Mathematics/Hydrodynamics/Geophysical Fluid Dynamics Refresher Course

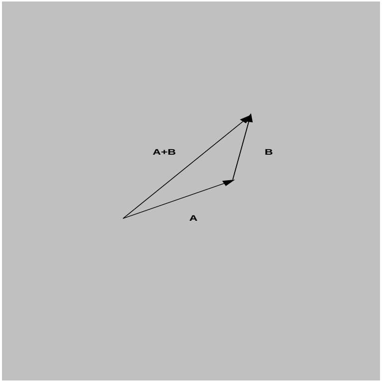
V. Zeitlin

M1 MOCIS

Vectors: definitions and superposition principle

Vector $\boldsymbol{A}$ is a coordinate-independent (invariant) object having a magnitude $|\boldsymbol{A}|$ and a direction. Alternative notation $\vec{A}$.

Adding/subtracting vectors:



Superposition principle: Linear combination of vectors is a vector

Products of vectors

Scalar product of two vectors:

Projection of one vector onto another:

$$\boldsymbol{A} \cdot \boldsymbol{B} := |\boldsymbol{A}|\,|\boldsymbol{B}| \cos \phi_{AB} \equiv \boldsymbol{B} \cdot \boldsymbol{A},$$

where ϕ_{AB} is an included angle between the two.

Vector product of two vectors:

$$\boldsymbol{A} \wedge \boldsymbol{B} := \hat{\boldsymbol{i}}_{AB}\,|\boldsymbol{A}|\,|\boldsymbol{B}| \sin \phi_{AB} = -\boldsymbol{B} \wedge \boldsymbol{A},$$

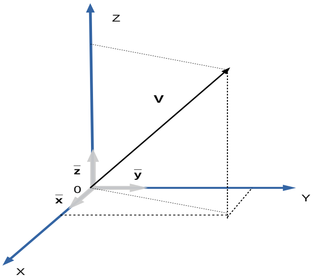
where $\hat{\boldsymbol{i}}_{AB}$ is a unit vector, $|\hat{\boldsymbol{i}}_{AB}| = 1$, perpendicular to both $\boldsymbol{A}$ and $\boldsymbol{B}$, with the orientation of a right-handed screw rotated from $\boldsymbol{A}$ toward $\boldsymbol{B}$.

$\times$ is an alternative notation for $\wedge$.

Distributive properties:

$(\boldsymbol{A}+\boldsymbol{B}) \cdot \boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{C} + \boldsymbol{B} \cdot \boldsymbol{C}$, $(\boldsymbol{A}+\boldsymbol{B}) \wedge \boldsymbol{C} = \boldsymbol{A} \wedge \boldsymbol{C} + \boldsymbol{B} \wedge \boldsymbol{C}$.

Vectors in Cartesian coordinates



Cartesian coordinates: defined by a right triad of mutually orthogonal unit vectors forming a basis:

$$(\hat{\boldsymbol{x}},\ \hat{\boldsymbol{y}},\ \hat{\boldsymbol{z}}) \equiv (\hat{\boldsymbol{x}}_1,\ \hat{\boldsymbol{x}}_2,\ \hat{\boldsymbol{x}}_3),$$

Tensor notation and Kronecker delta

$(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}) \rightarrow \hat{\boldsymbol{x}}_i, \; i = 1, 2, 3$. Ortho-normality of the basis:

$$\hat{\boldsymbol{x}}_i \cdot \hat{\boldsymbol{x}}_j = \delta_{ij},$$

where δ_{ij} is Kronecker delta-symbol, an invariant tensor of second rank (3×3 unit diagonal matrix):

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

The components V_i of a vector $\boldsymbol{V}$ are given by its *projections* on the axes $V_i = \boldsymbol{V} \cdot \hat{\boldsymbol{x}}$:

$$\boldsymbol{V} = V_1 \hat{\boldsymbol{x}}_1 + V_2 \hat{\boldsymbol{x}}_2 + V_3 \hat{\boldsymbol{x}}_3 \equiv \sum_{i=1}^{3} V_i \, \hat{\boldsymbol{x}}_i$$

Einstein's convention:
$\sum_{i=1}^{3} A_i \, B_i \equiv A_i \, B_i$ (self-repeating index is “dumb”).

Vector products by Levi-Civita tensor

Formula for the vector product:

$$\boldsymbol{A} \wedge \boldsymbol{B} = \begin{Vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{Vmatrix}$$

Tensor notation (with Einstein's convention):

$$(\boldsymbol{A} \wedge \boldsymbol{B})_i = \epsilon_{ijk} A_j B_k,$$

where

$$\epsilon_{ijk} = \begin{cases} 1, \text{ if } ijk = 123, 231, 312 \\ -1, \text{ if } ijk = 132, 321, 213 \\ 0, \text{ otherwise} \end{cases}$$

Magic identity:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}. \tag{1}$$

Scalar, vector, and tensor fields

Any point in space is given by its radius-vector $\boldsymbol{x} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$.
A field is an object defined at any point of space $(x, y, z) \equiv (x_1, x_2, x_3)$ at any moment of time t, i.e. a function of $\boldsymbol{x}$ and t.
Different types of fields:

- scalar $f(\boldsymbol{x}, t)$,
- vector $\boldsymbol{v}(\boldsymbol{x}, t)$,
- tensor $t_{ij}(\boldsymbol{x}, t)$

The fields are dependent variables, and x, y, z and t - independent variables.
Physical examples: scalar fields - temperature, density, pressure, geopotential, vector fields - velocity, electric and magnetic fields, tensor fields - stresses, gravitational field.

