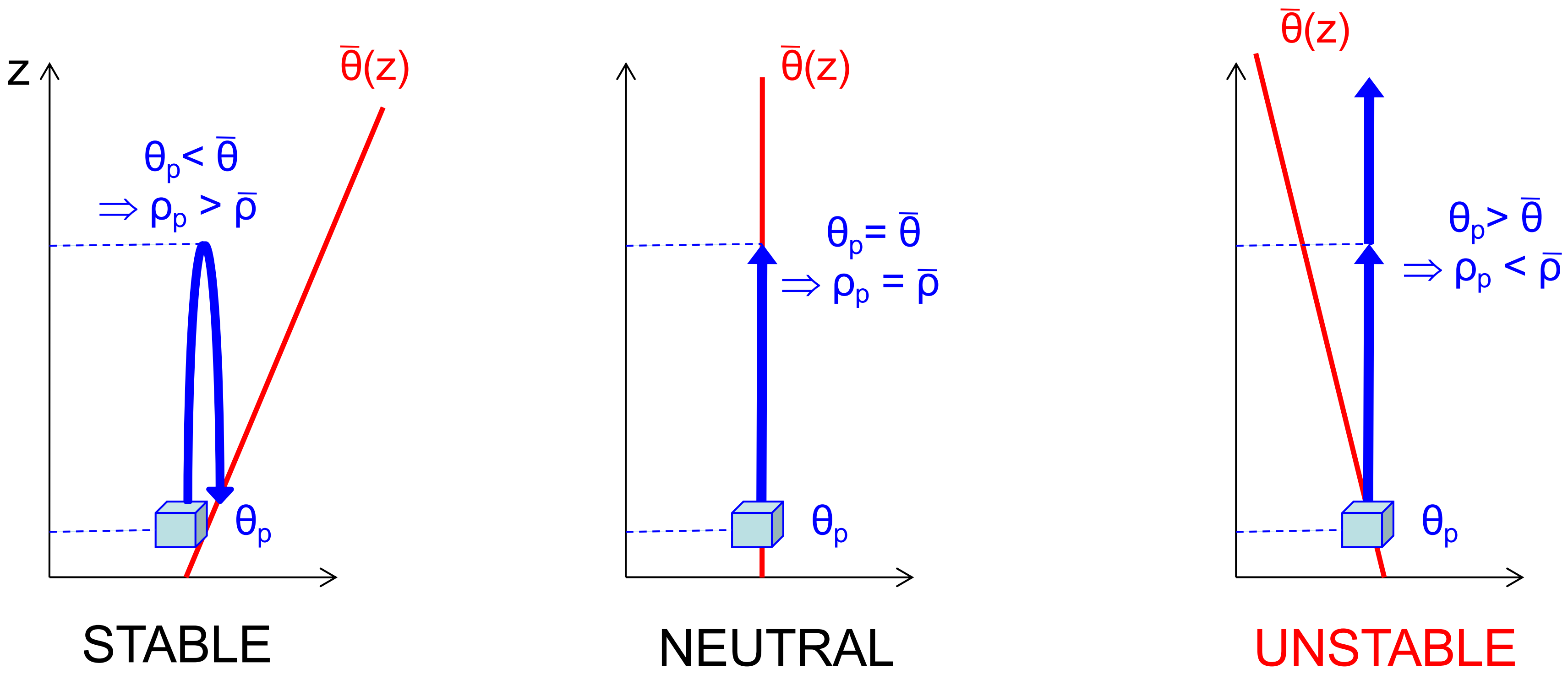
Atmospheric thermodynamics: instability

When is an atmosphere unstable to dry convection?

When potential temperature $\theta = T\,(p_0 / p)^{R/cp}$ decreases with height !

The parcel method:

Small vertical displacement of a fluid parcel adiabatic (=> θ = constant).
During movement, pressure of parcel = pressure of environment.



