Wave and Vortex Motions and Instabilities in Rotating Shallow Water Model

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M1 MOCIS

Parent equations

Fluid with constant density ρ_0 with a free surface $\leftrightarrow$ single isopycnal, simplest stratification. Thin layer $\to$ columnar motion $\equiv$ horizontal velocity $\boldsymbol{v}_h = \boldsymbol{v}_h(x, y, t)$.
Hydrostatic equations, $\boldsymbol{v} = \boldsymbol{v}_h + \hat{\boldsymbol{z}} w$:

$$\frac{\partial \boldsymbol{v}_h}{\partial t} + \boldsymbol{v}_h \cdot \boldsymbol{\nabla} \boldsymbol{v}_h + f \hat{z} \wedge \boldsymbol{v}_h = -\frac{\boldsymbol{\nabla}_h P}{\rho}, \quad \boldsymbol{\nabla}_h = (\partial_x, \partial_y)$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0, \quad g = -\frac{\partial_z P}{\rho}.$$

Boundary conditions

Horizontal boundary conditions:

Periodic, decay, or other

Vertical kinematic boundary conditions:

- Free surface $z = h(x, y, t)$:

$$w|_{z=h} = \frac{dh}{dt} = \partial_t h + \boldsymbol{v}_h \cdot \boldsymbol{\nabla}_h h.$$

  Meaning: material surface made of fluid parcels.

- Flat bottom:

$$w|_{z=0} = 0.$$

  Meaning: non-penetration through the boundary.

Vertical dynamic boundary condition:

Continuity of pressure: $P|_{z=h} = P_0 = \text{const.}$

Eliminating P and w

- Integrating hydrostatic equation:

$$P(x, y, z, t) = -\rho_0 \, g \, z + \mathcal{P}(x, y, t),$$

$\mathcal{P}(x, y, t)$ - integration "constant". Dynamic boundary condition: $\mathcal{P}(x, y, t) = \rho_0 g h(x, y, t) + P_0$.

- Integrating continuity equation:

$$\nabla_h \boldsymbol{v}_h(x, y, t) + \partial_z w(x, y, z, t) = 0 \rightarrow$$

$$w = -z \, \nabla_h \boldsymbol{v}_h(x, y, t) + \mathcal{W}(x, y, t).$$

$\mathcal{W}(x, y, t)$ - integration "constant". Bottom boundary condition: $\mathcal{W}(x, y, t) = 0$.

- Kinematic boundary condition at the surface:

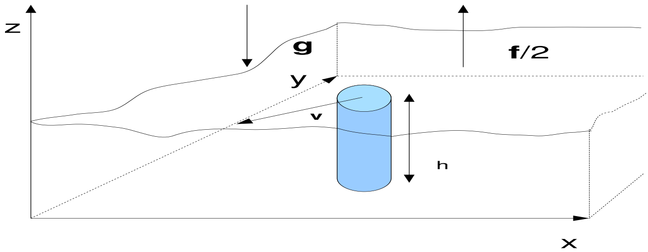
$$w|_{z=h} = -h \nabla_h \boldsymbol{v}_h(x, y, t) = \partial_t h + \boldsymbol{v}_h \cdot \nabla_h h.$$

Rotating shallow water (Saint-Venant) equations

$$\partial_t \boldsymbol{v}_h + \boldsymbol{v}_h \cdot \nabla \boldsymbol{v}_h + f\hat{\mathbf{z}} \wedge \mathbf{v}_h + g\nabla h = 0\,, \tag{1}$$

$$\partial_t h + \nabla \cdot (\mathbf{v}_h h) = 0\,, \tag{2}$$

Meaning: horizontal motion of columns of fluid of variable depth.