Differential operations on scalar fields

Partial derivatives:

$$\frac{\partial f}{\partial x} := \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x},$$

and similar for other independent variables. Differential operator nabla:

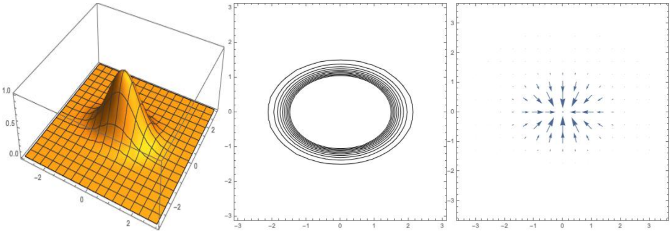
$$\boldsymbol{\nabla} := \hat{\boldsymbol{x}} \frac{\partial}{\partial x} + \hat{\boldsymbol{y}} \frac{\partial}{\partial y} + \hat{\boldsymbol{z}} \frac{\partial}{\partial z}$$

Gradient of a scalar field: the vector field

$$\text{grad}\, f \equiv \boldsymbol{\nabla} f = \hat{\boldsymbol{x}} \frac{\partial f}{\partial x} + \hat{\boldsymbol{y}} \frac{\partial f}{\partial y} + \hat{\boldsymbol{z}} \frac{\partial f}{\partial z}$$

Heuristic meaning: a vector giving direction and rate of fastest increase of the function f.

Visualizing gradient in 2D



From left to right: 2D relief, its contour map, and its gradient. Graphics by Mathematica©

Differential operations with vectors

- Scalar product: divergence

$$\mathrm{div}\,\boldsymbol{v} \equiv \boldsymbol{\nabla} \cdot \boldsymbol{v}(\boldsymbol{x}) = \frac{\partial v_i}{\partial x_i}$$

- Vector product: curl

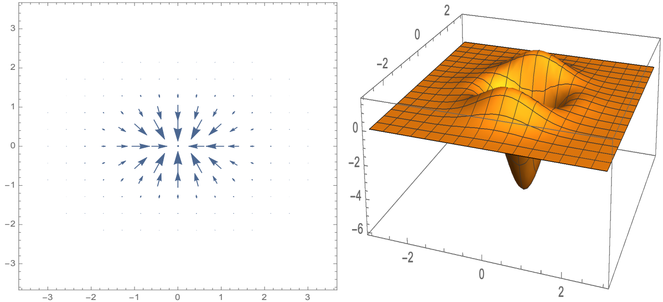
$$\mathrm{curl}\,\boldsymbol{v} \equiv \boldsymbol{\nabla} \wedge \boldsymbol{v}(\boldsymbol{x}); \quad (\mathrm{curl}\,\boldsymbol{v})_i = \epsilon_{ijk} \frac{\partial v_k}{\partial x_j}$$

- Tensor product:

$$\boldsymbol{\nabla} \otimes \boldsymbol{v}(\boldsymbol{x}); \quad (\boldsymbol{\nabla} \otimes \boldsymbol{v})_{ij} = \frac{\partial v_i}{\partial x_j}$$

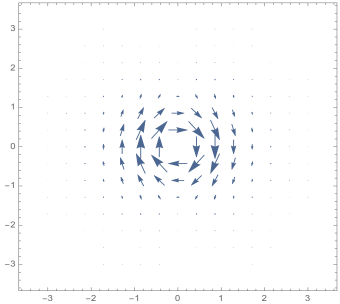
For any $\boldsymbol{v}$, f : $\mathrm{div}\,\mathrm{curl}\,\boldsymbol{v} \equiv 0$, $\mathrm{curl}\,\mathrm{grad}\, f \equiv 0$,
$\mathrm{div}\,\mathrm{grad}\, f = \boldsymbol{\nabla}^2 f$, $\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ - Laplacian.

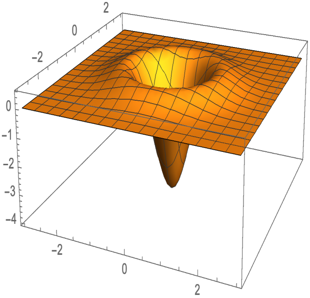
Visualizing divergence in 2D



From left to right: vector field $\boldsymbol{v}(x, y) = (v_1(x, y), v_2(x, y)$, and its divergence $\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}$. The curl $\hat{\boldsymbol{z}}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$ of this field is identically zero. (The field is a gradient of the previous example.) Graphics by Mathematica©

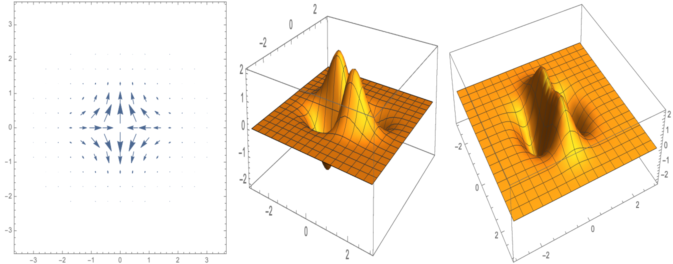
Visualizing curl in 2D





From left to right: vector field $\boldsymbol{v}(x, y) = (v_1(x, y), v_2(x, y)$, and its curl $\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$. The divergence $\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}$ of this field is identically zero, so the field is a curl of another vector field. Graphics by Mathematica©

Strain field with non-zero curl and divergence



From left to right: vector field, and its curl and divergence.

Graphics by Mathematica©

Useful identities

$$\nabla \wedge (\nabla \wedge \boldsymbol{v}) = \nabla(\nabla \cdot \boldsymbol{v}) - \nabla^2 \boldsymbol{v}, \tag{2}$$

$$\boldsymbol{v} \wedge (\nabla \wedge \boldsymbol{v}) = \nabla \left(\frac{\boldsymbol{v}^2}{2}\right) - (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v}, \tag{3}$$

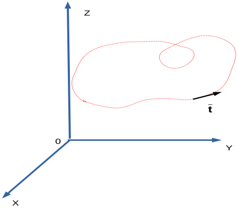
$$\nabla f \cdot (\nabla \wedge \boldsymbol{v}) = -\nabla \cdot (\nabla f \wedge \boldsymbol{v}). \tag{4}$$

Proofs: using tensor representation $(\nabla \wedge \boldsymbol{v})_i = \epsilon_{ijk}\partial_j v_k$, with shorthand notation $\frac{\partial}{\partial x_i} \equiv \partial_i$, exploiting the antisymmetry of ϵ_{ijk}, using that $\delta_{ij} v_j = v_i$, and applying the magic formula (1).

