Surface and Interface Waves. Kelvin-Helmholtz Instability

V. Zeitlin

M1 MOCIS

Infinitely deep water: equations of motion

Two-dimensional (homogeneous in y) motions of incompressible fluid with density $\rho_0 = \text{const}$.
Small scales $\Rightarrow$ rotation negligible, non-hydrostatic:

$$\frac{du}{dt} = -\frac{P_x}{\rho_0} \quad (1)$$

$$\frac{dw}{dt} + g = -\frac{P_z}{\rho_0}, \quad (2)$$

$$u_x + w_z = 0. \quad (3)$$

Here $\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}$
Boundary conditions at the free surface situated at $z = \eta(x, t)$:

- Kinematic condition : $\frac{d\eta}{dt} = \eta_t + u\eta_x = w(z = \eta)$,
- Dynamic condition : $P(z = \eta) = P_0$, $P_0 = \text{const}$ - pressure over the surface.
- Bottom condition at $z \to -\infty$: decay of all perturbations

Linearisation about the state of rest

State of rest in hydrostatic equilibrium
$P = P(z) = -\rho_0 gz + P_0$ - exact solution.
Linearisinf about this state:

$$u_t = -\frac{p_x}{\rho_0}, \quad (4)$$
$$w_t = -\frac{p_z}{\rho_0}, \quad (5)$$
$$u_x + w_z = 0. \quad (6)$$

Small perturbations sought in the form of harmonic waves propagating in the x- direction with a vertical structure to be determined:

$$(u, w, p, \eta) = (\hat{u}(z), \hat{w}(z), \hat{p}(z), \hat{\eta})\, e^{i(kx-\omega t)} + c.c.. \quad (7)$$

Wave solutions

Elimination of u, w by respective differentiations of (4) and (5) and use of (6):

$$p_{xx} + p_{zz} = 0, \;\Rightarrow \hat{p}''(z) - k^2\hat{p}(z) = 0. \qquad (8)$$

Solution obeying the bottom decay condition at $z \to -\infty$: $\hat{p}(z) = \hat{p}_0 e^{kz}$.
Linearised dynamic boundary condition at the surface:

$$-\rho_0 g\hat{\eta} + \hat{p}_0 = 0 \qquad (9)$$

Using (5) at $z = 0$: $-\omega\hat{w} = -\frac{k}{\rho_0}\hat{p}_0 = -gk\hat{\eta}$, taking into account (9).
Linearised kinematic boundary condition : $-i\omega\hat{\eta} = \hat{w} \Rightarrow$ dispersion relation for surface waves in very deep water:

$$\omega^2 = gk. \qquad (10)$$

Exercise:

Obtain polarisation relation for these waves and analyse them.

Finite depth

Decay condition $\rightarrow$ condition of non-penetration at the (flat) bottom situated at $z = -H$: $w(-H) = 0$. For wave solutions this means that $P_z(-H) = 0$, cf. (5). Equations for perturbations are the same, solution for $\hat{p}(z)$ with the modified b.c. becomes:

$$\hat{p}(z) = Ae^{kH}\left(e^{k(z-H)} + e^{-k(z-H)}\right), \qquad (11)$$

where A - amplitude to be determined from the continuity of pressure at the surface.. The same procedure as before gives the dispersion relation de:

$$\omega^2 = gk \tanh kH. \qquad (12)$$

Exercise:

Determine the phase and the group velocities of the surface waves.

Kelvin -Helmholtz (KH) instability

The model:

Non-hydrostatic Euler equations for two layers of incompressible fluid with $\rho_i = \text{const}$, $i = 1, 2$ without rotation ($Ro \to \infty$) in the vertical plane x, z.

Equations of motion:

$$
\begin{aligned}
u_t^{(i)} + u^{(i)} u_x^{(i)} + w^{(i)} u_z^{(i)} &= -\frac{1}{\rho_i} P_x^{(i)}, \\
w_t^{(i)} + u^{(i)} w_x^{(i)} + w^{(i)} w_z^{(i)} + g &= -\frac{1}{\rho_i} P_z^{(i)}, \\
u_x^{(i)} + w_z^{(i)} &= 0. \qquad (13)
\end{aligned}
$$

No summation over repeating indices.

