Wave and Vortex Motions and Instabilities in the Primitive Equations

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Oceanic PE

$$\frac{\partial \boldsymbol{v}_h}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}_h + f\hat{z} \wedge \boldsymbol{v}_h = -\frac{\boldsymbol{\nabla}_h \pi}{\rho} \equiv -\boldsymbol{\nabla}_h \phi, \tag{1}$$

$$\partial_t \sigma + \boldsymbol{v} \cdot \boldsymbol{\nabla} \sigma + w \rho_s'(z) = 0. \tag{2}$$

$$g\frac{\sigma}{\rho_0} = -\partial_z \phi, \quad \boldsymbol{\nabla}_h \cdot \boldsymbol{v}_h + \partial_z w = 0, \tag{3}$$

Eulerian conservation

- Horizontal momentum (modulo Coriolis force), with density $\rho_0 \boldsymbol{v}_h$
- Mass, with density ρ
- Energy, with density

$$e = \rho_0 \frac{\boldsymbol{v}_h^2}{2} + \rho g Z, \quad \rho = \rho_s + \sigma, \tag{4}$$

where Z is Lagrangian position of the fluid element.

Proof of energy conservation

$$\begin{aligned} \frac{d\,e}{dt} &= \rho_0 \boldsymbol{v}_h \cdot \frac{d\,\boldsymbol{v}_h}{dt} + g \frac{d\,\rho}{dt} Z + g\rho \frac{d\,Z}{dt} \\ &= -\rho_0 \boldsymbol{v}_h \cdot \boldsymbol{\nabla}_h \phi - \rho_0 \frac{\partial \phi}{\partial z} w \equiv -\boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{v} \phi), \end{aligned} \tag{5}$$

where hydrostatic relation between geopotential ϕ and density was used. Recalling that $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$, this gives:

$$\frac{\partial e}{\partial t} = -\boldsymbol{\nabla} \cdot (\boldsymbol{v}(\rho_0 \phi - e)).$$

Absolute vorticity in PE

Absolute vorticity:

$$\zeta_a = \zeta + \hat{\mathbf{z}} f, \quad \nabla \cdot \zeta_a = 0, \tag{6}$$

where relative vorticity under PE scaling is:

$$\zeta = -\partial_z v \hat{\mathbf{x}} + \partial_z u \hat{\mathbf{y}} + (\partial_x v - \partial_y u)\hat{\mathbf{z}} \tag{7}$$

Application of $\nabla \wedge$ to PE + "hydrodynamic identity":

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} = \frac{1}{2}\nabla \boldsymbol{v}^2 - \boldsymbol{v} \wedge (\nabla \wedge \boldsymbol{v}) \tag{8}$$

$\rightarrow$ equation for ζ_a:

$$\begin{aligned} \frac{d\zeta_a}{dt} &= \zeta_a \cdot \nabla \boldsymbol{v} + \frac{g}{\rho_0}\hat{\mathbf{z}} \wedge \nabla \sigma \rightarrow \\ \frac{\partial \zeta_a}{\partial t} &= \nabla \wedge (\boldsymbol{v} \wedge \zeta_a) + \frac{g}{\rho_0}\hat{\mathbf{z}} \wedge \nabla \sigma. \end{aligned} \tag{9}$$

Conservation of potential vorticity

$$q := \zeta_a \cdot \nabla\rho, \quad \rho = \rho_s(z) + \sigma, \quad \frac{dq}{dt} = 0. \tag{10}$$