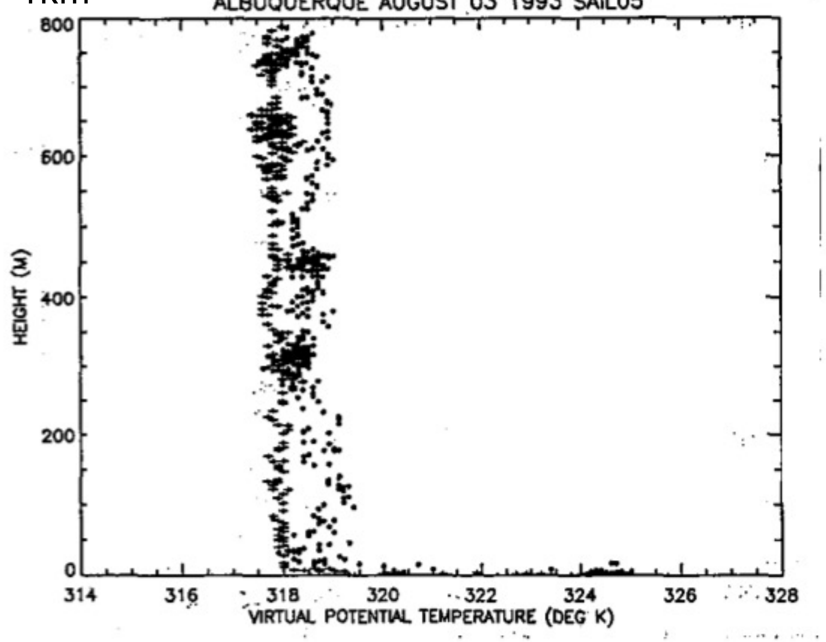
Atmospheric thermodynamics: instability

Convective adjustment time scales is very fast (minutes for dry convection) compared to destabilizing factors (surface warming, atmospheric radiative cooling…)

=> The observed state is very close to convective neutrality

Dry convective boundary layer over daytime desert

<1km



[Renno and Williams, 1995]

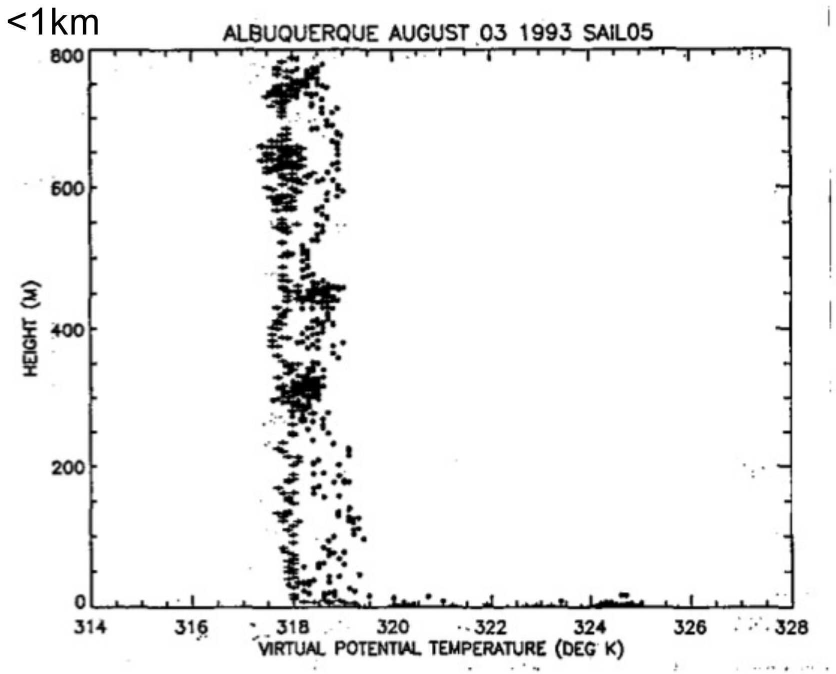
But above a thin boundary layer, not true anymore that θ = constant. Why?…

