# How large scale barotropic instabilities favor the formation of 

## anticyclonic vortices in the ocean

Gaele Perret *<br>Laboratoire Ondes et Milieux Complexes, Université du Havre, Le Havre, France<br>Thomas Dubos<br>Laboratoire de Météorologie Dynamique, Ecole Polytechnique, Palaiseau, France

Alexandre Stegner<br>Laboratoire de Météorologie Dynamique, ENS, Paris, France

[^0]
#### Abstract

Large-scale vortices, i.e. eddies whose characteristic length scale is larger than the local Rossby radius of deformation $R_{d}$, are ubiquitous in the oceans, with anticyclonic vortices more prevalent than cyclonic ones. Stability properties of already formed shallow-water vortices have been investigated to explain this cyclone-anticyclone asymmetry. Here the focus is on possible asymmetries during the process itself of generation of vortices through barotropic instabilty of a parallel flow. The initial stage and the nonlinear stage of the instability are studied by means of linear stability analysis and direct numerical simulations of the one-layer rotating shallow water equations, respectively. Several parallel flows are studied : isolated shears, the Bickley jet and a family of wakes obtained by combining two shears of opposite signs. The flows are characterized by a small Rossby number and by relative deviations of isopycnals ranging from small (quasi-geostrophic regime) to finite (frontal-geostrophic regime).


Results show that the barotropic instability of shears, jets and wake flows favors the formation of large-scale anticyclonic eddies. This cyclone-anticyclone asymmetry occurs either during the linear stage of the instability or only later, during the nonlinear stage. For instance in the frontal regime, an anticyclonic shear flow has higher linear growth rates and is much more unstable than a cyclonic shear flow. The nonlinear saturation then leads to the formation of almost axisymetric anticyclones, while the cyclones tend to be more elongated in the shear direction. However the coupling between shears of opposite signs may supress, at the linear stage of the instability, the cyclone-anticyclone asymmetry. If the distance separating two shear regions is larger than two or three deformation radii, the barotropic instability develops independently in each shear, leading in the frontal regime
to a significant cyclone-anticyclone asymmetry at the linear stage. Conversely, if the two shear regions are close to each other, the shears tend to be coupled at the linear stage. The most unstable pertubation then resembles the sinuous mode of the Bickley jet, making no distinction between regions of cyclonic or anticyclonic vorticity. Nevertheless, when the nonlinear saturation occurs, large-scale anticyclones tend to be axisymetric while the cyclonic structures are highly distorted and elongated along the jet meander.

## 1. Introduction

In-situ measurements and general circulation models have shown that large-scale vortices, i.e. eddies whose characteristic length scale is larger than the local deformation radius, are ubiquitous in the oceans (Olson (1991); McWilliams (1985)). A striking characteristic of these large-scale and long-lived structures is that anticyclonic vortices tend to be more prevalent than cyclonic ones. Large-scale anticyclones are frequently observed in the lee of oceanic archipelago such as Hawaii (Mitchum (1995); Flament et al. (2001)), the Canaria (Sangrá et al. (2005)) or in the vicinity of the Alghulas (Olson and Evans (1986)) or the Brazilian currents. For all these various configurations the coastal boundary of the archipelago or a cape induces significant shear flows in the open ocean far away from the coast.

In order to explain the predominance of anticyclones among large-scale eddies, several studies were devoted to the specific stability of anticyclonic vortices in rotating shallow-water flows (Arai and Yamagata (1994); Stegner and Dritschel (2000); Baey and Carton (2002)). Moreover, stable anticyclones tend to remain coherent among a turbulent flow (Polvani et al. (1994); Arai and Yamagata (1994); Linden et al. (1995)) and they were found to be more robust to external strain perturbations than cyclonic eddies (Graves et al. (2005)). Taking into account the weak beta effect, which may affect large-scale oceanic eddies, reveals that only anticyclones could resist for a long time to the Rossby wave dispersion (Matsuura and Yamagata (1982); Nycander and Sutyrin (1992); Stegner and Zeitlin (1995, 1996)).

However, very few studies investigate how the generation process by itself may induce an initial selection between cyclonic or anticyclonic eddies. Most of the mesoscale oceanic vortices are formed by the unstable meanders of shears, jets or wake flows occurring in
coastal regions or in the open sea. The instability of oceanic currents has often been modeled using Quasi-Geostrophic (QG) theory since they typically have small Rossby number. Using the QG equations, valid for small Rossby and small isopycnal displacement, there is no distinctions between cyclonic or anticyclonic vorticity regions. Therefore, the baroclinic or the barotropic instabilities of a jet or a wake flow generate eddies of both sign having the same size or intensity. Nevertheless, the departure from the QG regime (finite isopycnal displacement and/or finite Rossby number) may induce a specific asymmetry.

A recent study (Poulin and Flierl (2003)) has investigated the linear stability of a Bickley jet and its nonlinear evolution in the framework of rotating shallow-water (RSW) equations. At the linear stage, the most unstable mode is sinuous and the jet meanders with no distinction between the cyclonic and the anticyclonic side. However, beyond the quasi-geostrophic regime, when the Rossby number becomes finite and induce large isopycnal displacements due to the cyclogeostrophic balance, the nonlinear evolution of the instability lead to an asymmetric eddy formation: the cyclones tend to be elongated and stretched in comparison with almost circular anticyclones. Other recent papers have investigated, by means of laboratory experiments (Perret et al. (2006a)) or stability analysis (Perret et al. (2006b)), the large-scale wake of circular islands. For the frontal geostrophic regime (Cushman-Roisin (1986); Cushman-Roisin and Tang (1990)), i.e. small Rossby number and finite isopycnal displacement, a significant asymmetry occurs in the wake between cyclonic and anticyclonic vortices. For some extreme cases, coherent cyclones do not emerge at all, and only an anticyclonic vortex street appears several diameters behind the circular island. This asymmetry was first explained by the linear stability analysis of parallel wake flows in the framework of RSW equations (Perret et al. (2006b)). Indeed, in the frontal regime, the most unstable
mode is fully localized in the anticyclonic shear region. Hence, the anticyclonic perturbations, leading to large-scale anticyclones, have the fastest growth rates. Direct numerical simulations show that the nonlinearities exacerbate the dominance of the anticyclonic mode linearly selected. Here again, when they are formed, the cyclones tend to be stretched and elongated in comparison with the large circular anticyclones. Anticyclonic predominance was also found in a fully stratified simulation (ROMS model) when the island size becomes larger than the local deformation radius (Dong et al. (2007)). Hence, both the linear and/or the nonlinear instability of parallel jets or wake flows may induce an asymmetric eddy formation.

The main goal of the present work is to understand how the barotropic instability of various parallel flows (shears, jets and wakes) may favor the formation of large-scale anticyclonic eddies. In which dynamical regime will the anticyclonic predominance become significant? Why does this asymmetry appear at the linear stage for a parallel wake flow (Perret et al. (2006b)) and at the nonlinear stage for the Bickley jet (Poulin and Flierl (2003)) ?

In Sec.II we discuss the dynamical parameters governing the RSW model and the numerical methods used for the linear stability analysis or to compute the full nonlinear equations. As a first step, we study in Sec. III the linear and the nonlinear stability of localized cyclonic and anticyclonic shear flows. In Section IV, the stability of the Bickley jet is investigated from the quasi-geostrophic to the frontal geostrophic regimes. In section $V$, various wake profiles are constructed as a combination of two localized shears. At this stage we introduce a new dynamical parameter to quantify the distance between the two shears and analyze its impact on the stability of the parallel wake flows. We then discuss and summarize the results in Sec. VI.

## 2. Model

a. Rotating shallow water equations and dynamical regimes

As a first approximation of oceanic flows, we used a reduced gravity model, also known as a $11 / 2$-layer model, assuming a thin upper layer above a motionless deep bottom layer. According to Cushman-Roisin (Cushman-Roisin (1986)), if the ratio of the upper layer depth to the total depth of the ocean is smaller than $\min (1, \mathrm{Bu})$, the upper layer will not be affected by the bottom layer dynamics. Hence, the upper layer motion follows the RSW equations. We use below the dimensionless form of the RSW model on the $f$-plane :

$$
\begin{gather*}
R o\left(\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V} \cdot \nabla) \mathbf{V}\right)+f \mathbf{n} \times \mathbf{V}=-\frac{B u}{R o} \nabla \eta+\frac{R o}{R e} \nabla^{2} \mathbf{V}  \tag{1}\\
\left(\frac{\partial \eta}{\partial t}+\mathbf{V} \cdot \nabla \eta\right)+(1+\eta) \nabla \cdot \mathbf{V}=0 \tag{2}
\end{gather*}
$$

with $\mathbf{n}$ the upward-pointing unit vector, $\mathbf{V}=(u, v)$ the horizontal velocity scaled by the typical velocity $V_{0}$ and $\eta$ the surface deviation scaled by the unperturbed layer depth $h_{0}$. We define the Rossby number Ro, the relative surface deviation parameter $\lambda$, the Burger number $B u$ and the Reynolds number $R e$ as follows:

$$
R o=\frac{V_{0}}{f L}, \quad \lambda=\frac{R o}{B u}, \quad B u=\left(\frac{R_{d}}{L}\right)^{2}, \quad R e=\frac{V_{0} L}{\nu}
$$

where $L$ is the characteristic length scale of the flow, $R_{d}=\sqrt{g h_{0}} / f$ the deformation radius, $f=2 \Omega$ the Coriolis parameter, $g$ the gravity acceleration and $\nu$ the kinematic viscosity.