Using identities

$$\nabla A \cdot (\nabla \wedge \boldsymbol{B}) = -\nabla \cdot (\nabla A \wedge \boldsymbol{B}), \tag{11}$$

$$\boldsymbol{A} \wedge (\boldsymbol{B} \wedge \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B}), \tag{12}$$

and $\nabla \cdot \boldsymbol{v} = \nabla \cdot \zeta_a = 0$:

$$\begin{aligned}
\partial_t (\zeta_a \cdot \nabla\rho) &= (\partial_t \zeta_a) \cdot \nabla\rho + \zeta_a \cdot \nabla(\partial_t \rho) \\
&= \nabla\rho \cdot (\nabla \wedge (\boldsymbol{v} \wedge \zeta_a)) - \zeta_a \cdot \nabla(\boldsymbol{v} \cdot \nabla\rho) \\
&= -\nabla \cdot (\nabla\rho \wedge (\boldsymbol{v} \wedge \zeta_a)) - \zeta_a \cdot \nabla(\boldsymbol{v} \cdot \nabla\rho) \\
&= -\nabla \cdot (\boldsymbol{v}(\zeta_a \cdot \nabla\rho)) + \nabla \cdot (\zeta_a(\boldsymbol{v} \cdot \nabla\rho)) \\
&- \zeta_a \cdot \nabla(\boldsymbol{v} \cdot \nabla\rho) = -\boldsymbol{v} \cdot \nabla(\zeta_a \cdot \nabla\rho).
\end{aligned} \tag{13}$$

Spectrum of small perturbations - PE model

Linearised equations:

Perturbations about the state of rest: $\boldsymbol{v} = 0$ with constant stratification on the f- plane. Linearised equations:

$$
\begin{aligned}
u_t - fv + \phi_x &= 0, \\
v_t + fu + \phi_y &= 0, \qquad (14) \\
\phi_z + \frac{g}{\rho_0}\sigma = 0, \;\; \sigma_t + w\rho_s' &= 0, \\
u_x + v_y + w_z &= 0, \qquad (15)
\end{aligned}
$$

where u, v w - three components of velocity perturbation, ϕ - geopotential perturbation, σ - perturbation of the profile of background density ρ_s, with $\rho_s' = const$.

Elimination of σ and w:

Elimination of σ:

$$\phi_{zt} + wN^2 = 0, \tag{16}$$

where $N^2 = -\frac{g\rho_s'}{\rho_0}$, N - Brunt - Väisälä frequency

Elimination of w:

$$\begin{aligned} u_t - fv + \phi_x &= 0, \\ v_t + fu + \phi_y &= 0, \quad (17) \\ u_x + v_y - N^{-2}\phi_{zzt} &= 0, \quad (18) \end{aligned}$$

Method of Fourier

Solutions - harmonic waves:

$$(u, v, \phi) = (u_0, v_0, \phi_0) e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})} + \text{c.c.}, \tag{19}$$

where ω et $\boldsymbol{k} = (k_x, \, k_y, \, k_z)$ are frequency and wavenumber, respectively.

Algebraic system for (u_0, v_0, ϕ_0):

$$\begin{pmatrix} i\omega & -f & -ik_x \\ f & i\omega & -ik_y \\ -ik_x & -ik_y & i\frac{\omega}{N^2}k_z^2 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \\ \eta_0 \end{pmatrix} = 0, \tag{20}$$

Dispersion equation

Condition of solvability:

$$\det \begin{pmatrix} i\omega & -f & -ik_x \\ f & i\omega & -ik_y \\ -ik_x & -ik_y & i\frac{\omega}{N^2}k_z^2 \end{pmatrix} = 0, \tag{21}$$

which gives:

$$\omega\left(\omega^2 - \left(N^2\frac{k_x^2+k_y^2}{k_z^2} + f^2\right)\right) = 0. \tag{22}$$

Physical meaning of solutions

Three roots of this equation correspond to

- Stationary solutions $\omega = 0$
- Propagative waves with dispersion relation:

$$\omega^2 = N^2 \frac{k_x^2 + k_y^2}{k_z^2} + f^2 \qquad (23)$$

Internal inertia-gravity waves: IGW.