Conservation laws and acoustic analogy

Eulerian conservation laws

Equations (1), (2) express the local conservation of the horizontal momentum and mass.
By direct calculation using (1), (2), for energy density:

$$e = h\frac{\mathbf{v}^2}{2} + g\frac{h^2}{2} \quad (3)$$

we get

$$\partial_t e = -\nabla \cdot \left(\mathbf{v} h \left(\frac{\mathbf{v}^2}{2} + gh \right) \right) \Rightarrow \quad (4)$$

total energy, $E = \int dxdy\, e =$ const, for isolated system.

Acoustic analogy

Equation (2) is a continuity equation for "density" h.
Equations. (1) are 2-dimensional Euler equations in a rotating frame for a barotropic fluid with density h and pressure $P = \frac{g\,h^2}{2}$.

Vorticity and Potential vorticity

Only the vertical component of relative vorticity counts in RSW:

$$\zeta = v_x - u_y$$

Relative vorticity: vorticity measured in the rotating frame.
Absolute vorticity: vorticity measured in a fixed frame

$$\zeta_a = \zeta + f$$

Planetary vorticity f: vorticity due to rotation of the system.
Potential vorticity (PV):

$$q = \frac{\zeta + f}{h}. \tag{5}$$

Lagrangian conservation of PV:

$$\frac{dq}{dt} \equiv (\partial_t + \boldsymbol{v} \cdot \nabla)\, q = 0, \tag{6}$$

is obtained by combining equations of vorticity:

$$\frac{d(\zeta + f)}{dt} + (\zeta + f)\nabla \cdot \boldsymbol{v} = 0, \tag{7}$$

and mass conservation:

$$\frac{dh}{dt} + h\nabla \cdot \boldsymbol{v} = 0 \ : \tag{8}$$

$$\frac{d}{dt}\frac{\zeta + f}{h} = \frac{1}{h}\frac{d}{dt}(\zeta + f) - \frac{\zeta + f}{h^2}\frac{d}{dt}h = 0, \tag{9}$$

Eulerian expression:

Conservation of PV leads to independence of time of any integral:

$$\int dxdy\, h\mathcal{F}(q), \tag{10}$$

over the whole flow, with $\mathcal{F}$ - arbitrary function.

Qualitative image of the RSW dynamics:

Two-dimensional motion of the fluid columns of variable depth, each preserving its potential vorticity.

Spectrum of small perturbations to the state of rest on the f-plane

Method of small perturbations

State of rest $\boldsymbol{v} = 0, \; h = H_0 = const$ - exact solution. Consider small perturbations $\boldsymbol{v} = (u, \; v) \, h = H_0 + \eta$, such that $||u||, \; ||v||, \; ||\eta||$ are all $\ll 1$, which allows to neglect the nonlinear terms $\rightarrow$ linearization.

Linearized RSW equations :

Linearized equations in the approximation $f = const$:

$$
\begin{aligned}
u_t - fv + g\eta_x &= 0, \\
v_t + fu + g\eta_y &= 0, \\
\eta_t + H_0(u_x + v_y) &= 0,
\end{aligned}
\qquad (11)
$$

Method of Fourier

Solutions - harmonic waves:

$$(u, v, \eta) = (u_0, v_0, \eta_0)e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})} + \text{c.c.}, \tag{12}$$

where ω and $\boldsymbol{k}$ are frequency and wavenumber, respectively $\Rightarrow$
algebraic system for (u_0, v_0, η_0):

$$\begin{pmatrix} i\omega & -f & -igk_x \\ f & i\omega & -igk_y \\ -iH_0k_x & -iH_0k_y & i\omega \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \\ \eta_0 \end{pmatrix} = 0, \tag{13}$$

Dispersion equation

Condition of solvability:

$$\det \begin{pmatrix} i\omega & -f & -igk_x \\ f & i\omega & -igk_y \\ -iH_0 k_x & -iH_0 k_y & i\omega \end{pmatrix} = 0, \tag{14}$$

which gives:

$$\omega \left(\omega^2 - gH_0 \boldsymbol{k}^2 - f^2 \right) = 0. \tag{15}$$

Physical meaning of solutions

3 roots of the equation correspond to

- Stationary solutions $\omega = 0 \leftrightarrow$ linearized PV-conservation equation:

$$\partial_t \left(\frac{\partial_x v - \partial_y u}{H_0} - \frac{f\,\eta}{H_0^2} \right) = 0$$