The Rossby number is assumed small and fixed at $R o=0.1$ in the whole study. Therefore, the geostrophic balance is satisfied at leading order, imposing the surface deviation to balance the Coriolis term in Eq. (1). Thus, the dominant balance imposes $\eta \simeq \lambda$, meaning that the amplitude of the (dimensional) surface deviation scales like $L V_{0} f / g$.

Depending on the value of the Burger number, one distinguishes two dynamical regimes at small Rossby number. The classical quasi-geostrophic regime (Pedlosky (1987)) is defined when $B u=\mathcal{O}(1)$ or equivalently $\lambda=\mathcal{O}(R o) \ll 1$. In that limit the symmetry $\mathbf{V} \rightarrow$ $-\mathbf{V}$ and $\eta \rightarrow-\eta$ holds at the leading order meaning that cyclones and anticyclones obey the same equation. Even if the Rossby number is not asymptotically small in this study ( $R o=0.1$ ), cyclones and anticyclones are expected to follow roughly the same evolution when $B u=\mathcal{O}(1)$. The second regime corresponds to the frontal regime and is defined by $B u=\mathcal{O}(R o) \ll 1$ or equivalently $\lambda=\mathcal{O}(1)$. In this regime, the flow is expected to follow at the leading order the frontal geostrophic (FG) asymptotic model (Cushman-Roisin (1986); Cushman-Roisin and Tang (1990)). In that model the surface deviations are order unity, the symmetry $\mathbf{V} \rightarrow-\mathbf{V}$ and $\eta \rightarrow-\eta$ does not hold and the evolution of cyclonic and anticyclonic structures should differ strongly.

For a given steady and parallel velocity (Fig. 1a) or vorticity profile (Fig.1b), the relative surface deviation (Fig.1c) satisfies the geostrophic balance and increases when the dynamical regime is varied from a quasi-geostrophic to a frontal regime. Moreover, when the width $L$ of the shear becomes larger than the deformation radius, the amplitude of the potential vorticity fluctuation is also modified (Fig.1d).

In order to study the stability in a wide range of parameters from the QG to the FG regime, the parameter $\lambda$ is varied from $\lambda=0.1$ to $\lambda=1.0$ while the Rossby number is kept fixed at $R o=0.1$.

## b. Stability criteria for the $Q G$ and the $F G$ balanced models

The QG model (3) and the FG model (4) corresponds to the quasi-geostrophic ( $\lambda=$ $\mathcal{O}(R o) \ll 1)$ or the frontal regime $(\lambda \simeq 1, R o \ll 1)$ asymptotic expansions truncated at the first order in Rossby number.

$$
\begin{gather*}
\frac{\partial}{\partial t}\left(\Delta \eta-B u^{-1} \eta\right)-J\left(\eta, \Delta \eta-B u^{-1} \eta\right)=0  \tag{3}\\
\frac{\partial \eta}{\partial t}-J\left(\eta,(1+\lambda \eta) \Delta \eta+\frac{1}{2}(\nabla \eta)^{2}\right)=0 \tag{4}
\end{gather*}
$$

Notice that we distinguish in the present paper between the dynamical regimes and the balanced asymptotic models. Unlike the RSW equations which could induce ageostrophic fast wave motions, the standard QG and FG models corresponds to balanced equations where inertia-gravity waves and other ageostrophic motions are filtered out. The simplification of the balanced models allows relatively simple stability criteria. The Rayleigh inflexion-point criterion for two dimensional parallel flows (Drazin and Reid (1981)) can be extended to both the QG model and the FG model (see Appendix A). In both regimes, it is found that a parallel flow $\bar{U}(y)=-\frac{1}{\lambda} \frac{\partial \bar{\eta}}{\partial y}$ obeying the geostrophic balance will be linearly stable if the corresponding potential vorticity $q$ is monotonic, i.e. if:

$$
\begin{equation*}
\forall y \quad \frac{\partial q}{\partial y} \neq 0 \tag{5}
\end{equation*}
$$

where $q=\Delta \eta-B u^{-1} \eta$ in the QG model and $q=1 / \eta$ in the FG model. As can be seen in Fig.1d, the potential vorticity of a parallel shear flow is non-monotonic for both QG and FG models. Hence, such global stability criteria will not give useful indications on the stability
properties of parallel flows (shears, jets and wakes) in the FG model. A complete stability analysis is therefore needed to understand the cyclone-anticyclone asymmetry of large scale flows.

## c. Numerical resolution

For each parallel flow we perform a temporal stability analysis and compute the nonlinear evolution of the perturbed flow in the framework of the RSW equations.

## 1) Linear stability analysis

To perform the temporal stability analysis, the shallow water equations are linearized around a parallel basic state in geostrophic balance $(\bar{U}(y), \bar{\eta}(y))$. Assuming that the instability mechanism is inviscid, we consider the RSW equations without viscous terms. The perturbation field is expanded in normal form: $(\tilde{u}(x, y, t), \tilde{v}(x, y, t), \tilde{\eta}(x, y, t))=\left(\hat{u(y)}, \hat{v(y)}, \hat{\eta(y))} e^{i(k x-\omega t)}\right.$, with $\omega=\omega_{r}+i \sigma$ the complex eigenfrequency and $k$ the real wavenumber. We then obtain the following eigenvalue problem:

$$
\underbrace{\left(\begin{array}{ccc}
-k \lambda^{-1} \frac{\partial \bar{\eta}}{\partial y} & i\left(\lambda^{-1} \frac{\partial^{2} \bar{\eta}}{\partial y^{2}}+R o^{-1}\right) & R o^{-1} \lambda^{-1} k \\
-i R o^{-1} & -k \lambda^{-1} \frac{\partial \bar{\eta}}{\partial y} & i R o^{-1} \lambda^{-1} \frac{\partial}{\partial y} \\
(1+\bar{\eta}) k & -i\left(\frac{\partial \bar{\eta}}{\partial y}+(1+\bar{\eta}) \frac{\partial}{\partial y}\right) & -k \frac{\partial \bar{\eta}}{\partial y}
\end{array}\right)}_{\mathcal{L}}\left(\begin{array}{l}
\hat{u} \\
\hat{v} \\
\hat{\eta}
\end{array}\right)=\omega\left(\begin{array}{l}
\hat{u} \\
\hat{v} \\
\hat{\eta}
\end{array}\right)
$$

An approximation $\mathcal{L}_{N}$ of the matrix of the linear operator $\mathcal{L}$ is computed in a spectral basis with periodic boundary conditions. With $N$ Fourier modes, $\mathcal{L}_{N}$ is a $3 N \times 3 N$ matrix which
we diagonalize using the LAPACK linear algebra package (Anderson et al. (1999)). This provides the full eigenvalue spectrom of $\mathcal{L}_{N}$ ( $3 N$ complex eigenvalues), among which only 0 , 1 or 2 have a positive real part $\sigma_{N}$.

The convergence of solutions is obtained for a resolution varying between $N=256$ and $N=2048$, depending of the basic state and the dynamical regime. Starting with $N=256$, we double $N$ and monitor the relative difference between successive approximations of $\sigma$. Specifically we compute

$$
\varepsilon_{N}=\frac{\sum_{j}\left(\sigma_{2 N}\left(k_{j}\right)-\sigma_{N}\left(k_{j}\right)\right)^{2}}{\sum_{j} \sigma_{2 N}\left(k_{j}\right)^{2}}
$$

where the $k_{j}$ are a number of wavevectors spanning an interval $\left[k_{\min }, k_{\max }\right]$ of interest. We stop doubling $N$ when $\varepsilon_{N}$ becomes of the order of $10^{-4}$.