Remark: at each fixed k_z - dispersion relation of RSW with $\sqrt{gH_0} \to \frac{N}{|k_z|}$

Non-hydrostatic Boussinesq equations

$$g\frac{\sigma}{\rho_0} = -\phi_z \rightarrow \frac{dw}{dt} + g\frac{\sigma}{\rho_0} = -\phi_z. \qquad (24)$$

Elimination of $b = -g\frac{\sigma}{\rho_0}$ and w:

$$b = \phi_z + w_t, \quad -\left(\partial_{tt} + N^2\right)(u_x + v_y) + \phi_{zzt} = 0 \Rightarrow \qquad (25)$$

$$u_t - fv = -\phi_x, \qquad (26)$$

$$v_t + fu = -\phi_y, \qquad (27)$$

$$\left(\partial_{tt} + N^2\right)(u_x + v_y) - \phi_{zzt} = 0, \qquad (28)$$

Dispersion relation:

$$\omega\left[\omega^2 - \left(N^2\frac{k_x^2 + k_y^2}{k_x^2 + k_y^2 + k_z^2} + f^2\frac{k_z^2}{k_x^2 + k_y^2 + k_z^2}\right)\right] = 0 \qquad (29)$$

Typically in the atmosphere and ocean

$$N^2 > f^2 \Rightarrow f^2 \leq \omega^2 \leq N^2 \qquad (30)$$

PE, oceanic case

Characteristic scales

- Typical horizontal velocity: U
- Typical horizontal scale: L
- Time-scale: $T \sim L/U$ -turn-over time
- Typical vertical scale: H
- Typical vertical velocity:W $\frac{W}{H} \sim \lambda \frac{U}{L}$ -to confirm *aposteriori*
- Presssure scale: $\rho_0 g H$

Parameters

- Rossby number: $Ro = \frac{U}{f_0 L} \equiv \epsilon$
- Typical dimensionless deviation of the isopycnal surfaces: λ
- Dimensionless gradient of the Coriolis parametre: $\tilde{\beta}$
- Stratification parameter: $S = \frac{\text{variable part of density}}{\text{constant part of density}}$
- Burger number: $Bu = \frac{R_d^2}{L^2}$, where R_d baroclinic deformation radius with reduced gravity $g' = S\,g$.

Pressure and density related via hydrostatics:

$$\begin{aligned} \rho &= \rho_0 \left[1 + S\left(\rho_s(z) + \lambda\sigma(x, y, z; t)\right)\right], \Rightarrow \\ P &= \rho_0 g H \left[(1 - z) + S\left(p_s(z) + \lambda\pi(x, y, z; t)\right)\right] \end{aligned} \quad (31)$$

Non-dimensional equations

$$\epsilon \frac{d}{dt}\boldsymbol{v}_h + (1 + \tilde{\beta} y)\hat{z} \wedge \boldsymbol{v}_h = -\boldsymbol{\nabla}_h \pi. \tag{32}$$

$$\frac{d}{dt}\sigma + \rho_s' w = 0, \quad \partial_z \pi + \sigma = 0. \tag{33}$$

$$\boldsymbol{\nabla}_h \cdot \boldsymbol{v}_h + \lambda \partial_z w = 0; \tag{34}$$

where

$$\frac{d}{dt} = \partial_t + \boldsymbol{v}_h \cdot \nabla_h + \lambda w \partial_z \tag{35}$$

Boundary conditions - rigid lid/flat bottom, for simplicity:

$$w|_{z=0,1} = 0. \tag{36}$$

Thermal wind

QG regime

$$\lambda \sim \tilde{\beta} \sim \epsilon \ll 1, \quad \Rightarrow \quad L \sim R_d, \; \frac{W}{H} = \lambda \frac{U}{L}. \qquad (37)$$

Leading order in small parameters

Geostrophic + hydrostatic equilibria:

$$u = -\partial_y \pi, \; v = \partial_x \pi, \; \sigma = -\partial_z \pi \Rightarrow \partial_z v = -\partial_x \sigma, \; \partial_z u = +\partial_y \sigma \qquad (38)$$

Horizontal density gradient $\leftrightarrow$ vertical shear of the horizontal velocity. Atmosphere: $\sigma \to -\theta$.
Leading-order vertical velocity:

$$w = -\frac{1}{\rho_s'(z)} \frac{d_g \sigma}{dt} = \frac{1}{\rho_s'(z)} \frac{d_g \partial_z \pi}{dt}, \qquad (39)$$

where $\frac{d_g}{dt} = \partial_t + \partial_x \pi \partial_y - \partial_y \pi \partial_x$ - geostrophic advection.