- Propagative waves with the dispersion relation:

$$\omega^2 - gH_0 \boldsymbol{k}^2 - f^2 = 0. \qquad (16)$$

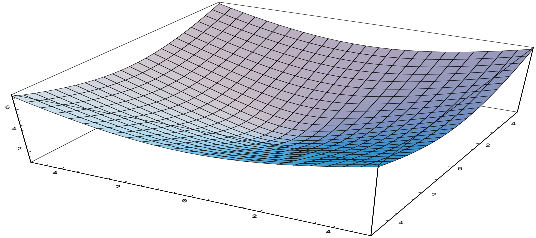
Inertia-gravity waves.

Dispersion relation (16) is isotropic. No-rotation limit:

$$\omega = \pm\sqrt{gH_0}|\boldsymbol{k}| \rightarrow$$

acoustic waves with "speed of sound" $c = \sqrt{gH_0}$.

Dispersion relation



Dispersion relation for inertia-gravity waves.
$c = \sqrt{gH_0} = 1$, $f = 1$, the part with $\omega < 0$ is not presented.

Horizontal motion equations

$$\frac{\partial \boldsymbol{v}_h}{\partial t} + \boldsymbol{v}_h \cdot \boldsymbol{\nabla}_h \boldsymbol{v}_h + f\hat{z} \wedge \boldsymbol{v}_h = -g\boldsymbol{\nabla}_h h. \qquad (17)$$

$$f = f_0(1 + \beta y), \quad H = H_0 + \eta \qquad (18)$$

“Vortex” scaling

- Velocity $\boldsymbol{v}_h = (u, v), \; u, v \sim U$
- Unique horizontal spatial scale L,
- Vertical scale $H_0 << L$,
- Time-scale: tirn-over time $T \sim L/U$.

Characteristic parameters

Intrinsic scale of the system: deformation (Rossby) radius:

$$R_d = \frac{\sqrt{gH_0}}{f_0} \tag{19}$$

- Rossby number: $Ro = \frac{U}{f_0 L}$,
- Burger number: $Bu = \frac{R_d^2}{L^2}$,
- Typical amplitude of the free surface elevations = non-linearity parametre: $\lambda = \Delta H / H_0$, where ΔH is the typical value of η,
- Non-dimensional meridional gradientof f: $\tilde{\beta} \sim \beta L$

Non-dimensional RSW equations

$$Ro\,(\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\mathbf{v}) + (1+\tilde{\beta}y)\hat{\mathbf{z}}\wedge\mathbf{v} = -\frac{\lambda Bu}{Ro}\nabla\eta\,, \tag{20}$$

$$\lambda\partial_t\eta + \nabla\cdot(\mathbf{v}(1+\lambda\eta)) = 0\,. \tag{21}$$

Geostrophic equilibrium

Equilibrium between the force of Coriolis and pressure force $\rightarrow$ geostrophic wind:

$$f\hat{\mathbf{z}}\wedge\mathbf{v}_g = -g\nabla h \tag{22}$$

Quasi-geostrophic approximation

Conditions of realization of the geostrophic equilibrium:

- $Ro \to 0$,
- $\lambda\ Bu \sim Ro$,
- $\tilde{\beta} \to 0$.