Example: proof of (2).

$$\epsilon_{ijk}\partial_j \epsilon_{klm}\partial_l v_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l v_m = \partial_i\partial_j v_j - \partial_j\partial_j v_i.$$

Integration of a field along a (closed) 1D contour

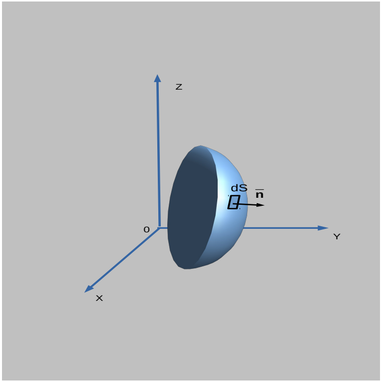


Summation of the values of the field at the points of the contour times oriented line element $d\boldsymbol{l} = \hat{\boldsymbol{t}}\, dl$:

$$\oint d\boldsymbol{l}\,(\ldots),$$

where $\hat{\boldsymbol{t}}$ is unit tangent vector, and dl is a length element along the contour. Positive orientation: anti-clockwise.

Integration of a field over a 2D surface

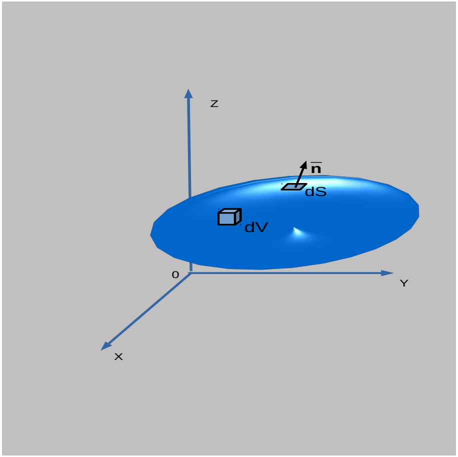


Summation of the values of the field at the points of the surface times oriented surface element $d\boldsymbol{s} = \hat{\boldsymbol{n}}\, ds$:

$$\int\int d\boldsymbol{s}\,(...) \equiv \int_S d\boldsymbol{s}\,(...),$$

where $\hat{\boldsymbol{n}}$ is unit normal vector. Positive orientation for closed surfaces: outwards.

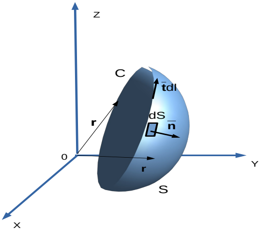
Integration of a field over a 3D volume



Summation of the values of the field at the points in the volume times volume element dV.

$$\int\int\int dV\,(\ldots) \equiv \int_V dV\,(\ldots).$$

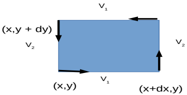
Linking contour and surface integrations: Stokes theorem



$$\oint_C d\boldsymbol{l} \cdot \boldsymbol{v}(\boldsymbol{x}) = \int_{S_C} d\boldsymbol{s} \cdot (\nabla \wedge \boldsymbol{v}(\boldsymbol{x})). \quad (5)$$

Left-hand side: circulation of the vector field over the contour C. Right-hand side: curl of $\boldsymbol{v}$ integrated over any surface S_C having the contour C as a base.

Stokes theorem: the idea of proof



Circulation of the vector $\boldsymbol{v} = v_1 \hat{\boldsymbol{x}} + v_2 \hat{\boldsymbol{y}}$ over an elementary contour, with $dx \to 0$, $dy \to 0$, using first-order Taylor expansions:

$$v_1(x, y)dx + v_2(x + dx, y)dy - v_1(x, y + dy)dx - v_2(x, y)dy = \frac{\partial v_2}{\partial x} dx\, dy - \frac{\partial v_1}{\partial y} dx\, dy,$$

with a z-component of curl$\boldsymbol{v}$ multiplied by the z-oriented surface element arising in the right-hand side.

Linking surface and volume integrations: Gauss theorem

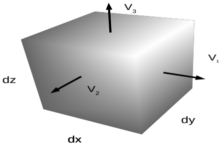
$$\oint_{S_V} d\boldsymbol{s} \cdot \boldsymbol{v}(\boldsymbol{x}) = \int_V dV\, \boldsymbol{\nabla} \cdot \boldsymbol{v}(\boldsymbol{x}). \qquad (6)$$

Left-hand side: flux of the vector field through the surface S_V which is a boundary of the volume V. Right-hand side: volume integral of the divergence of the field.

Important. The theorem is also valid for the scalar field:

$$\oint_{S_V} d\boldsymbol{s} \cdot f(\boldsymbol{x}) = \int_V dV\, \boldsymbol{\nabla} f(\boldsymbol{x}). \qquad (7)$$

Gauss theorem: the idea of proof



Flux of the vector $\boldsymbol{v} = v_1\hat{\boldsymbol{x}} + v_2\hat{\boldsymbol{y}} + v_3\hat{\boldsymbol{z}}$ over a surface of an elementary volume, taking into account the opposite orientation of the oriented surface elements:

$$[v_1(x+dx,y,z) - v_1(x,y,z)]\,dydz+$$
$$[v_2(x,y+dy,z) - v_2(x,y,z)]\,dxdz+$$
$$[v_3(x,y,z+dz) - v_3(x,y,z)]\,dxdy = \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) dx\,dy\,dz$$