Boundary conditions:

- Dynamic b.c.:

$$P^{(1)}\Big|_{z=\eta} = P^{(2)}\Big|_{z=\eta}, \qquad (14)$$

- Kinematic b.c.:

$$\eta_t + u^{(i)}\eta_x = w^{(i)}\Big|_{z=\eta}, \quad i = 1, 2. \qquad (15)$$

where $\eta(x, t)$ - position of the interface between the layers 1 (superior) and 2 (inferior).

Stationary solution: velocity shear across the interface

$$w^{(i)} = 0;\; u^{(i)} = U_i = \text{const};\; \eta = 0;\; P_z^{(i)} = -g\rho_i,\; i = 1, 2. \qquad (16)$$

Linearisation about this solution:

- Equations for perturbations:

$$\begin{aligned} u_t^{(i)} + U_i u_x^{(i)} &= -\frac{1}{\rho_i} p_x^{(i)}, \\ w_t^{(i)} + U_i w_x^{(i)} &= -\frac{1}{\rho_i} p_z^{(i)}, \\ u_x^{(i)} + w_z^{(i)} = 0 \ \Rightarrow \nabla^2 p^{(i)} &= 0. \end{aligned} \tag{17}$$

- Boundary conditions:

$$p^1\Big|_{z=0} - p^2\Big|_{z=0} = g(\rho_1 - \rho_2)\eta. \tag{18}$$

Solution of the Laplace equation:

$$p^{(1)} = \bar{p}_1 e^{-kz} e^{i(kx-\omega t)}, \;\; p^{(2)} = \bar{p}_2 e^{+kz} e^{i(kx-\omega t)} \tag{19}$$

Separation of variables in $w^{(i)}$:

$$w^{(i)} = \bar{w}_i(z) e^{i(kx-\omega t)} \Rightarrow \tag{20}$$

$$\bar{w}_1 = -i \frac{k\bar{p}_1 e^{-kz}}{\rho_1(kU_1-\omega)}, \;\; \bar{w}_2 = i \frac{k\bar{p}_2 e^{kz}}{\rho_2(kU_2-\omega)}. \tag{21}$$

Kinematic b.c.:

$$\eta = \bar{\eta} e^{i(kx-\omega t)} \Rightarrow \; -i(\omega - kU_i)\bar{\eta} = \left.\bar{w}_i\right|_{z=0}, \; \Rightarrow \tag{22}$$

$$\bar{p}_1 = -\frac{\bar{\eta}}{k}\rho_1(\omega - kU_1)^2, \;\; \bar{p}_2 = +\frac{\bar{\eta}}{k}\rho_2(\omega - kU_2)^2 \tag{23}$$

Dynamic b.c.:

$$\rho_2(\omega - kU_2)^2 + \rho_1(\omega - kU_1)^2 = kg(\rho_2 - \rho_1) \equiv kg\Delta\rho, \quad \Delta\rho > 0. \Rightarrow \tag{24}$$

Dispersion relation:

$$(\rho_1 + \rho_2)\omega^2 - 2k(U_1\rho_1 + U_2\rho_2)\omega + \left[k^2(\rho_1 U_1^2 + \rho_2 U_2^2) - kg\Delta\rho\right] = 0. \tag{25}$$

Solution in the moving frame $U_2 = 0$, $U_1 = U$:

$$c = \frac{\omega}{k} = \frac{U\rho_1 \pm \sqrt{(\rho_1 + \rho_2)\frac{g\Delta\rho}{k} - \rho_1\rho_2 U^2}}{\rho_1 + \rho_2} \tag{26}$$

Instability of short waves:

$$k > \frac{g\Delta\rho}{U^2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right). \tag{27}$$

Shear instability in homogeneous fluid

Particular case $\Delta\rho = 0$:

$$c = \frac{\omega}{k} = U\frac{\rho_1 \pm i\sqrt{\rho_1\rho_2}}{\rho_1 + \rho_2} \Rightarrow \tag{28}$$

always unstable

Exercise:

In the limit $U \to 0$ stable waves on the interface result. Compare their properties with those of surface waves.

Example of KH instability