## 2) Nonlinear Evolution

The nonlinear evolution of the instability is studied by computing the nonlinear evolution of the perturbed parallel flows. The rotating shallow-water equations are discretized in space with a pseudo-spectral method and in time with a second-order Leapfrog scheme. In order to lower the computation time, the domain is reduced to two wavelengths of the most unstable mode in the streamwise direction, as determined from the linear stability analysis. The resolution, in the spanwise direction is 256 and the boundary conditions are periodic. The basic state is parallel with the same velocity profile extended to the entire domain. The Reynolds number is fixed at $R e=9000$ to reduce viscous effects. When a numerical simulation is initialized with a parallel basic velocity profile, without external perturbation,
the flow stays parallel and the velocity diffuses slowly. The basic state velocity is then initially perturbed with a random perturbation field whose fluctuation level is about $0.1 \%$ times the velocity amplitude. Furthermore the initial surface deviation is in geostrophic balance with the initial velocity deviation in order to avoid gravity wave emission.

## 3. Localized shear flow

## a. Basic flow

The hyperbolic tangent velocity profile $\left(V_{x}(y)=V_{0} \tanh (y / L)\right)$ was often used as a generic flow to study the stability of a single two dimensional shear flow (Drazin and Reid (1981); Johnson (1963)). However, in the framework of the RSW equations the stability of such shear flow may strongly depends on the domain size. Indeed, due to the geostrophic balance, the constant velocities values, away from the shear zone, control the surface deviation in the central shear zone. If the domain size is too large outcropping (vanishing layer depth) may occur. In order to avoid this unpleasant influence of the domain size, the velocity field should vanish far away from the shear. Therefore, instead of using a standard hyperbolic tangent shear flow, we study in this section the stability of a localized shear flow defined, in a dimensionless form, by:

$$
\begin{aligned}
& \bar{\eta}(y)=\frac{V_{x}(y)}{V_{0}}=c y e^{-2 y^{2}} \\
& \bar{U}(y)=\frac{\eta(y)}{h_{0}}=c \frac{\lambda}{4} e^{-2 y^{2}}
\end{aligned}
$$

where $c=2 e^{1 / 2}$. With this normalization coefficient, the localized shear flow is anticyclonic and extremum velocity values $V_{x} / V_{0}=1$ and $V_{x} / V_{0}=-1$ occur at $y=1 / 2$ and $y=-1 / 2$.

In this case, the minimal vorticity value $\frac{1}{f} \frac{\partial U}{\partial y}(0)=-c R o<0$ is negative and centered in $y=0$, while two positive vorticity peaks (having weaker amplitudes) correspond to the lateral shears (Fig. 18b). We easily get a cyclonic shear when $c=-2 e^{1 / 2}$. In Appendix B, we show that these lateral shears have little influence on the growth rate of the most unstable mode.
b. Linear instability in the $Q G$ and $F G$ regimes

A resolution of $N=256$ was shown to be sufficient to get convergence $\left(\varepsilon_{256}=1 \cdot 6 \cdot 10^{-4}\right)$. However, as the unstable normal mode may present strong discontinuities in a frontal regime (Fig. 2 f ), it was necessary to increase the number of collocation points for these computations, up to $N=2048$. Fig. 2 compares the unstable growth rates (b,e) and the leading normal modes (d,f) of an anticyclonic (a) and a cyclonic (d) localized shear flow.

In the quasi-geostrophic regime, corresponding to $\lambda=0.1$, the growth rates and the wavelengths of the most unstable perturbations in the cyclonic and the anticyclonic shear are very close. The slight differences in the growth rate are due to small ageostrophic effects since the Rossby number in our simulation is small but finite $R o=0.1$. The stability calculation performed with a standard QG model (i.e. $R o \rightarrow 0$ ) using the same stability solver showed no discrepancy between the stability of the cyclonic and the anticyclonic localized shear flow.

In a frontal regime, corresponding to $\lambda=1$, the dimensionless growth rate of the cyclonic and the anticyclonic shears differ greatly. While the growth rate in the anticyclonic shear decreases by only $30 \%$ from its quasi-geostrophic value, the unstable growth within the
cyclonic shear strongly decreases. For the latter the maximum growth rate in the FG regime is only $6 \%$ of the maximum growth rate in the QG regime (Fig.2d). For an easier comparison with oceanic observations we compute, for these geostrophically balanced shear flows ( $R o=$ 0.1 ), the exponential growth time of the perturbations. This latter will stay around 2 or 3 days for the anticyclonic flow while for the cyclonic one the growth time will goes up to 35 days when the isopycnal displacement become finite $(\lambda=1)$. Hence, in the linear stage of the instability, large-scale (i.e. $B u=0.1$ ) cyclonic and anticyclonic shears have different stability properties. The frontal regime stabilizes the cyclonic barotropic shear and therefore, favors the development of unstable perturbations in the anticyclonic shear. In all the cases, the leading unstable mode extends throughout the whole shear zone (Fig.2c,f) and its shape is not affected by the deformation radius variations.

For the incompressible two-dimensional shear instability (Drazin and Reid (1981)), the unstable wavelength is controlled by the width of the shear layer $L$. For a barotropic shear in the RSW model, the unstable wavelength could be controlled, at least, by two horizontal scales: $L$ and the deformation radius $R_{d}$. We plot in Fig.4a (resp. 4b) the most unstable wavelength normalized by $L$ (resp. $R_{d}$ ). According to Fig. 4a the unstable wavelength of the anticyclonic shear is proportional to $L$ and is weakly affected by the variation of the deformation radius from $B u=1$ (i.e. $\lambda=0.1$ ) to $B u=0.1$ (i.e. $\lambda=1$ ) while the unstable wavelength of the cyclonic shear increases significantly. According to Fig. 4b, the most unstable wavelength is always larger than the deformation radius and this ratio increases from $2 \pi / k=3.5 R_{d}$ in the quasi-geostrophic regime up to $2 \pi / k \simeq 10 R_{d}$ for the anticyclonic shear and $2 \pi / k \simeq 18 R_{d}$ for the cyclonic shear in the frontal regime.

## c. Nonlinear evolution

The nonlinear evolution of the instability and the vortex formation are studied by direct numerical simulations. The nonlinear evolution of both shear flows are computed in the quasi-geostrophic regime (Fig. 5) and in the frontal regime (Fig. 6). In agreement with the linear analysis and as expected for the quasi-geostrophic regime $(\lambda=0.1)$, the nonlinear saturation of the shear instability does not induce a cyclone-anticyclone asymmetry. The distance between two emerging vortices $4 L$ is very close to the most unstable wavelength $2 \pi / k=3.5 L$.

In a frontal regime (Fig. 6), the growth of the perturbations differs significantly between the cyclonic and the anticyclonic shear. Vortices appear in the anticyclonic shear at $t / T_{0}=$ 20 whereas the emergence of cyclonic vortices only occurs at $t / T_{0}=200 \sim 250$, where $T_{0}=\frac{2 \pi}{\Omega}$ is the rotation period. Besides, once they are formed within the unstable shear, cyclonic eddies tend to have a larger extension along the shear than anticyclonic ones as expected from the linear stability analysis. Indeed, in the frontal regime, according to figures 2 b and 2 e the wavelength of the most unstable cyclonic mode is twice longer than the anticyclonic one. An axisymetrization process generally occurs during the nonlinear saturation but here, the coherent structures remain elongated in the cyclonic shear while axisymetric eddies are formed in the anticyclonic shear (Fig. 6). According to previous studies (Arai and Yamagata (1994); Graves et al. (2005)), the axisymetrization process was found to be much weaker for elliptical cyclones beyond the QG regime. The mean cyclonic shear may amplify this tendency. Morever, if we analyze the vorticity field during the nonlinear stage of the instability, we can detect some differences between the cyclonic
and the anticyclonic structures (Fig. 7). Indeed, when the isopycnal deviations become finite, the vorticity seems to be strongly perturbed in the core of cyclonic eddies while anticyclonic eddies remain coherent and circular (Fig. 7b). This could be a signature of the cyclonic sensitivity to external strain perturbations, induced here by the mean shear and the neighboring vortices, as suggested by the recent study of Graves et al. (Graves et al. (2005))

## 4. Bickley jet

a. Basic flow

The barotropic instability of two-dimensional Bickley jet has been extensively studied (Lipps (1962); Howard and Drazin (1964); Maslowe (1991); Engevik (2004)). For large or finite Burger numbers (i.e. two-dimensional or QG models) unstable jet generates cyclonic and anticyclonic eddies symmetric in their size, strength and shape. For finite Rossby numbers, the recent study of Poulin and Flierl (Poulin and Flierl (2003)) exhibit a significant cyclone-anticyclone asymmetry . In order to explore the large-scale effect on the jet stability, we study here the linear and the non-linear destabilization of a barotropic jet in the frontal regime (small Rossby $R o=0.1$ but finite isopycnal deviation $\lambda=0.5$ ). In order to allow easier comparisons with previous stability analysis and analytical results, we use the geostrophically balanced Bickley jet (Fig. 8) defined as follows:

$$
\begin{aligned}
& \bar{U}(y)=\cosh ^{-2} y+u_{0} \\
& \bar{\eta}(y)=-\lambda\left(\tanh y-\frac{y}{y_{b}} \tanh y_{b}\right)
\end{aligned}
$$

where the lateral extent of the domain is $2 y_{b}=20$. Due to geostrophic balance and periodic boundary conditions, the mean velocity must vanish and the central jet is surrounded by a weak reverse flow $u_{0}=-\frac{\tanh y_{b}}{2 y_{b}}$. The velocity, the vorticity, the surface deviation and the potential vorticity of the Bickley jet for $\lambda=0.1$ and $\lambda=0.5$ are plotted on Fig. 8 .