Atmospheric thermodynamics: instability

Convective adjustment time scales is very fast (minutes for dry convection) compared to destabilizing factors (surface warming, atmospheric radiative cooling…)

=> The observed state is very close to convective neutrality

Dry convective boundary layer over daytime desert



[Renno and Williams, 1995]

But above a thin boundary layer, not true anymore that θ = constant. Why?…

Most atmospheric convection involves phase change of water
Significant latent heat with phase changes of water = Moist Convection

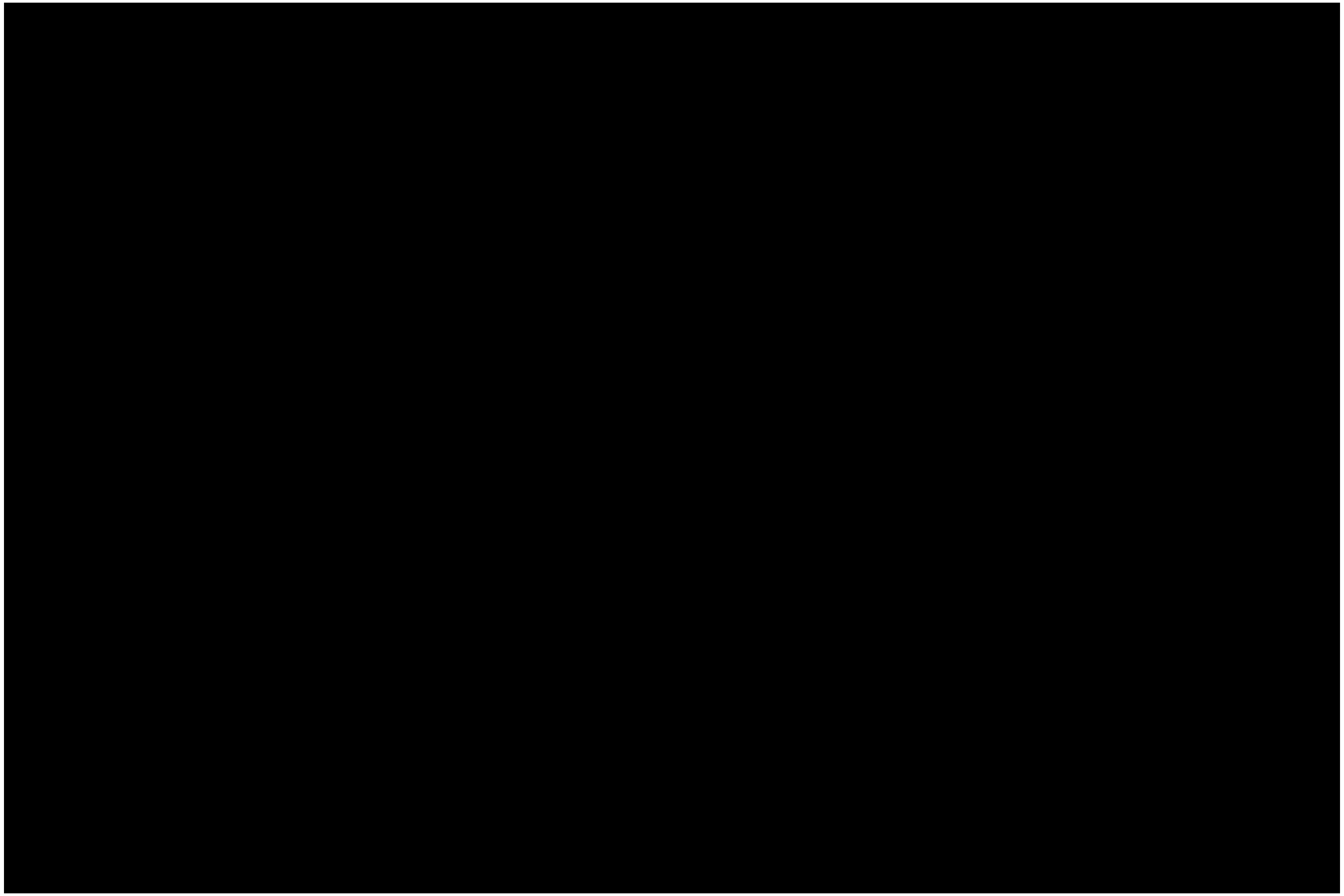
Remark on cloud formation: upward motion and cooling