Fourier series for periodic functions

Consider $f(x) = f(x + 2\pi)$, a periodic smooth function on the interval $[0, 2\pi]$. Fourier series:

$$f(x) = \sum_{n=0}^{\infty} \left[a_n \cos(n\,x) + b_n \sin(n\,x)\right].$$

The expansion is unique du to ortogonality of the basis functions:

$$\int_0^{2\pi} dx\, \cos(n\,x)\cos(m\,x) = \int_0^{2\pi} dx\, \sin(n\,x)\sin(m\,x) = \pi\delta_{nm},$$

$$\int_0^{2\pi} dx\, \sin(n\,x)\cos(m\,x) \equiv 0.$$

The coefficients of expansion, thus, are uniquely defined:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} dx\, f(x)\cos(n\,x), \quad b_n = \frac{1}{\pi}\int_0^{2\pi} dx\, f(x)\sin(n\,x)$$

Complex exponential form

$$e^{inx} = \cos(n\,x) + i\sin(n\,x) \Rightarrow$$

$$\cos(n\,x) = \frac{e^{inx} + e^{-inx}}{2},\ \sin(n\,x) = \frac{e^{inx} - e^{-inx}}{2i}$$

Hence

$$f(x) = \sum_{n=0}^{\infty} \frac{(a_n - ib_n)}{2} e^{inx} + c.c \equiv \sum_{-\infty}^{\infty} A_n e^{inx},\ A_n^* = A_{-n}$$

Orthogonality:

$$\int_0^{2\pi} dx\, e^{inx} e^{-imx} = 2\pi\delta_{nm}$$

Expression for coefficients

$$A_n = \frac{1}{2\pi}\int_0^{2\pi} dx\, f(x)\, e^{-inx}$$

Fourier integral

Fourier series on arbitrary interval L: $\sin(nx)$, $\cos(nx) \to \sin(\frac{2\pi}{L}nx)$, $\cos(\frac{2\pi}{L}nx)$, $\int_0^{2\pi} dx \to \int_0^L dx$, normalization $\frac{1}{\pi} \to \frac{1}{L}$. In the limit $L \to \infty$: $\sum_{-\infty}^{\infty} \to \int_{-\infty}^{\infty}$.

Fourier-transformation and its inverse:

$$f(x) = \int_{-\infty}^{\infty} dk\, F(k)\, e^{ikx}, \quad F(k) = \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx}.$$

Based on orthogonality:

$$\int_{-\infty}^{\infty} dx\, e^{ikx} e^{-ilx} = \delta(k-l),$$

where $\delta(x)$ - Dirac's delta-function, continuous analog of Kronecker's δ_{nm}, with properties:

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1, \quad \int_{-\infty}^{\infty} dy\, \delta(x-y)\, F(y) = F(x).$$

Multiple variables and differentiation

$$f(x,y,z) = \int_{-\infty}^{\infty} dk\, dl\, dm\, F(k,l,m)\, e^{i(kx+ly+mz)},$$

$$F(k,l,m) = \int_{-\infty}^{\infty} dx\, dy\, dz\, f(x,y,z)\, e^{-i(kx+ly+mz)}.$$

Physical space $(x,y,z) \longrightarrow (k,l,m)$, Fourier space.
Radius-vector $\boldsymbol{x} \to \boldsymbol{k}$, "wavevector",

$$f(\boldsymbol{x}) = \int_{-\infty}^{\infty} d\boldsymbol{k}\, F(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

Main advantage: differentiation in physical space $\to$ multiplication by the corresponding component of the wavevector in Fourier space $\frac{\partial}{\partial x} \to ik$:

$$\frac{\partial}{\partial x} f(\boldsymbol{x}) = \int_{-\infty}^{\infty} d\boldsymbol{k}\, ik\, F(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}},$$

and similarly for other variables.

Simplest wave equation

$$u_t + cu_x = 0. \qquad (8)$$

$u(x, t)$ - dynamical variable, defined for all x:
$-\infty < x < +\infty$, and t: $0 \le t < \infty$, $c = \text{const}$.
Notation: $(...)_x = \frac{\partial(...)}{\partial x}$, $(...)_t = \frac{\partial(...)}{\partial t}$
Methode of solution 1: change of variables.

$$(x,\, t) \to (\xi_+,\, \xi_-) = (x + ct,\, x - ct)\,. \qquad (9)$$

$$\frac{\partial \xi_\pm}{\partial x} = 1, \quad \frac{\partial \xi_\pm}{\partial t} = \pm c \;\Rightarrow \qquad (10)$$

$$\frac{\partial u}{\partial t} = c\left(\frac{\partial u}{\partial \xi_+} - \frac{\partial u}{\partial \xi_-}\right), \qquad (11)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi_+} + \frac{\partial u}{\partial \xi_-} \qquad (12)$$

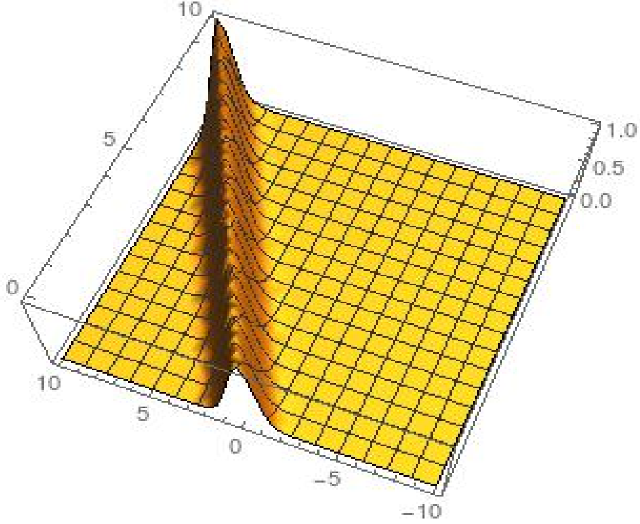
Simplification of the equation

$$u_t + cu_x = 0 \rightarrow 2c\frac{\partial u}{\partial \xi_+} = 0 \;\Rightarrow u = u(\xi_-). \qquad (13)$$