## b. Linear stability in $Q G$ and $F G$ regimes

In order to check the convergence of the stability analysis, we compute the growth rate for different resolutions in the classical quasi-geostrophic regime $B u=1.0$ and $R o=0.1$ (Fig. 9). For $N=256$ collocation points, we recover the curve obtained by Poulin and Flierl (Poulin and Flierl (2003)) for the same parameters (see their Fig. 3). A first bump is observed at dimensionless wavenumber $k=1.5$ and a second one at $k=1.8$. The maximum dimensionless growth rate $\sigma$ is slightly different, $\sigma_{\max }=0.062$ in our stability analysis while they obtain $\sigma_{\max }=0.05$. This difference may be due to the different boundary conditions. Indeed, in our case, we impose periodic boundary conditions while Poulin and Flierl impose no-slip conditions. Moreover, decreasing the total streamwise extent, from $2 y_{b}=20$ to $2 y_{b}=10$ increases the growth rate. Therefore the confinement of the jet as well as the boundary conditions may have an influence on the growth rate.

When the spatial resolution is increased, the bumps observed for $N=256$ disappear. Therefore, we assume that these bumps do not have a physical signification, but are due to a lack of resolution. In what follows the simulations are performed with $N=1024$ collocation points.

In Appendix C, we derive analytically the neutral eigenmodes of the QG asymptotic
model along the lines of previous work (Lipps (1962); Howard and Drazin (1964); Maslowe (1991); Engevik (2004)). This leads to two neutral eigenmodes: $k=\sqrt{1-B u^{-1}}$ and $k=$ $\sqrt{4-B u^{-1}}$. In the non rotating case $(B u=\infty)$ there exists a sinuous branch of instability for $k<2$ and a varicose branch for $k<1$. The above results show that in the quasigeostrophic case, these bounds are modified into $\sqrt{4-B u^{-1}}$ and $\sqrt{1-B u^{-1}}$, respectively. These bounds provide a good test of the accuracy of the numerical procedure, since it is wellknown that near-neutral modes are almost singular and difficult to approximate numerically. For $B u=1(\lambda=0.1)$, the neutral mode wavenumber $k R_{d}=1.733$ satisfies the QG prediction $k R_{d}=\sqrt{4-B u^{-1}}$ with a very good accuracy ( $0.1 \%$. deviation). This further illustrates the stabilizing effect of a small Burger number $B u=\left(R_{d} / L\right)^{2}$, i.e. a small deformation radius. In the framework of the QG model, the varicose branch of instability disappears for $B u<1$ while the sinuous branch disappears for $B u<0.25$. In this work we let $R o=0.1$ hence we expect the varicose branch to disappear at $\lambda=R o / B u>0.1$ and the Bickley jet to be completely stabilized if $\lambda>0.4$.

As for the localized shear layer we first study the linear stability properties of the flow within the RSW model from the quasi-geostrophic regime ( $R o=0.1, \lambda=0.1$ ) to the frontal regime ( $R o=0.1, \lambda \simeq 1$ ). As expected, for the whole range of the parameter studied we get only one unstable mode, corresponding to the sinuous mode, This normal mode is mainly localized in the center of the jet and despite a small shift of the secondary PV peak the unstable mode keeps the same shape (Fig. 10b). As expected from the QG analysis, the Bickley jet is strongly stabilized in the RSW model when the frontal regime is reached. The growth rate strongly decreases and the unstable wavelength increases when the surface deviation parameter $\lambda$ becomes finite (Fig. 10b). In the RSW model, the Bickley jet is
completely stabilized when $\lambda>0.6$.
As far as linear stability is concerned, the Bickley jet is always destabilized by a symmetric sinuous mode and there is no signature of a selective destabilization of anticyclonic vorticity region in the frontal regime.

The evolution of the dimensionless growth time $1 /\left(\sigma_{\max } T_{0}\right)$ and the dimensionless wavelength $2 \pi /\left(k R_{d}\right)$ as a function of the relative surface deviation $\lambda$ are plotted in Fig. 11. The growth time of the sinuous mode of the barotropic Bickley jet increases from 20 to 250 days when the relative deviation becomes finite $\lambda=0.5$. We should note here that these values overestimate the observed growth of large scale oceanic jet. The Gulf Stream, for instance, exhibits a short growth time between 3 to 6 days (Watts and Johns (1982); Kontoyiannis and Watts (1994)) and a longer one about 12 to 25 days (Lee and Cornillon (1996)). This is probably due to the barotropic limitation of the RSW model. The baroclinicity of oceanic jets has a strong impact on their stability. In a wide range of parameters Baey, Riviere and Carton (Baey et al. (1999)) show that the baroclinic instability can be more efficient to create vortices than the barotropic instability. Hence, if the frontal regime stabilizes the barotropic jets, we could expect that the stability of large scale oceanic jets are controlled only by the baroclinic processes.

## c. Nonlinear evolution

As for the localized shear flow, the nonlinear evolution of the jet instability and the vortex formation are studied by direct numerical simulations. The dynamical evolution of the unstable Bickley jet are computed in the quasi-geostrophic regime (Fig. 12a) and in
the frontal regime (Fig. 12b). As expected, for the quasi-geostrophic regime $(\lambda=0.1)$, the nonlinear saturation of the sinuous mode does not induce any cyclone-anticyclone asymmetry. Vortices of both signs are formed with the same size, shape and amplitude. However, for finite isopycnal deviation $(\lambda=0.5)$ once coherent vortices are formed, the cyclonic eddies tend to be stretched in an elongated boomerang shape while the large-scale anticyclone remains coherent and almost circular (Fig. 12b). A similar behavior was found by Poulin and Flierl (Poulin and Flierl (2003)) when the Rossby number reaches finite values. In this cyclogeostrophic regime the surface deviation becomes finite as for the frontal regime. Hence, even if the linear growth of unstable sinuous mode does not lead to cyclone-anticyclone asymmetry the nonlinear evolution of the system does.

## 5. Parallel wake flows

a. Basic flow

Unlike the Bickley jet, parallel wake flows may exhibit a significant cyclone-anticyclone asymmetry at the linear stage of the instability. According to both numerical (Perret et al. (2006b); Dong et al. (2007)) and laboratory studies (Perret et al. (2006a)) when the deformation radius become smaller than the typical width of the wake (i.e. in the frontal regime) the wake flow tends to have the same stability properties as two independent shear layers. The most unstable mode is localized in the anticyclonic shear and the convectively unstable flow behave as a noise amplifier (Perret et al. (2006b)). Hence, in the following stability analysis, we construct various parallel wake flows as a combination of two localized shear
flows having the same amplitude. The typical width of the wake $D$ is defined as the distance between the two shears, i.e. the distance between the two vorticity extrema. For an oceanic or a laboratory wake, the typical width $D$ will corresponds to the mean island diameter or the cylinder diameter. As in section 3 the extent of each localized shear is $L$ and we introduce the dimensionless wake parameter $\delta=D / L$. The parallel wake flow is divided here in three regions (Fig. 13a): two lateral shears separated by a central region of extent $d=D-L$ having a constant velocity $V_{x} / V_{0}=-1$. Hence we study a family of wake flow profiles having various widths $\delta$ according to:

$$
\begin{aligned}
y & \leq-\frac{\delta-1}{2} \\
|y| \leq \frac{\delta-1}{2} & \bar{U}(y)=-c(y+\delta / 2) e^{-2(y+\delta / 2)^{2}} \\
y & \geq \frac{\delta-1}{2}
\end{aligned} \quad \bar{U}(y)=-10=c(y-\delta / 2) e^{-2(y-\delta / 2)^{2}}
$$

where $c=2 e^{1 / 2}$ is a normalization coefficient. The relative surface deviation of this parallel wake flow family is derived from $\bar{U}(y)$ according to the geostrophic balance condition (Eq. 1). The typical variations of the relative surface deviation and the potential vorticity, for a wake profile corresponding to $\delta=2.5$, are plotted in Fig. 13 for the quasi-geostrophic regime ( $\lambda=0.1$ ) and the frontal regime $(\lambda=0.5)$.