$\Rightarrow$ condensation of water vapor into liquid/ice

Courtesy : Octave Tessiot

Remark on cloud formation: upward motion and cooling

$\Rightarrow$ condensation of water vapor into liquid/ice



Courtesy : Octave Tessiot

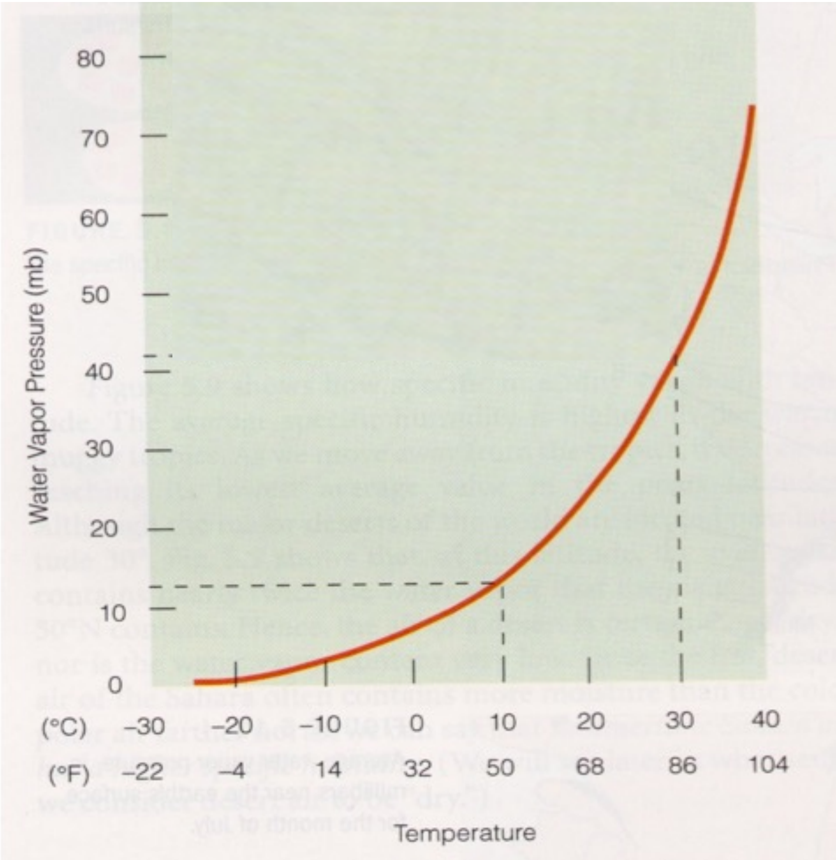
Atmospheric thermodynamics: instability

Clausius Clapeyron $$\frac{\mathrm{d}e_s}{\mathrm{d}T} = \frac{L_v(T)e_s}{R_v T^2}$$

where:

- e_s is saturation vapor pressure,
- T is a temperature,
- L_v is the specific latent heat of evaporation,
- R_v is water vapor gas constant.

$e_s(T)$



e_s depends only on temperature

e_s increases roughly exponentially with T

Warm air can hold more water vapor than cold air

Atmospheric thermodynamics: instability

When is an atmosphere unstable to moist convection ?