Function u determined from initial conditions:

$$\text{c.l.} : u_{t=0} = u_0(x) \Rightarrow u = u_0(x - ct). \qquad (14)$$

Spatio-temporal evolution of a localized initial perturbation, as follows from (8)



Fourier transform

Methode of solution 2: Fourier- transformation

$$u(x,t) = \frac{1}{2\pi} \int dk\, d\omega\, e^{i(kx-\omega t)} \hat{u}k,\omega + c.c.. \qquad (15)$$

Inverse transformation:

$$\hat{u}(k,\omega) = \frac{1}{2\pi} \int dx\, dt\, e^{-i(kx-\omega t)} u(x,t) + c.c.. \qquad (16)$$

Transformation $\times$ Inverse ransformation $= \mathbf{1}$, as

$$\int_{-\infty}^{\infty} dk\, e^{ik(x-x')} = \delta(x-x'),\ \int_{-\infty}^{\infty} d\omega\, e^{i\omega(t-t')} = \delta(t-t'), \qquad (17)$$

δ - Dirac's delta.
Fourier-modes: $\hat{u}(k,\omega)e^{i(kx-\omega t)} \leftrightarrow$ monochromatic waves.
Amplitude: $|\hat{u}|$; Phase: $\Phi = kx - \omega t + \Phi_0$, $\hat{u} = |\hat{u}|e^{i\Phi}$.

Superposition principle

Method of Fourier ⇔ superposition principle, valid for linear systems.

$$u_t + cu_x = 0 \Rightarrow i\,(kc - \omega)\,\hat{u}(k,\omega),\ \hat{u}(k,\omega) \neq 0 \Rightarrow \qquad (18)$$

$$\omega = c\,k, \text{dispersion relation.} \qquad (19)$$

General solution:

$$u(x,t) = \frac{1}{2\pi} \int dk\ e^{ik(x-ct)} \hat{u}(k) + c.c. \rightarrow \qquad (20)$$

superposition (sum or integral) of elementary Fourier-modes.

Phase velocity

Speed of propagation of the phase of a monochromatic wave: phase velocity:

$$c_{ph} = \frac{\omega}{k}. \quad (21)$$

Dispersion: dependence $c = c(k) \Rightarrow$ simple wave is non-dispersive: $c_{ph} = c = \text{const}$.

Groupe velocity:

$$c_g = \frac{\partial \omega}{\partial k} \quad (22)$$

- speed of propagation of modulations = speed of propagation of information.

Simple wave: $c_{ph} = c_g$ (like acoustic or electromagnetic waves).

Second-order wave equation

$$u_{tt} - c^2 u_{xx} = 0. \tag{23}$$

Same change of independent variables as in the 1st-order equation:

$$(x,\, t) \to (\xi_+,\, \xi_-) = (x + ct,\, x - ct)$$

.

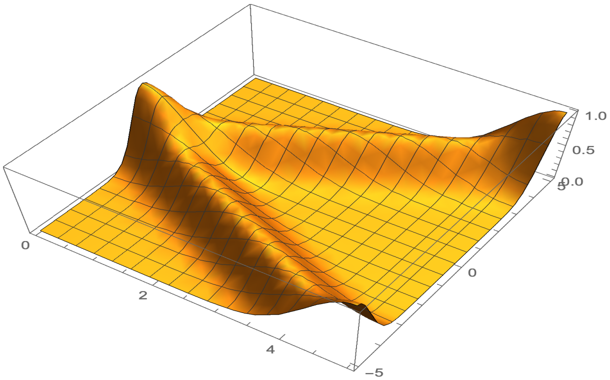
$$u_{tt} - c^2 u_{xx} = 0 \to 4c^2 \frac{\partial^2 u}{\partial \xi_+ \partial \xi_-} = 0 \;\Rightarrow \tag{24}$$

General solution:

$$u = u_-(\xi_-) + u_+(\xi_+), \tag{25}$$

where $u_- + u_+$ - arbitrary functions, to be determined from initial conditions. (2nd order $\Rightarrow$ 2 initial conditions required.)

Spatio-temporal evolution of the initial localized perturbation



Solution in the domain $-5 < x < 5$, $0 < t < 5$. Initial Gaussian perturbation propagates along a pair of characteristic lines with slopes $\pm c$. Graphics by Mathematica©

Introducing the simplest dispersion

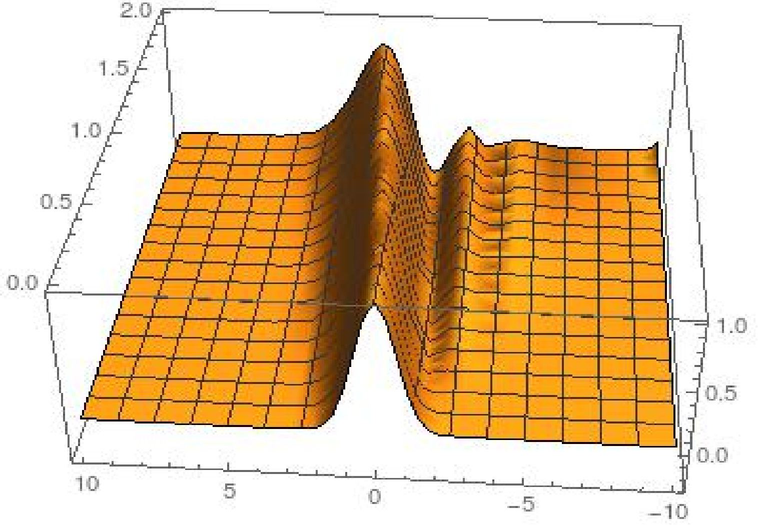
Dispersion - more derivatives.
In the case of unidirectional propagation - only odd-order derivatives to respect the symmetry of the initial equation with respect to reflexions. Simplest case:adding 3rd space derivative:

$$u_t + cu_x = 0 \to u_t + cu_x + \alpha u_{xxx} = 0 \quad \alpha = \text{const} \qquad (26)$$

Corresponds to waves in shallow channels.
Dispersion relation:

$$\omega = ck - \alpha k^3$$

Spatio-temporal evolution of a localized initial perturbation, as follows from (26)



Non-linearity

$$u_t + cu_x = 0 \to u_t + uu_x + cu_x = 0 \Rightarrow \qquad (27)$$

no more superposition principle. Produces steepening and wave breaking .