## b. Linear stability in the $F G$ and $Q G$ regimes

The linear stability of a parallel wake profile corresponding to $\delta=2.5$ is studied for various Burger numbers while the Rossby number is kept small $R o=0.1$. In the quasi-geostrophic regime ( $B u=1$ corresponding to $\lambda=0.1$ ) the two most unstable modes have roughly the same growth rates: $\sigma_{a} L / V_{0} \simeq 0.52$ and $\sigma_{c} L / V_{0} \simeq 0.45$. Theses values are very close to the
growth rates of the anticyclonic and the cyclonic localized shear layers found in section 3 (Fig. 2). The differences between the growth rates $\sigma_{a}$ and $\sigma_{c}$ are probably due to small ageostrophic effects induced by the small but nevertheless finite $R o=0.1$. The eigenmode called mode A (mode C) corresponds here to an unstable perturbation preferentially localized in the anticyclonic (cyclonic) shear region of the wake profile as shown in Fig. 14. However, in this quasi-geostrophic regime, each eigenmodes (A or C) extend spatially in both shears. Hence, unstable perturbations will grow exponentially at the same rate on both sides of the wake. On the other hand, for a frontal regime $B u=0.2$, corresponding to significant surface deviation $\lambda=0.5$, the most unstable eigenmode (mode A) is strictly localized in the anticyclonic shear region (Fig. 14d) and the mode C has a reduced growth rate $\sigma_{c} L / V_{0} \simeq 0.14$. Hence, in this case, the unstable perturbations will grow much faster on the anticyclonic side of the wake. We recover here the results previously found (Perret et al. (2006b)): the linear stability of large-scale wake flow induces a selective destabilization of regions with anticyclonic vorticity.

If we reduce the central region ( $d \rightarrow 0$ corresponding to $\delta \rightarrow 1$ ) the parallel wake flow becomes similar to a parallel jet flow. In this case, as for the Bickley jet, we expect the most unstable eigenmode to be a sinuous mode and therefore the cyclone-anticyclone asymmetry should disappear at the linear stage of the instability. Hence, in order to study the influence of the width parameter on the wake flow stability, we compare the previous case where $\delta=2.5$ with the case where $\delta=1.22$ (Fig. 15). As expected, Fig. 15b shows for the quasigeostrophic regime $(\lambda=0.1)$ that the most unstable branch corresponds to a symmetric mode (i.e. a sinuous perturbation) and the second branch to an antisymmetric mode (i.e. varicose perturbations). In this regime, the maximum sinuous growth rate $\sigma_{s} L / V_{0} \simeq 0.6$ is significantly higher than the maximum varicose growth rate $\sigma_{v} L / V_{0} \simeq 0.35$. Hence, like
the Bickley jet, the most unstable mode corresponds to a symmetric perturbation even if the growth rates of the antisymmetric perturbations (varicose mode) are not negligible. When the isopycnal displacement becomes finite the wake enters the frontal regime ( $\lambda=0.5$ corresponding to $B u=0.2$ ). In this case, the symmetry properties of the two most unstable eigenmodes change. The unstable modes corresponding to the sinuous branch (resp. varicose branch) remain almost symmetric (resp. antisymmetric) in the center of the wake, but not on the border of the jet. The amplitude of the most unstable modes of the sinuous branch are amplified on the anticyclonic side of the wake, while the unstable modes of the varicose branch tend to be amplified on the cyclonic side of the wake. Therefore, at the linear stage of the wake instability a cyclone/anticyclone asymmetry starts to appear in the frontal regime, but the amplitude of the asymmetry depends on the relative width of the wake $\delta$. When the width of the wake is large $(\delta=2.5)$ the cyclonic and anticyclonic shears of the wake are less connected and for smaller deformation radius (frontal regime) the most unstable perturbation is fully localized within the anticyclonic shear (Fig. 14d). For smaller width ( $\delta=1.22$ ) even for large-scale flows $(B u \ll 1)$ the unstable perturbation may destabilize both sides of the wake (Fig. 15d). Hence, the cyclone-anticyclone selection of the unstable modes at the linear stage of the instability depends both on the Burger number $B u$ and the width parameter of the wake $\delta$.

## c. Coupling of the cyclonic and anticyclonic shears

In order to quantify more accurately the coupling between the unstable modes on the anticyclonic and the cyclonic sides of the wake, we introduce the correlation coefficient R :

$$
R\left(q_{1}, q_{2}\right)=\frac{\operatorname{Re}\left(\int q_{\text {mode } 1}^{*} q_{\text {mode } 2} d y\right)}{\sqrt{\int\left|q_{\text {mode } 1}\right|^{2} d y \int\left|q_{\text {mode } 2}\right|^{2} d y}}
$$

where $q^{*}$ indicates the complex conjugate, $R e$ the real part and $q_{1}(y)$ and $q_{2}(y)$ are the spatial distribution of the potential vorticity of the two most unstable eigenmodes. When $R=0$ the two modes are uncorrelated spatially, while for $R=1$ the two eigenmodes are spatially identical. If $R>0.2$ (this threshold is arbitrary) we say that the two eigenmodes are coupled, while if $R<0.2$ we say that they are uncoupled. Moreover, we define the parameter $\gamma=d / R_{d}=(\delta-1) / B u^{1 / 2}$ which quantifies the relative extent of the central zone of the wake separating the two shears on both sides of the wake. According to Fig. 16 the boundary between the coupled and the uncoupled modes depends mainly on $\gamma$. At the first order of approximation, the line $\gamma=2.5$ defines reasonably well the separation between the coupled and the uncoupled eigenmodes for a wide range of Burger numbers. When the two shear layers of the wake profile are separated by more than two deformation radius $(\gamma>2)$ the unstable eigenmodes tend to be uncoupled at the linear stage of the instability. If they are not coupled, we can use the results of the first section on each localized shear flows indicating that the anticyclonic shear of the wake is the most unstable when the isopycnal displacement become finite.

## d. Nonlinear evolution

The nonlinear evolution of the instability of parallel wake flows and the vortex formation are studied by direct numerical simulations (Fig.17). In agreement with our previous results on the stability of parallel wake flows, the nonlinear interactions between vortices of opposite sign enhance the cyclone-anticyclone asymmetry of large scale wakes (Perret et al. (2006b)). Cyclonic vortices are stretched or strongly deformed into triangular shapes in comparison with the anticyclonic vortices which remain robust and circular in the frontal regime $(\lambda=0.5)$ . For a small width parameter $(\delta=1.22)$ the first meanders lead to the formation of vortices of both sign with a slight cyclone anticyclone asymmetry (Fig. 17f), then the nonlinear interactions between vortices lead to a strong distortion of cyclonic structures (Fig. 17h). Moreover, for both the quasi-geostrophic and the frontal regimes, the opposite sign vortices are aligned along the same line, unlike the standard Karman vortex street. For a larger width parameter ( $\delta=2.5$ ) the first meander leads to a significant cyclone-anticyclone asymmetry in the vortex formation (Fig. 17b). This is the signature of a selective destabilization of the anticyclonic shear region of the wake at the linear stage of the instability. Indeed, for this case $(\delta=2.5, \gamma \simeq 3.3)$, according to Fig. 14 and Fig. 17, the perturbations growing in each shear layer are not coupled. Coherent vortices emerge first on the anticyclonic side of the wake and once they are formed, the nonlinear interactions with the cyclonic shear lead to the formation of distorted cyclones.

## 6. Conclusion

The stability of various parallel flows was investigated in the context of the RSW model. This simple model describes the barotropic surface motion in the upper thermocline and neglects the baroclinic interactions of the surface and the deep oceanic flows. However, this model captures the non-QG dynamics and the fast inertia-gravity wave motion. Several aspects of the cyclone-anticyclone asymmetry of large-scale and parallel flows (shear, jets and wakes) were then studied.

According to the various stability analysis performed on a wide variety of parallel flows, we emphasize that the barotropic instability of oceanic shears, jets and wake flows favors the formation of large-scale anticyclonic eddies. In the frontal regime (small Rossby number and finite isopycnal displacements), an anticyclonic shear flow will have higher growth rates and be much more unstable than a cyclonic one. The linear stage of the instability induces here a strong cyclone-anticyclone asymmetry and favor the development of unstable perturbations in the anticyclonic shear. The nonlinear saturation leads to the formation of coherent and almost axisymmetric anticyclones, while the cyclones tend to be more elongated in the shear direction once they are formed. Besides, other studies on the stability of isolated eddies have shown that, beyond the QG regime, anticyclones tend to be more stable and coherent than their cyclonic counterparts (Arai and Yamagata (1994); Baey and Carton (2002); Stegner and Dritschel (2000); Graves et al. (2005)). Hence, the opposite stability properties of anticyclonic shear regions (unstable) and anticyclonic eddies (stable) may explain the predominance of large-scale and long-lived anticyclones in the oceans.