Equivalent potential temperature $\theta_e = T\,(p_0 / p)^{R/cp}\ e^{\,Lv\,qv / (cp\,T)}$ is conserved under adiabatic displacements :

1st law thermodynamics if air saturated (q_v=q_s) :

d(internal energy) = Q (latent heat) – W (work done by parcel)

$$c_v\,dT = -\ L_v\,dq_s - \ p\,d(1/\rho)$$

$$\Rightarrow d \ln T - R / c_p\, d \ln p = d\ \ln (T / p^{R/cp}) = -L_v / (c_p\,T)\,dq_s$$

$$\Rightarrow\ T / p^{R/cp}\ e^{\,Lv\,qs / (cp\,T)} \sim \text{constant}$$

Note: Air saturated => q_v=q_s
Air unsaturated => q_v conserved

Hence

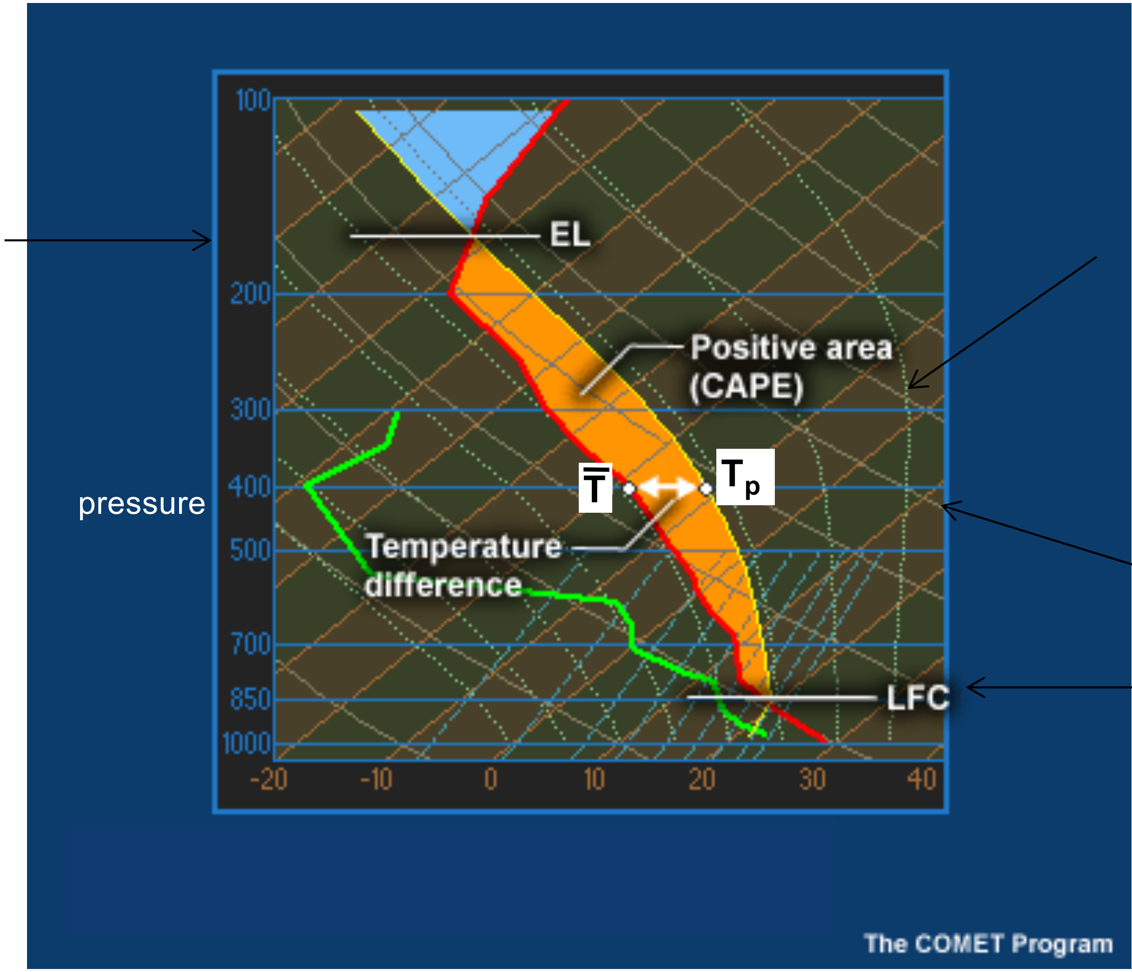
$\theta_e = T\,(p_0 / p)^{R/cp}\ e^{\,Lv\,qv / (cp\,T)}$ equivalent potential temperature is conserved

Atmospheric thermodynamics: instability

When is an atmosphere unstable to moist convection ?