Qualitative explanation : $c \to c + u \Rightarrow$ the larger the amplitude the larger the speed: a maximum moves faster than surrounding and "catches up" with the preceding part.

Korteweg - deVries equation: mutual compensation of dispersion and nonlinearity

Dispersion + non-linearity:

$$u_t + cu_x = 0 \to u_t + uu_x + cu_x + \alpha u_{xxx} = 0 \qquad (28)$$

Produces steady solitary waves.

Equations of motion

Eulerian description: in terms of fluid velocity field $\mathbf{v}(\mathbf{x}, t)$, and scalar density and pressure fields $\rho(\mathbf{x}, t)$, $P(\mathbf{x}, t)$, defined at each point $\mathbf{x}$ of the volume occupied by the fluid at any time t.

Euler equations

Local conservation of momentum in the presence of forcing $\mathbf{F}$:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}\right) = -\boldsymbol{\nabla}P + \mathbf{F}, \tag{29}$$

Continuity equation

Local conservation of mass:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0. \tag{30}$$

Equation of state: baroclinic fluid

Fluid: thermodynamical system $\Rightarrow$ equation of state relating P and ρ and closing the system (29), (30) (4 equations for 5 dependent variables).
General equation of state:

$$P = P(\rho, s), \tag{31}$$

$s(\mathbf{x}, t)$ is entropy per unit mass $\Rightarrow$ evolution equation for s required. Perfect fluid:

$$\frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \nabla s = 0. \tag{32}$$

Equation of state: barotropic fluid

$$P = P(\rho) \leftrightarrow s = \text{const}, \tag{33}$$

sufficient to close the system (29), (30).
Particular case: incompressible fluid. Conservation of volume per unit mass $\Rightarrow$ zero divergence:

$$\nabla \cdot \boldsymbol{v} = 0, \ \Rightarrow \tag{34}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot \nabla \rho = 0, \ \text{and} \ \nabla \cdot (\boldsymbol{v} \cdot \nabla \boldsymbol{v}) = -\nabla \cdot \left(\frac{\nabla P}{\rho} \right) \Rightarrow \tag{35}$$

Pressure entirely determined by density and velocity distributions.

Lagrangian view of the fluid: momentum balance

Fluid $\equiv$ ensemble of fluid parcels with time-dependent positions $\mathbf{X}(\mathbf{x}_0, t)$, $\mathbf{X}(\mathbf{x}_0, 0) = \mathbf{x}$.
Euler - Lagrange duality: continuity of the fluid $\Rightarrow$ any point in the flow $\mathbf{x}$ is, at the same time, a position of some fluid parcel $\Rightarrow$ Eulerian velocity at the point $\mathbf{v}(\mathbf{x}) =$ velocity of the parcel $\mathbf{v}(\mathbf{X}, t) = \frac{d\mathbf{X}}{dt} \equiv \dot{\mathbf{X}}$. Lagrangian (material) derivative in Eulerian terms by chain differentiation:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial \boldsymbol{x}}{\partial t} \cdot \boldsymbol{\nabla} \equiv \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla. \quad (36)$$

$\Rightarrow$ Newton's second law for the parcel

$$\rho(\mathbf{X}, t) \frac{d^2\mathbf{X}}{dt^2} = -\boldsymbol{\nabla}_{\mathbf{X}} P(\mathbf{X}, t) + \mathbf{F}, \quad (37)$$

$\Leftrightarrow$ Euler equation (29).

Lagrangian view of the fluid: mass balance

Mass conservation in Lagrangian terms:

$$\rho_i(\mathbf{x}) d^3\mathbf{x} = \rho(\mathbf{X}, t) d^3\mathbf{X}, \leftrightarrow \rho_i(\mathbf{x}) = \rho(\mathbf{X}, t)\mathcal{J} \qquad (38)$$

where ρ_i is the initial distribution of density, and $d^3\mathbf{x}$ and $d^3\mathbf{X}$ are initial and current elementary volumes. The Jacobi determinant (Jacobian) in this formula is defined as the determinant:

$$\mathcal{J} = \begin{vmatrix} \frac{\partial X}{\partial x} & \frac{\partial X}{\partial y} & \frac{\partial X}{\partial z} \\ \frac{\partial Y}{\partial x} & \frac{\partial Y}{\partial y} & \frac{\partial Y}{\partial z} \\ \frac{\partial Z}{\partial x} & \frac{\partial Z}{\partial y} & \frac{\partial Z}{\partial z} \end{vmatrix} = \frac{\partial(X, Y, Z)}{\partial(x, y, z)}$$

Incompressibility in Lagrangian terms: $\mathcal{J} = 1$. Taking Lagrangian time-derivative of this relation, we obtain the incompressibility condition of zero velocity divergence in Eulerian terms. Advection of entropy (32) $\Leftrightarrow$ conservation of entropy by each fluid parcel $\dot{s} = 0$.