The second result of this paper is to confirm that the cyclone-anticyclone asymmetry
of parallel flows may occur respectively at the linear stage or at the nonlinear stage of the instability. Two different mechanisms are then involved. Indeed, unlike a localized shear flow the stability analysis of a Bickley jet, in the frontal regime, reveals no distinction between regions of positive (cyclonic) or negative (anticyclonic) vorticity. The most unstable perturbation corresponds here to a sinuous mode leading to the meandering of the whole jet. However, when the nonlinear saturation occurs and coherent structures are formed, large-scale anticyclones tend to be axisymmetric while the cyclonic structures are be highly distorted and elongated along the jet meander. We recover here the patterns of non-QG turbulent flows where several coherent structures interact together with a predominance of large-scale (larger than the deformation radius) and robust anticyclones (Arai and Yamagata (1994); Polvani et al. (1994); Linden et al. (1995)). The nonlinear mechanism involved here could be the strong sensitivity of cyclonic structures to external strain perturbations studied by Graves et al. (Graves et al. (2005)). The Bickley jet favors the emergence of robust anticyclonic eddies only at the nonlinear stage.

Moreover, we demonstrate how the coupling between opposite shears may supress, at the linear stage of the instability, the cyclone-anticyclone asymmetry. Assuming that the deformation radius is a characteristic length scale of interaction between localized structures, two localized shear flows will not "feel" each other if they are distant enough. We found that if the distance $D-L=\gamma R_{d}$ separating two shear regions is larger than two or three deformation radius ( $\gamma \simeq 2.5$ ) the two localized shears will be linearly uncoupled. In such case, if the width $L$ of the shears are large enough the flow will exhibit a significant cycloneanticyclone asymmetry. Hence, for large-scale wakes flows the linear perturbations will grow much faster in the anticyclonic vorticity region. However, if the two shear regions are too
close each other $(\gamma<2.5)$ the opposite shears will be coupled at the linear stage and the most unstable pertubation will then be a sinuous mode. In such case, no distinction occurs between the cyclonic or the anticyclonic vorticity region in the linear stage of instability, as for the Bickley jet.

Even if the baroclinic instability is not taken into account, this study may contribute to a better understanding of the preferred formation of large-scale anticyclones. In the frontal regime (small Rossby number and finite isopycnal displacements) only large-scale anticyclones will emerge in a reasonable time if the barotropic shear instability is the dominant mechanism of eddy formation. For realistic large-scale wakes, such as the Hawaiian or the Canaria archipelago wakes the eddy formation is mainly governed by the barotropic shear instability. However, for theses specific cases, both large-scale anticyclonic shear (width larger than the deformation radius) and smaller cyclonic shear coexists. The large-scale anticyclonic shear will corresponds to the FG regime but the cyclonic shear will obey to the QG regime and eddies of both signs but different size will be formed. The cyclonic eddies will then be smaller than the anticyclonic ones. Therefore, a more detailed study, taking into acount the interactions and the relative stability between shears of different width and intensity is needed to provide quantitative predictions on typical vortex shedding frequencies.

## Acknowledgments.

We greatfully acknowledge Jean-Marc Chomaz for enlightning discussions.

## APPENDIX A

## Rayleigh criterion for the frontal geostrophic model

Here we extend the classical Rayleigh criterion for instability of parallel incompressible two-dimensional flows to the frontal geostrophic model. The dimensionless frontal model is written:

$$
\begin{equation*}
\frac{\partial \eta}{\partial t}-J\left(\eta,(1+\eta) \Delta \eta+\frac{1}{2}(\nabla \eta)^{2}\right)=0 \tag{A1}
\end{equation*}
$$

with $\eta$ the surface deviation and $J(a, b)=\partial_{x} a \partial_{y} b-\partial_{y} a \partial_{x} b$ is the Jacobian operator (CushmanRoisin (1986)). Any parallel flow $\eta_{0}(y)$ is a stationary solution of (A1). To study the stability of that flow, we decomposed the surface deviation $\eta$ as follows:

$$
\eta=\eta_{0}(y)+\tilde{\eta}(x, y, t)
$$

where $\tilde{\eta}$ is a small perturbation. The linearized frontal equation is then:

$$
\begin{equation*}
\frac{\partial \tilde{\eta}}{\partial t}-\left(1+\eta_{0}\right)\left[J\left(\eta_{0}, \Delta \tilde{\eta}\right)+J\left(\tilde{\eta}, \Delta \eta_{0}\right)\right]-J\left(\eta_{0}, \nabla \eta_{0} . \nabla \tilde{\eta}\right)-\frac{1}{2} J\left(\tilde{\eta},\left(\nabla \eta_{0}\right)^{2}\right)=0 \tag{A2}
\end{equation*}
$$

The perturbation $\tilde{\eta}$ may be decomposed into normal modes:

$$
\begin{equation*}
\tilde{\eta}(x, y, t)=\hat{\eta}(y) e^{i(k x-\omega t)} \tag{A3}
\end{equation*}
$$

with $k$ prescribed and $\omega$ and $\hat{\eta}(y)$ and complex unknowns. The solution has to be bounded as $x \rightarrow \infty$, implying that $k$ is real. Introducing (A3) into (A2) yields:

$$
\begin{equation*}
\omega \hat{\eta}+k\left(1+\eta_{0}\right)\left[\frac{d \eta_{0}}{d y}\left(k^{2} \hat{\eta}-\frac{d^{2} \hat{\eta}}{d y^{2}}\right)+\hat{\eta} \frac{d^{3} \eta_{0}}{d y^{3}}\right]-k\left[\left(\frac{d \eta_{0}}{d y}\right)^{2} \frac{d \hat{\eta}}{d y}+\hat{\eta} \frac{d^{2} \eta_{0}}{d y^{2}} \frac{d \eta_{0}}{d y}\right]=0 \tag{A4}
\end{equation*}
$$

Setting the phase velocity $c=\omega / k=c_{r}+i c_{i}$, one can write:

$$
\begin{equation*}
\left(1+\eta_{0}\right) \frac{d \eta_{0}}{d y} \frac{d^{2} \hat{\eta}}{d y^{2}}+\left(\frac{d \eta_{0}}{d y}\right)^{2} \frac{d \hat{\eta}}{d y}-\left[c+\left(1+\eta_{0}\right)\left(\frac{d \eta_{0}}{d y} k^{2}+\frac{d^{3} \eta_{0}}{d y^{3}}\right)-\frac{d^{2} \eta_{0}}{d y^{2}} \frac{d \eta_{0}}{d y}\right] \hat{\eta}=0 \tag{A5}
\end{equation*}
$$

Assuming that the basic state profile is monotonic, $d \eta_{0} / d y \neq 0$, then one can divide the previous equation by $d \eta_{0} / d y$ :

$$
\begin{aligned}
& \left(1+\eta_{0}\right) \frac{d^{2} \hat{\eta}}{d y^{2}}+\frac{d \eta_{0}}{d y} \frac{d \hat{\eta}}{d y}-\left[\frac{c}{d \eta_{0} / d y}+\left(1+\eta_{0}\right)\left(k^{2}+\frac{d^{3} \eta_{0} / d y^{3}}{d \eta_{0} / d y}\right)-\frac{d^{2} \eta_{0}}{d y^{2}}\right] \hat{\eta} \\
= & \frac{d}{d y}\left(\left(1+\eta_{0}\right) \frac{d \hat{\eta}}{d y}\right)-\left[\frac{c}{d \eta_{0} / d y}+\left(1+\eta_{0}\right)\left(k^{2}+\frac{d^{3} \eta_{0} / d y^{3}}{d \eta_{0} / d y}\right)-\frac{d^{2} \eta_{0}}{d y^{2}}\right] \hat{\eta}=0
\end{aligned}
$$

Assuming, moreover, that $\hat{\eta}$ is a localized perturbation, then $\int_{-\infty}^{+\infty}|\hat{\eta}|^{2} d y$ exists and $\lim _{y \rightarrow \infty}|\hat{\eta}|=$ 0 . Multiplying the equation by the conjugate of the perturbation $\hat{\eta}^{*}$ and integrating between $-\infty$ and $\infty$, we get:

$$
\begin{equation*}
\int_{-\infty}^{+\infty} \hat{\eta}^{*} \frac{d}{d y}\left[\left(1+\eta_{0}\right) \frac{d \hat{\eta}}{d y}\right] d y-\int_{-\infty}^{\infty}\left[\frac{c}{d \eta_{0} / d y}+\left(1+\eta_{0}\right)\left(k^{2}+\frac{d^{3} \eta_{0} / d y^{3}}{d \eta_{0} / d y}\right)-\frac{d^{2} \eta_{0}}{d y^{2}}\right]|\hat{\eta}|^{2} d y=0 \tag{A6}
\end{equation*}
$$