Skew T diagram (isoT slanted), atmospheric T in red

EL equilibrium level



pressure

Moist adiabat θ_e= cstt

Dry adiabat θ = cstt

LFC level of free convection (= LCL lifted condensation level for simplicity)

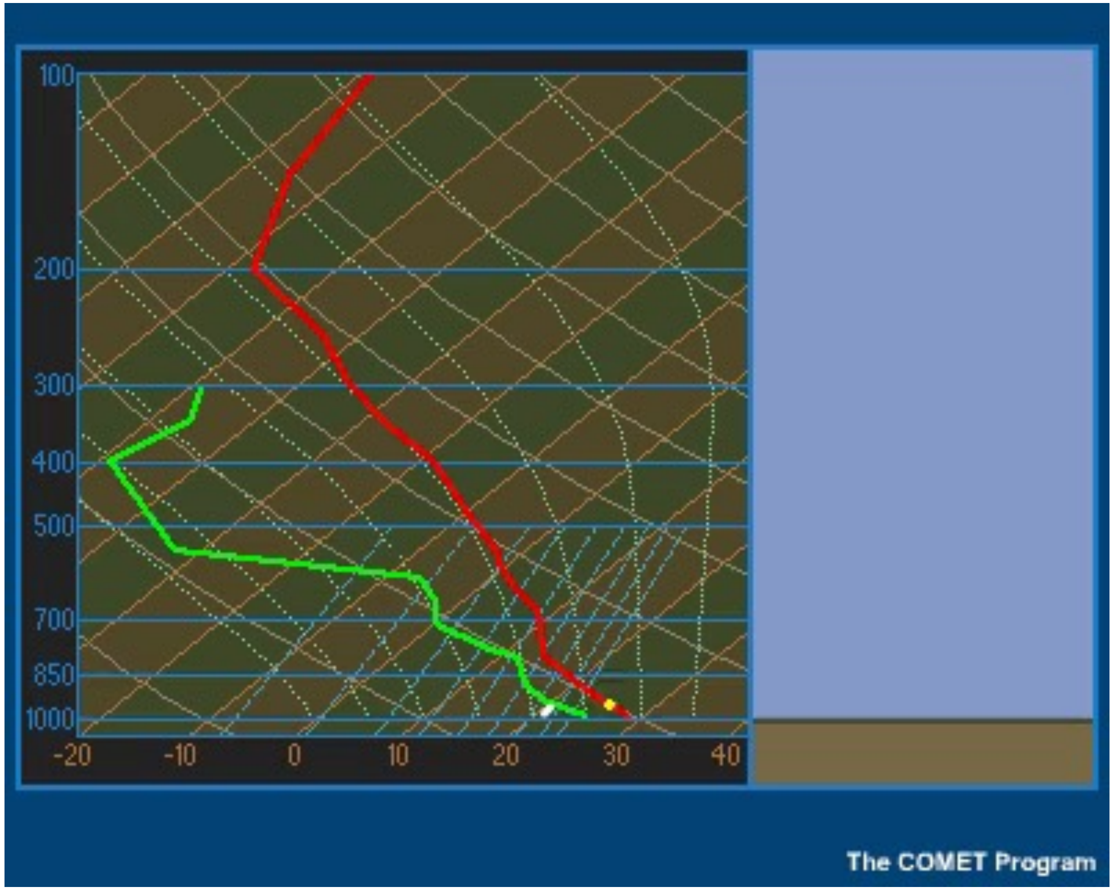
CAPE: convective available potential energy