1st principle of thermodynamics

Reversible processes in one-phase systems:

$$\delta\epsilon = T\delta s - P\delta v, \tag{39}$$

ϵ - internal energy per unit mass, $v = \frac{1}{\rho}$ - specific volume.Enthalpy per unit mass: $h = \epsilon + Pv \Rightarrow$

$$\delta h = T\delta s + v\delta P. \tag{40}$$

Energy density: sum of kinetic and internal parts:

$$e = \frac{\rho \mathbf{v}^2}{2} + \rho\epsilon. \tag{41}$$

Local conservation of energy :

$$\frac{\partial e}{\partial t} + \nabla \cdot \left[\rho \mathbf{v} \left(\frac{\mathbf{v}^2}{2} + h\right)\right] = 0. \tag{42}$$

Barotropic fluid:

$$\delta h = \frac{\delta P}{\rho} \quad \Rightarrow \frac{\nabla P}{\rho} = \nabla h. \tag{43}$$

Kelvin theorem

Circulation of velocity around a contour Γ consisting of fluid parcels, and moving with the fluid:

$$\gamma = \int_{\Gamma} \mathbf{v} \cdot d\mathbf{l} = \int_{S_\Gamma} (\nabla \wedge \mathbf{v}) \cdot d\mathbf{l}, \quad (44)$$

Kelvin theorem states that

- for barotropic fluids

$$\frac{d\gamma}{dt} = 0, \quad (45)$$

- for baroclinic fluids

$$\frac{d\gamma}{dt} = -\int_{\Gamma} \frac{\nabla P}{\rho} \cdot d\mathbf{l}. \quad (46)$$

Proof: direct calculation of the time-derivative of the circulation using the equations of motion, and the Lagrangian nature of Γ.

Perfect vs real fluids

Perfect fluid approximation: macroscopic fluxes of mass, momentum and energy. Real fluids: corrections to these fluxes due to molecular transport. Simplest way to include them: flux-gradient relations following from Le Chatelier principle: molecular fluxes tend to restore the thermodynamical equilibrium. For any thermodynamical variable A

$$\mathbf{f}_A = -k_A \nabla A,$$

where $\mathbf{f}_A$ is related molecular flux, and k_A is molecular transport coefficient.

Viscosity, diffusivity, and thermal conductivity

- Viscosity corrections to the Euler equation in the incompressible case, giving the Navier - Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\frac{\nabla P}{\rho} + \nu \nabla^2 \boldsymbol{v}, \ \nabla \cdot \boldsymbol{v} = 0. \tag{47}$$

- Diffusivity corrections to the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = D \nabla^2 \rho. \tag{48}$$

- Thermal conductivity corrections to the heat/temperature advection giving the heat equation

$$\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T = \chi \nabla^2 T. \tag{49}$$

ν, D, χ are kinematic viscosity, diffusivity, and thermo-conductivity, the molecular transport coefficients for momentum, mass, and energy, respectively, all with dimension $\left[\frac{L^2}{T}\right]$

Dimensional/scale analysis. Reynolds number

Molecular transport coefficients: dimensional, value varies with changes if units. Only *non-dimensional parameters* are relevant. Typical space and velocity scales in the incompressible fluid flow: L, U. Time-scale $T = L/U$. Pressure scale: ρU^2.
Scaled NS equation:

$$\frac{U^2}{L}\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla P\right) = \frac{U\,\nu}{L^2}\nabla^2\mathbf{v} \rightarrow \qquad (50)$$

Non-dimensional NS equation

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla P + \frac{1}{Re}\nabla^2\mathbf{v} \qquad (51)$$

$Re = \frac{U\,L}{\nu}$ - Reynolds number, the true measure of viscosity. Similar, Pecklet number for diffusivity.

Motion in a rotating frame

Material point in a frame rotating with angular velocity $\boldsymbol{\Omega}$:

$$m\frac{d\boldsymbol{v}}{dt} + 2m\boldsymbol{\Omega} \wedge \boldsymbol{v} + m\boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \boldsymbol{x}) = \boldsymbol{F}, \quad \boldsymbol{v} = \frac{d\boldsymbol{x}}{dt} \qquad (52)$$

m- mass, $\boldsymbol{x}$-current position of the point, $\boldsymbol{F}$ - sum of forces acting on the point

Euler equations in the rotating frame +gravity:

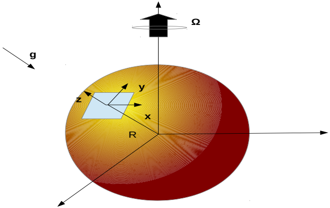
Fluid under the influence of gravity: $m \to \rho$, $\frac{d}{dt} \to \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla$, forces: pressure + gravity $\Rightarrow$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + 2\Omega \wedge \boldsymbol{v} = -\frac{\nabla P}{\rho} + \boldsymbol{g}^* \qquad (53)$$

Effective gravity: gravity + centrifugal acceleration (also potential)

$$\boldsymbol{g}^* = \boldsymbol{g} + \boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \boldsymbol{x}) \qquad (54)$$

Tangent plane approximation



$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} + f \hat{z} \wedge \boldsymbol{v} = -\frac{\boldsymbol{\nabla} P}{\rho} + \boldsymbol{g} \qquad (55)$$

f - plane: $f = const$; β - plane: $f = f + \beta y$; f - Coriolis parameter: $f = 2\Omega \sin \phi$, where ϕ - latitude

Hydrostatics. Stratification

The state of rest $\boldsymbol{v} \equiv 0$ is solution of (55) if hydrostatic equilibrium holds:

$$0 = -\frac{\nabla P}{\rho} + \boldsymbol{g}$$

The continuity equation:

$$\frac{d\rho}{dt} + \rho \nabla \cdot \boldsymbol{v} = 0$$

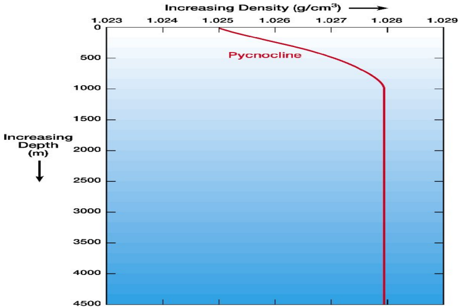
is satisfied by time-independent ρ in a state of rest. Statically stable states: $\rho = \rho_0(z)$, $\rho_0'(z) \leq 0 \rightarrow$

$$P = P_0(z) = -\int dz\, g\, \rho_0(z)$$

Dependence of ρ_0 on z is called stratification. Surfaces of constant ρ: isopycnals.