PV in QG approximation

QG PV

$$q = (\boldsymbol{\zeta} + \hat{\boldsymbol{z}} f) \cdot \left(\rho_s'(z)\hat{\boldsymbol{z}} + \boldsymbol{\nabla}\sigma\right) \propto$$

$$(\epsilon\boldsymbol{\zeta} + \hat{\boldsymbol{z}}(1+\epsilon y)) \cdot \left((\rho_s'(z) + \epsilon\partial_z\sigma)\hat{\boldsymbol{z}} + \epsilon\frac{H}{L}\boldsymbol{\nabla}_h\sigma\right)$$

Using (38), and $H \ll L$:

$$q \propto \rho_s' + \epsilon\left[(\boldsymbol{\nabla}_h^2\pi + y)\rho_s' - \partial_{zz}^2\pi\right] + \mathcal{O}(\epsilon^2)$$

QG advection

$$\frac{d}{dt} = \partial_t + \boldsymbol{v}_h \cdot \boldsymbol{\nabla}_h + \epsilon w \partial_z = \frac{d_g}{dt} + \epsilon\left(\frac{1}{\rho_s'(z)}\frac{d_g\partial_z\pi}{dt}\partial_z + \boldsymbol{v}_h^a \cdot \boldsymbol{\nabla}_h\right) + \mathcal{O}(\epsilon^2),$$

where $\boldsymbol{v}_h^a$ is ageostrophic horizontal velocity.

QG with continuous stratification

QG PV conservation

$$\frac{d_g q}{dt} \propto \frac{d_g}{dt}\left[(\nabla_h^2\pi + y)\rho_s' - \partial_{zz}^2\pi\right] + \frac{\rho_s''}{\rho_s'(z)}\frac{d_g\partial_z\pi}{dt}$$
$$= \rho_s'\frac{d_g}{dt}\left[(\nabla_h^2\pi + y) - \partial_z\left(\frac{1}{\rho_s'}\partial_z\pi\right)\right] = 0. \quad (40)$$

Equation (40): quasi-geostrophic approximation to primitive equations.

Boundary conditions

Evolution equations (dynamics!)

$$w|_{z=0,1} = 0 \Rightarrow \frac{d_g}{dt}\partial_z\pi\Big|_{z=0,1} = 0. \quad (41)$$

Baroclinic Rossby waves

Formal linearisation

$$\partial_t \left[\nabla_h^2 \pi - \partial_z \left(\frac{1}{\rho_s'(z)} \partial_z \pi \right) \right] + \partial_x \pi = 0, \quad \partial_{tz}^2 \pi \Big|_{z=0,1} = 0. \tag{42}$$

Separation of variables

$$\pi(x, y, z; t) = p(x, y; t) S(z) \Rightarrow \tag{43}$$

$$\partial_t \nabla_h^2 p(x, y; t) S(z) - \partial_t p(x, y; t) \left[\frac{1}{\rho_s'(z)} S'(z) \right]' + \partial_x p(x, y; t) S(z) = 0 \Rightarrow$$

Equations in z and in x, y, t:

- $$\frac{1}{S(z)}\left[\frac{1}{\rho_s'(z)}S'(z)\right]' = \kappa^2 \tag{44}$$

- $$\partial_t \nabla_h^2 p(x,y;t) - \kappa^2 \partial_t p(x,y;t) + \partial_x p(x,y;t) = 0, \tag{45}$$

κ - separation constant

Vertical modes

Sturm - Liouville problem:

$$\left[\frac{1}{\rho_s'(z)}S'(z)\right]' - \kappa^2 S(z) = 0, \quad S'(z)\big|_{z=0,1} = 0 \qquad (46)$$

Eigenfunctions $S_n(z)$ and eigenvalues κ_n, $n = 0, 1, 2, ...$

Example: linear stratification $\rho_s = -N^2 z$

$$S''(z) + (N\kappa)^2 S(z) = 0, \quad S_n \propto \cos(\pi n z), \; \kappa_n = \frac{\pi n}{N}. \qquad (47)$$

Horizontal motion

Wave solutions: $p(x, y; t) \propto e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \rightarrow$

$$\omega = -\frac{k_x}{\boldsymbol{k}^2 + \kappa_n^2} - \text{Rossby waves.} \qquad (48)$$

$n \nearrow$ (stronger vertical shear) $\Rightarrow c_{phase} \searrow$.