Atmospheric thermodynamics: instability

Moist convection

Parcel = yellow dot

EL equilibrium level



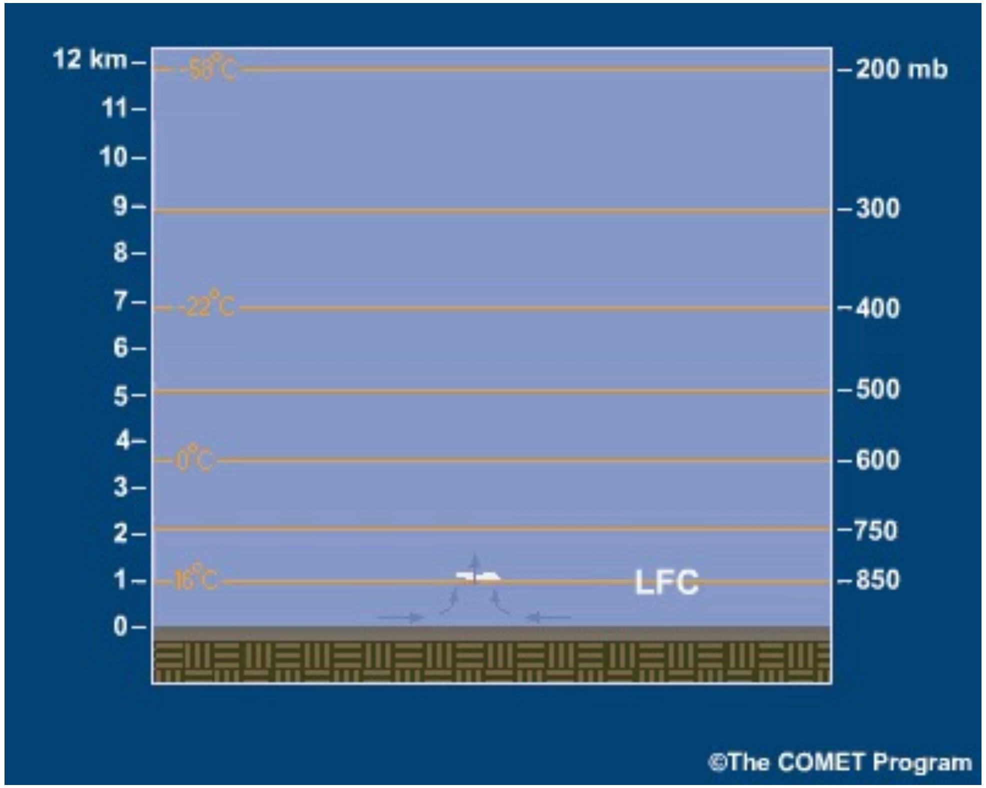
LFC level of free convection

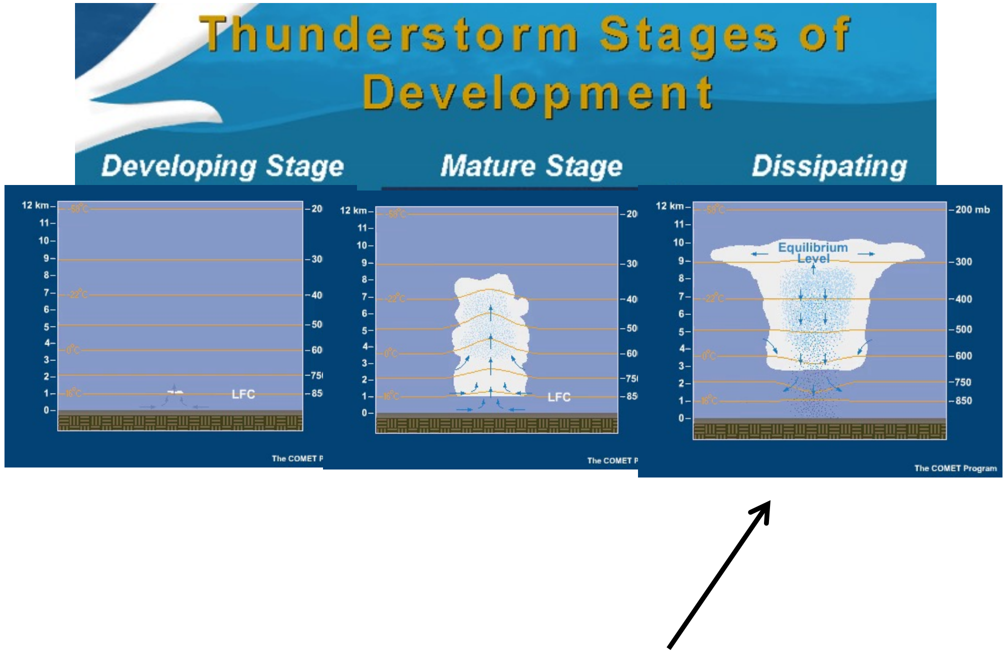
CAPE: convective available potential energy