Quasi- geostrophy (QG):

$$Ro \equiv \epsilon \ll 1, \lambda \sim Ro, \Rightarrow Bu \sim 1, \Rightarrow L \sim R_d,\ \tilde{\beta} \sim Ro \quad (23)$$

Potential vorticity in QG approximation

Non-dimensional potential vorticity:

$$q = \frac{f_0}{H_0}\frac{\epsilon(v_x - u_y) + (1+\epsilon y)}{1+\epsilon\eta}$$
$$= \frac{f_0}{H_0}\left(\epsilon(v_x - u_y) + (1+\epsilon y)\right)(1 - \epsilon\eta + ...)$$
$$= \frac{f_0}{H_0}\left[1 + \epsilon\left(v_x - u_y + y - \eta\right) + \mathcal{O}\left(\epsilon^2\right)\right]. \quad (24)$$

Non-dimensional geostrophic wind:

$$v = \eta_x \quad u = -\eta_y \Rightarrow v_x - u_y = \nabla^2\eta \quad (25)$$

Advection by the geostrophic wind:

$$\partial_t ... + u\partial_x ... + v\partial_y ... \to \partial_t ... + \mathcal{J}(\eta, ...) \quad (26)$$

$\mathcal{J}(A, B) = A_x B_y - A_y B_x$ - Jacobian.

QG equation

$$\partial_t \left(\nabla^2 \eta - \eta \right) + \mathcal{J}(\eta, \nabla^2 \eta - \eta) + \partial_x \eta = 0. \quad (27)$$

Physical meaning: conservation of (non-dimensional) geostrophic PV $q_G = \nabla^2 \eta - \eta + y$. Restitution of dimensions:

$$\nabla^2 \eta - \eta \to \nabla^2 \eta - \frac{1}{R_d^2} \eta, \; \partial_x \eta \to \beta \partial_x \eta. \quad (28)$$

Formal linearization:

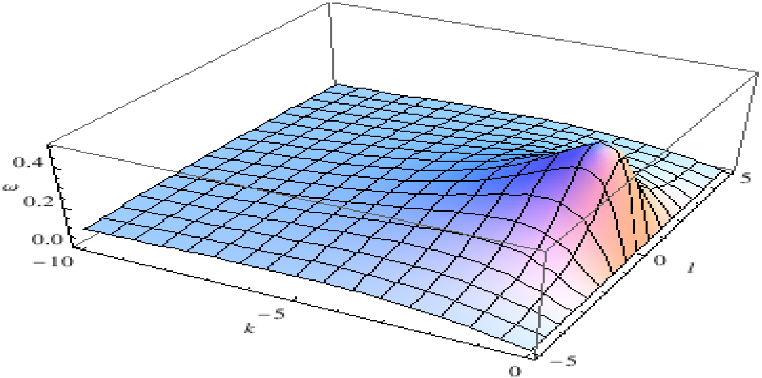
$$\partial_t \eta - \nabla^2 \partial_t \eta - \partial_x \eta = 0. \quad (29)$$

Wave solutions: $\eta \propto exp^{i(kx+ly-\omega t)} \to$ dispersion relation:

$$\omega = -\frac{k}{k^2 + l^2 + 1}. \quad (30)$$

Rossby waves: strongly dispersive, with anisotropic dispersion $\leftrightarrow$ vorticity waves.

Dispersion diagram for Rossby waves



Rossby waves over a mean flow

Zonal flow: $(u, v) = (U(y), 0)$. Corresponding geopotential anomaly:

$$\eta = \eta_0(y) = -\int^y dy' U(y') \Rightarrow \nabla^2 \eta = -U'(y). \qquad (31)$$

Linearization of (27) about η_0, $\eta \to \eta_0 + \eta$:

$$(\partial_t + U(y)\partial_x)\left(\nabla^2\eta - \eta\right) + \partial_x\eta(-U''(y) + U(y)) + \partial_x\eta = 0. \Rightarrow \qquad (32)$$

PV gradient of the mean flow $(-U''(y) + U(y))$ plays the same rôle as β (last term in (32)).

If $U = const$, equation (32) has constant coefficients $\rightarrow$ Fourier -transform$\rightarrow$ dispersion relation in the limit $k^2 + l^2 \gg 1$:

$$\omega = Uk - \frac{k}{k^2 + l^2} \qquad (33)$$

absolute frequency . $\omega = -\frac{k}{k^2+l^2}$ - intrinsic frequency .