The first term may be integrated by parts:

$$
\begin{equation*}
\int_{-\infty}^{+\infty} \hat{\eta}^{*} \frac{d}{d y}\left[\left(1+\eta_{0}\right) \frac{d \hat{\eta}}{d y}\right] d y=\left[\left(1+\eta_{0}\right) \hat{\eta}^{*} \frac{d \hat{\eta}}{d y}\right]_{-\infty}^{+\infty}-\int_{-\infty}^{+\infty}\left(1+\eta_{0}\right)\left|\frac{d \hat{\eta}}{d y}\right|^{2} d y \tag{A7}
\end{equation*}
$$

The first term in the right hand side tends to zero at infinity, the equation A6 then becomes:
$c \int_{-\infty}^{+\infty} \frac{|\hat{\eta}|^{2}}{d \eta_{0} / d y} d y+\int_{-\infty}^{\infty}\left(1+\eta_{0}\right)\left|\frac{d \hat{\eta}}{d y}\right|^{2} d y+\int_{-\infty}^{\infty}\left[\left(1+\eta_{0}\right)\left(k^{2}+\frac{d^{3} \eta_{0} / d y^{3}}{d \eta_{0} / d y}\right)-\frac{d^{2} \eta_{0}}{d y^{2}}\right]|\hat{\eta}|^{2} d y=0$

There is only one imaginary term in this equation, which must then vanish:

$$
\begin{equation*}
c_{i} \int_{-\infty}^{+\infty} \frac{|\hat{\eta}|^{2}}{d \eta_{0} / d y} d y=0 \tag{A9}
\end{equation*}
$$

Therefore, to be unstable $\left(c_{i} \neq 0\right)$, the flow has to have a non-monotonic basic state surface deviation :

$$
\exists y_{0} \quad / \quad \frac{d \eta_{0}}{d y}\left(y_{0}\right)=0
$$

## APPENDIX B

# Sensitivity of an isolated shear to details of the 

## velocity profile

The isolated shear defined in section 3 has two lateral vorticity peaks (Fig. 18b). In order to test the influence of the lateral shears on the stability of a localized shear flow, we compare two parallel flows having the same central shear but different lateral ones (Fig. 18a, b). According to Fig. 18c,d, the growth rate and the wavelength of the most unstable modes are weakly affected by the lateral shears. Hence, if we remain far from an outcropping configuration, the central shear controls the instability of this localized barotropic flow and the results of the following stability analysis can be extended to a large variety of shear flows.

## APPENDIX C

# Neutral eigenmodes of the Bickley jet in a quasi-geostrophic model 

Any profile $\eta_{0}(y)$ is a stationary solution of the quasi-geostrophic equations (3). Inserting $\eta=\eta_{0}(y)+\psi(x, y, t)$ into (3) and neglecting terms quadratic in $\psi$ yields the linearized QG equation :

$$
\left(\frac{\partial}{\partial t}+U \frac{\partial}{\partial x}\right)\left(\Delta \psi-B u^{-1} \psi\right)-\frac{\partial \psi}{\partial x}\left(U^{\prime \prime}-B u^{-1} U\right)=0
$$

where $U=-d \eta_{0} / d y$ and $U^{\prime \prime}=d^{2} U / d y^{2}$. Considering a normal mode $\psi(x, y, t)=\phi(y) \exp (i k(x-$ ct)) yields the Rayleigh-Kuo equation:

$$
\begin{equation*}
(U-c)\left(\phi^{\prime \prime}-\left(k^{2}+B u^{-1}\right) \phi\right)-\left(U^{\prime \prime}-B u^{-1} U\right) \phi=0 . \tag{C1}
\end{equation*}
$$

¿From now on $U(y)=\cosh ^{-2} y$ is the Bickley profile. A neutral eigenmode has by definition a zero growth rate or equivalently a real phase velocity $c$. Hence we look for a real phase velocity $c$ and a real wave-number $k$ such that Eq. (C1) admits a bounded solution $\phi(y)$.

Neutral eigenmodes of the Bickley jet have been found in the context of $\beta$-plane, incompressible dynamics (Engevik (2004)). In that case the normal modes satisfy

$$
\begin{equation*}
(U-c)\left(\phi^{\prime \prime}-k^{2} \phi\right)-\left(U^{\prime \prime}-\beta\right) \phi=0 . \tag{C2}
\end{equation*}
$$

Eq. (C2) is singular at the abscissa $y_{c}$ of the critical layer such that $U\left(y_{c}\right)=c$, unless
$U^{\prime \prime}\left(y_{c}\right)=\beta$, i.e. $\beta=(4-6 c) c$. Eq. (C2) then admits at least the following solutions :

$$
\begin{aligned}
& \phi_{1}=e^{-m y}\left(3 \tanh ^{2} y+3 m \tanh y+m^{2}-1\right), \\
& \phi_{2}=e^{m y}\left(3 \tanh ^{2} y-3 m \tanh y+m^{2}-1\right),
\end{aligned}
$$

where $m^{2}=k^{2}+4-6 c$. For these solutions to be bounded when $y \rightarrow \pm \infty$ we need $3-3 m+m^{2}-1=0$, i.e. either $m=1$ or $m=2$, in which case $\phi_{1}$ is a multiple of $\phi_{2}$. This leads to two neutral eigenmodes for each value of $c$ (Engevik (2004)).

Now (C1) is of the form (C2) with $\beta=B u^{-1} c$. The quasi-geostrophic neutral eigenmodes are found using simple substitution into the $\beta$-plane results. Since $\beta=B u^{-1} c$, the solutions of $\beta=(4-6 c) c$ are $c=0$ (critical layer at infinity) and $c=2 / 3-B u^{-1} / 6$. The latter case, the only one we consider here, leads to $m^{2}=k^{2}+B u^{-1}$. Finally $m=1$ and $m=2$ leads to two neutral eigenmodes: one with $k=\sqrt{1-B u^{-1}}$ and one with $k=\sqrt{4-B u^{-1}}$.

## REFERENCES

Anderson, E., Z. Bai, and C. B. et. al., 1999: LAPACK Users'Guide. Society for Industrial and Applied Mathematics, Philadelphia, PA, third edition.

Arai, M. and T. Yamagata, 1994: Asymmetric evolution of eddies in rotating shallow water. Chaos, 4 (2), 163-175.

Baey, J. M. and X. Carton, 2002: Vortex multipoles in two-layer rotating shallow-water flows. J. Fluid Mech., 460, 151-175.

Baey, J. M., P. Riviere, and X. Carton, 1999: Oceanic jet instability : a model comparison. Third International Workshop on Vortex Flows and Related Numerical Methods., Vol. 7, 12-23.

Cushman-Roisin, B., 1986: Frontal geostrophic dynamics. J. Phys. Ocean., 16, 132-143.

Cushman-Roisin, B. and B. Tang, 1990: Geostrophic turbulence and emergence of eddies beyond the radius of deformation. J. Phys. Ocean., 20, 97-113.

Dong, C., J. C. McWilliams, and A. F. Shchepetkin, 2007: Island wakes in deep water. J. Phys. Ocean., 37, 962-981.

Drazin, P. and W. H. Reid, 1981: Hydrodynamic stability. Cambridge University Press.

Engevik, L., 2004: A note on the barotropic instability of the bickley jet. J. Fluid Mech., 499, 315-326.

Flament, P., R. Lumpkin, J. Tournadre, and L. Arni, 2001: Vortex pairing in an unstable anticyclonic shear flow : discrete subharmonics of one pendulum day. J. Fluid Mech., 440, 401-409.

Graves, L. P., J. C. McWilliams, and M. T. Montgomery, 2005: Vortex evolution due to straining: a mechanism for dominance of strong, interior anticyclones. submitted to Geophys. Astrophys. Fluid Dynamics.

Howard, L. and P. G. Drazin, 1964: On stability of parallel flow of inviscid fluid in a rotating system with variable coriolis parameter. J. Maths and Phys., 43, 83-99.

Johnson, J., 1963: Stability of shearing motion in rotating fluid. J. Fluid Mech., 17, 337-352.

Kontoyiannis, H. and D. R. Watts, 1994: Observations on the variability of the gulf stream path between 74w and 70w. J. Phys. Ocean., 24, 1999-2013.