Atmospheric thermodynamics: instability

If enough atmospheric instability present, cumulus clouds are capable of producing serious storms!!!

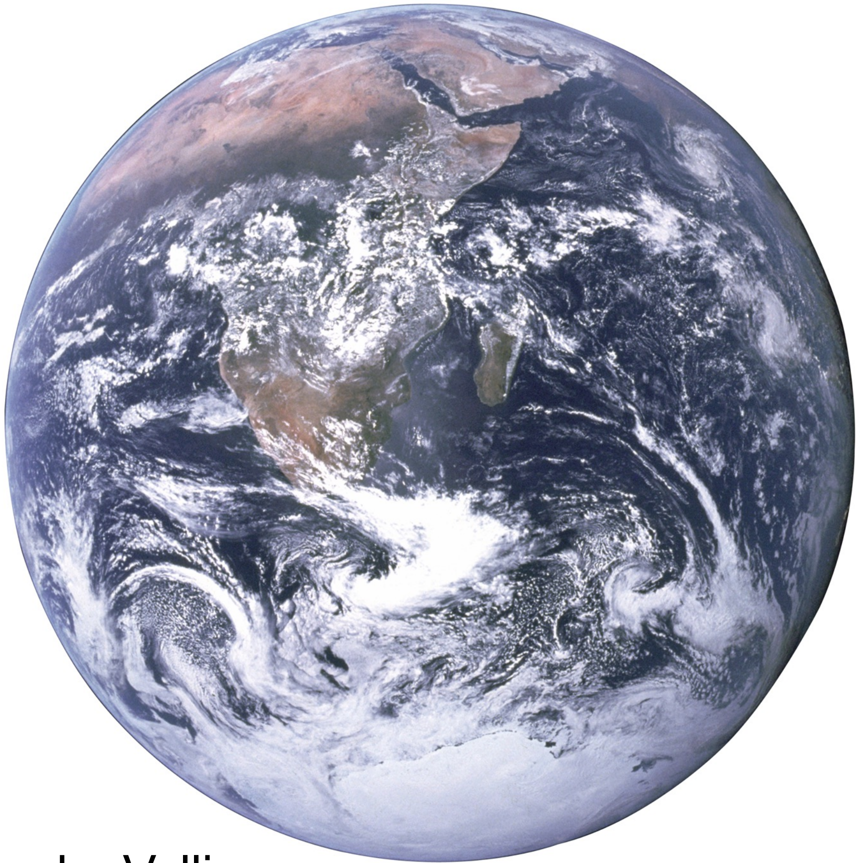
Strong updrafts develop in the cumulus cloud => mature, deep cumulonimbus cloud. Associated with heavy rain, lightning and thunder.





Evaporative driven cold pools

Thank you for your attention!



Caroline Muller

References:

« Atmospheric and Oceanic Fluid Dynamics » by Vallis

« Physics of Climate » by Peixoto & Oort

« Introduction to Geophysical Fluid Dynamics » by Cushman-Roisin