Eady model

QG with constant stratification $N = \text{const}$ on the f-plane

$$\frac{d^{(0)}}{dt}\left(\partial_x^2\pi + \partial_y^2\pi + \frac{1}{N^2}\partial_z^2\pi\right) = 0, \quad \frac{d^{(0)}}{dt}\partial_z\pi\Big|_{z=0,1} = 0 \tag{49}$$

Thermal wind

Exact solution: $\vec{v} = U_0(z)\hat{\boldsymbol{x}},\ U_0 = z$.

Linearisation: $\pi = -U_0(z)y + \phi(x,y,z;t),\ ||\phi|| \ll 1$:

$$\begin{aligned}(\partial_t + U_0(z)\partial_x)\left(\partial_x^2\phi + \partial_y^2\phi + \frac{1}{N^2}\partial_z^2\phi\right) &= 0\\ \left[(\partial_t + U_0(z)\partial_x)(-y\partial_z U_0(z) + \partial_z\phi) - \partial_x\phi\partial_z U_0(z)\right]\Big|_{z=0,1} &= 0.\end{aligned}$$

Solution by separation of variables

Fourier modes

B.C. in y-direction: zonal channel $-1 \le y \le 1 \Rightarrow$
$\partial_x \phi|_{y=\pm 1} = 0$.

$$\phi = A(z) \cos l_n y \, e^{ik(x-ct)}, \quad l_n = (n + \frac{1}{2})\pi, n = 0, 1, 2, \dots$$

Equations and b.c.:

$$(z-c)\left(A''(z) - \mu^2 A(z)\right) = 0, \quad \mu^2 = (k^2 + l_n^2)N^2, \quad (50)$$

$$cA'(0) + A(0) = 0, \quad (c-1)A'(1) + A(1) = 0. \quad (51)$$

Solution

Non-singular solutions $\Leftrightarrow$ absence of critical layers z_c:
$c = U_0(z_c) \Rightarrow A''(z) - \mu^2 A(z) = 0.$
General solution: $A(z) = a \cosh \mu z + b \sinh \mu z$:

$$\begin{aligned} a + c\mu b &= 0, \\ a\left[(c-1)\mu \sinh \mu + \cosh \mu\right] &+ \\ b\left[(c-1)\mu \cosh \mu + \sinh \mu\right] &= 0. \end{aligned} \qquad (52)$$

Dispersion relation:

$$c^2 - c + \frac{\coth \mu}{\mu} - \frac{1}{\mu^2} = 0 \Rightarrow \qquad (53)$$

$$c = \frac{1}{2} \pm \left(\frac{1}{4} + \frac{1}{\mu^2} - \frac{\coth \mu}{\mu} \right)^{\frac{1}{2}} \qquad (54)$$

Analysis of dispersion relation

Identity: $\coth\mu = \frac{1}{2}(\tanh\frac{\mu}{2} + \coth\frac{\mu}{2})$:

$$c = \frac{1}{2} \pm \frac{1}{\mu}\left[\left(\frac{\mu}{2} - \coth\frac{\mu}{2}\right)\left(\frac{\mu}{2} - \tanh\frac{\mu}{2}\right)\right]^{\frac{1}{2}}. \qquad (55)$$

$\forall x \ \tanh x \leq x \Rightarrow$ instability at
$\coth\frac{\mu}{2} > \frac{\mu}{2} \Rightarrow \mu < \mu_c \approx 2.4 \Rightarrow$ instability of long waves.