Lee, T. and P. Cornillon, 1996: Propagation and growth of he gulf-stream meanders between 75 and 45w. J. Phys. Ocean., 26, 225-241.

Linden, P. F., B. M. Boubnov, and S. B. Dalziel, 1995: Source-sink turbulence in a rotating stratified fluid. J. Fluid Mech., 298, 81-112.

Lipps, F. B., 1962: The barotropic instability of the mean winds in the atmosphere. J. Fluid Mech., 12, 397-407.

Maslowe, S., 1991: Barotropic instability of the bickley jet. J. Fluid Mech., 229, 417-426.

Matsuura, T. and T. Yamagata, 1982: On the evolution of nonlinear planetary eddies larger than the radius of deformation. J. Phys. Ocean., 12, 440-456.

McWilliams, J. C., 1985: Submesoscale, coherent vortices in the ocean. Rev. of Geophys., 23(2), 165-182.

Mitchum, G. T., 1995: The source of 90-day oscillations at wake island. J. Geophys. Res., 100, 2459-2475.

Nycander, J. and G. G. Sutyrin, 1992: Stationary translating anticyclones on the beta-plane. Dyn. Atm. Ocean., 16, 473-498.

Olson, D. B., 1991: Rings in the ocean. Annu. Rev. Planet. Sci., 19, 283-311.

Olson, D. B. and R. J. Evans, 1986: Rings of the agulhas. Deep-Sea Res., 33, 42.

Pedlosky, J., 1987: Geophysical Fluid Dynamics. Springer.

Perret, G., A. Stegner, M. Farge, and T. Pichon, 2006a: Cyclone-anticyclone asymmetry of large-scale wakes in the laboratory. Phys. Fluids, 18(3).

Perret, G., A. Stegner, D. T., C. J. M., and M. Farge, 2006b: Stability of parallel wake flows in quasi-geostrophic and frontal regime. Phys. Fluids, 18(12).

Polvani, L. M., J. C. McWilliams, M. A. Spall, and R. Ford, 1994: The coherent structures of shallow-water turbulence: deformation-radius effects, cyclone/anticyclone asymmetry and gravity-wave generation. Chaos, 4(2), 177-186.

Poulin, F. J. and G. R. Flierl, 2003: The nonlinear evolution of barotropically unstable jets. J. Phys. Ocean., 33, 2173-2192.

Sangrá, P., et al., 2005: Life history of an anticyclonic eddy. J. Geophys. Res., 110, C030 201,doi:10.1029/2004JC002 526.

Stegner, A. and D. G. Dritschel, 2000: A numerical investigation of the stability of isolated shallow water vortices. J. Phys. Ocean., 30, 2562-2573.

Stegner, A. and V. Zeitlin, 1995: What can asymptotic expansions tell us about large-scale quasi-geostrophic anticyclonic vortices ? Nonlinear Processes in Geophysics., 2, 186-193.

Stegner, A. and V. Zeitlin, 1996: Asymptotic expansions and monopolar solitary rossby vortices in barotropic and two-layer models. Geophys. Astro. Fluid Dyn., 83(3-4), 159194.

Watts, D. R. and W. E. Johns, 1982: Gulf stream meanders : Observations on propagation and growth. J. Geophys. Res., 87, 9467-9475.

## List of Figures

1 Anticyclonic shear basic state profiles : velocity (a), vorticity (b), relative surface deviation (c) and potential vorticity (d) for $\lambda=0.1$ and $\lambda=1.0$.

Stability properties of the anticyclonic (a) and the cyclonic (d) localized shear flow. The central panels (b), (e) show the unstable growth rates, calculated both for the quasi-geostrophic (filled dots) and the frontal regime (open dots). The potential vorticity of the most unstable modes are given in the bottom panel (c), (f).

3 Evolution of the dimensionless growth time for anticyclonic $(\bullet)$ and cyclonic $(\square)$ shear as a function of the surface deviation $\lambda$.

4 Evolution of the leading unstable wavelength for the anticyclonic ( $\bullet$ ) and the cyclonic ( $\square$ ) shear as a function of the surface deviation $\lambda=R o / B u$.

5 Nonlinear evolution of potential vorticity for the cyclonic and anticyclonic shear at $\lambda=0.1$ and $R o=0.1$. The extremum of potential vorticity anomaly is $\pm 0.3 f /\left(g h_{0}\right)$ and the contour interval is $3.10^{-2} f /\left(g h_{0}\right)$ where $f$ is the Coriolis parameter and $g h_{0}$ the mean geopotential.

6 Nonlinear evolution of potential vorticity (PV) for the anticyclonic and cyclonic shear at $\lambda=1.0$ and $R o=0.1$. The maximum cyclonic PV anomaly is $1.4 f /\left(g h_{0}\right)$ and minimum of anticyclonic PV anomaly is $-0.6 f /\left(g h_{0}\right)$. The contour interval is $6.10^{-2} f /\left(g h_{0}\right)$ for anticyclonic shear and $0.14 f /\left(g h_{0}\right)$ for cyclonic shear where $f$ is the Coriolis parameter and $g h_{0}$ the mean geopotential. 45

Vorticity for anticyclonic shear at $T=100 T_{0}$ and cyclonic shear at $T=500 T_{0}$ for $\lambda=0.1$ and $R o=0.1$. The minimum of anticyclonic vorticity is $-0.3 f$, contour interval is 0.03 , the maximum of cyclonic vorticity is $0.17 f$, contour interval is 0.017 .

8 The Bickley jet basic velocity (a), vorticity (b), relative surface deviation (c) and potential vorticity (d) for $\lambda=0.1$ and $\lambda=0.5$.

9 Growth rate of the bickley jet at different resolution for $\mathrm{Bu}=1.0$ and $\mathrm{Ro}=0.1$
10 (a) Growth rate of the bickley jet for various deviation of the free surface $\lambda$ at $R o=0.1$. The dashed line with square spots indicate $\left(k_{\max }, \sigma_{\max }\right)$ for intermediate values of $\lambda$. (b) Potential vorticity anomaly of the most unstable mode.

11 (a) Semi-log plot of the dimensionless growth time (a) and the dimensionless wavelength (b) for the Bickley jet as a function of the relative surface deviation $\lambda$.

12 Nonlinear evolution of potential vorticity (PV) for the Bickley jet at $R o=0.1$ and $\lambda=0.1$ on the left column and $\lambda=0.5$ on the right column. The maximum and minimum PV anomaly is $\pm 0.15 f /\left(g h_{0}\right)$ for $\lambda=0.1$ and $\pm 0.5 f /\left(g h_{0}\right)$ for $\lambda=0.5$ where $f$ is the Coriolis parameter and $g h_{0}$ the mean geopotential. The contour interval is $5 \cdot 10^{-3} f /\left(g h_{0}\right)$ for $\lambda=0.1$ and $2.10^{-2} f /\left(g h_{0}\right)$ for $\lambda=0.5$.

13 Velocity profile (a), vorticity (b) for a parallel wake flow of typical width $\delta=$ 2.5. The corresponding relative surface deviation (c) and potential vorticity (d) are plotted for $\lambda=0.1$ (solid line) and $\lambda=0.5$ (dashed line).

14 The two most unstable branches growth rate of wake flow for $\delta=2.5$ in a quasi-geostrophic regime (a) and a frontal regime (c) and PV of the most unstable modes for QG (b) and FG (d) regime.

15 The two most unstable branches growth rate of wake flow for $\delta=1.22$ in a quasi-geostrophic regime (a) and frontal regime (c) and PV of the most unstable modes for QG (b) and FG (d) regime.

16 Coupling of the two most unstable modes in the $\mathrm{Bu}-\gamma$ parameter space.
17 Nonlinear evolution of potential vorticity field for two wake profiles corresponding to $\delta=2.5$ and $\delta=1.22$ at $\lambda=0.1$ and $\lambda=0.5$ and $\mathrm{Ro}=0.1$. The range of potential vorticity is $\pm 0.4$ for $\delta=2.5$ and the contour interval is 4.0. $10^{-2}$ for $\lambda=0.1$ and $\pm 0.8$ with a contour interval of $8 \cdot 0.10^{-2}$.

18 Comparison between a standard localized shear profile (solid line) and a localized shear flow with less steep lateral shears (dashed line): (a) velocity profile, (b) vorticity profile, growth rates of the unstable mode for the quasigeostrophic regime $\lambda=0.1$ (c) and the frontal regime $\lambda=1.0$ (d).











$$
\lambda=0.1
$$























[^0]:    * Corresponding author address: Gaele Perret, Laboratoire Ondes et Milieux Complexes, 53, rue Prony, BP540, 76058 Le Havre Cedex, France.

    E-mail: perretg@univ-lehavre.fr