Rossby waves over mean flow on the f-plane

Equation (32) on the f-plane, in the limit $R_d \to \infty$:

$$\nabla^2 \eta_t + U(y) \nabla^2 \eta_x - \eta_x U''(y) = 0. \qquad (34)$$

Partial Fourier-transform: $\eta(x, y, t) \to \hat{\eta}(y) e^{ik(x-ct)} \Rightarrow$

$$\hat{\eta}''(y) - \left[k^2 + \frac{U''(y)}{U(y) - c} \right] \hat{\eta}(y) = 0. \qquad (35)$$

Boundary conditions: free-slip in the zonal channel $y_1 \le y \le y_2$

$$v|_{y=y_{1,2}} = \eta_x|_{y=y_{1,2}} = 0, \Rightarrow \hat{\eta}|_{y=y_{1,2}} = 0$$

Integral estimate

Integration over y

$$\int_{y_1}^{y_2} dy \left[\hat{\eta}^*(y) \left(\hat{\eta}''(y) - \left[k^2 + \frac{U''(y)}{U(y) - c} \right] \hat{\eta}(y) \right) \right] = 0 \tag{36}$$

Integration by parts + boundary conditions:

$$\int_{y_1}^{y_2} dy \left(\hat{\eta}^{*\prime}(y)\hat{\eta}'(y) + \left[k^2 + \frac{U''(y)}{U(y) - c} \right] \hat{\eta}^*(y)\hat{\eta}(y) \right) = 0 \tag{37}$$

Imaginary part:

Only in phase velocity $\Rightarrow$

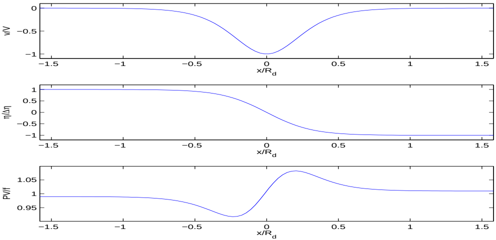
$$c_i \int_{y_1}^{y_2} dy \, \frac{U''(y)}{|U(y) - c|^2} \hat{\eta}^*(y)\hat{\eta}(y) = 0$$

Rayleigh criterion of barotropic instability

$$\int_{y_1}^{y_2} dy \, \frac{U''(y)}{|U(y) - c|^2} \hat{\eta}^*(y)\hat{\eta}(y) = 0, \quad \text{if} \;\; c_i \neq 0 \Rightarrow$$

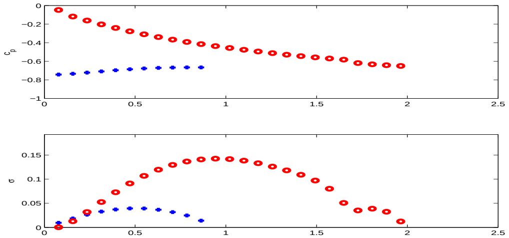
In the absence of critical levels ($U(y) - c \neq 0$), if the flow is unstable, then $U(y)$ has inflexion point $\exists y_0 : U''(y_0) = 0$.

Example: barotropic instability of a meridional jet on the f-plane in RSW



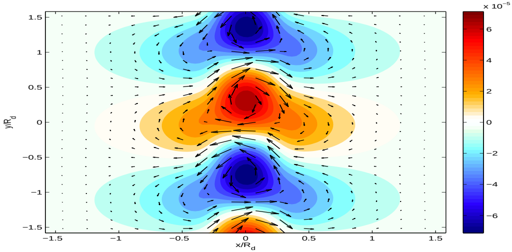
Jet in geostrophic equilibrium. From top to bottom: meridional velocity, geopotential, and PV.

Dispersion relation and growth rate



Phase velocity (top) and growth rate (bottom) of two most unstable modes.

The most unstable mode



Pressure and velocity anomalies of the most unstable mode.

Nonlinear evolution of the instability (relative vrticity)