Oceanic stratification

Typical density profile:



$$\rho(\vec{x}, t) = \rho_0 + \rho_s(z) + \sigma(x, y, z; t), \quad \rho_0 \gg \rho_s \gg \sigma. \qquad (56)$$

Hydrostatic approximation for large-scale motions:

$$g\rho + \partial_z P = 0, \ \Rightarrow P = P_0 + P_s(z) + \pi(x, y, z; t), \qquad (57)$$

Further approximations.

Boussinesq approximation

Deviations of density from ρ_0 neglected in the horizontal $\rightarrow$

$$\frac{\partial \boldsymbol{v}_h}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}_h + f \hat{z} \wedge \boldsymbol{v}_h = -\frac{\boldsymbol{\nabla}_h \pi}{\rho} \approx -\boldsymbol{\nabla}_h \phi, \qquad (58)$$

where $\phi = \frac{\pi}{\rho_0}$ - geopotential.

Incompressibility of water

Continuity equation splits in two:

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0, \quad \boldsymbol{v} = \boldsymbol{v}_h + \hat{\boldsymbol{z}} w. \qquad (59)$$

$$\partial_t \rho + \boldsymbol{v} \cdot \boldsymbol{\nabla} \rho = 0. \qquad (60)$$

Full set of oceanic PE

$$\frac{\partial \boldsymbol{v}_h}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}_h + f\hat{z} \wedge \boldsymbol{v}_h = -\frac{\boldsymbol{\nabla}_h \pi}{\rho} \equiv -\boldsymbol{\nabla}_h \phi, \tag{61}$$

$$\partial_t \sigma + \boldsymbol{v} \cdot \boldsymbol{\nabla} \sigma + w \rho_s'(z) = 0. \tag{62}$$

$$g \frac{\sigma}{\rho_0} = -\partial_z \phi, \quad \boldsymbol{\nabla}_h \cdot \boldsymbol{v}_h + \partial_z w = 0, \tag{63}$$

Remark

Hydrostatic approximation $\leftrightarrow$ anisotropic scaling proper for mesoscale motions:

$$W \ll U, \quad H \ll L, \quad \frac{W}{H} \sim \frac{U}{L}$$

where L, H and U, W are horizontal and vertical spatial and velocity scales, respectively.

Vertical boundary conditions

Most often sufficient for our purposes: rigid lid and flat bottom:

$$w|_{z=0} = w|_{z=H} = 0 \tag{64}$$

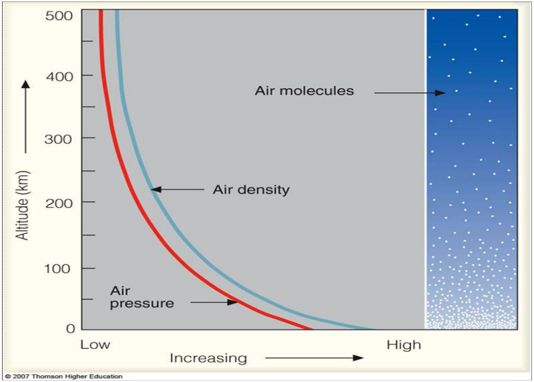
Non-trivial bathymetry : fluid parcels follow the bottom profile

$$w|_{z=b(x,y)} = \frac{db}{dt} = \mathbf{v} \cdot \nabla b$$

Free surface: fluid parcels move with the surface:

$$w|_{z=h(x,y;t)} = \frac{dh}{dt} = \frac{\partial h}{\partial t} + \mathbf{v} \cdot \nabla h$$

Atmosphere: pressure coordinates



Altitude $\leftrightarrow$ Pressure $\Rightarrow$ vertical coordinate.

Thermodynamics of the dry atmosphere

Equation of state - ideal gas:

$$P = \rho RT, \quad c_{P,V} = T\left(\frac{\partial s}{\partial T}\right)_{P,V} = const, \quad c_p - c_v = R. \tag{65}$$

Entropy:

$$s = c_p \ln T - R \ln P + const. \tag{66}$$

Adiabatic process:

$$s = \text{const} \Rightarrow c_p \frac{dT}{T} - R\frac{dP}{P} = 0, \Rightarrow T = T_s \left(\frac{P}{P_s}\right)^{\frac{R}{c_p}}. \tag{67}$$

Potential temperature :

$$\theta = T\left(\frac{P_s}{P}\right)^{\frac{R}{c_p}}, \; s = c_p \ln \theta + \text{const}. \tag{68}$$

Geopotential and hydrostatics

Geopotential variation: work to lift a unit mass against gravity: $\delta\phi = g\delta z$.
$z = z(p)$ becomes a thermodynamical variable.
Hydrostatic approximation:

$$\delta\phi = -\frac{RT}{P}\delta P \Rightarrow \tag{69}$$

$$\frac{\partial\phi}{\partial P} = -\frac{RT}{P} = -\frac{1}{\rho}. \tag{70}$$

Useful relation for small variations ρ, P, θ with respect to background ρ_0, P_0, θ_0:

$$\theta = \theta_0 \left[\frac{\left(1 - \frac{R}{c_p}\right) P}{P_0} - \frac{\rho}{\rho_0} \right] \tag{71}$$

Elimination of ρ in Euler equations

"Triangular" relation :

$$\left(\frac{\partial P}{\partial x}\right)_z \left(\frac{\partial x}{\partial z}\right)_P \left(\frac{\partial z}{\partial P}\right)_x = -1 \ \Rightarrow \tag{72}$$

$$\left(\frac{\partial P}{\partial x}\right)_z = -\left(\frac{\partial P}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_P = \rho \left(\frac{\partial \phi}{\partial x}\right)_P . \tag{73}$$

Incompressibility in pressure coordinates

Lagrangian volume element in pressure coordinates:

$$\rho dxdydz = -\frac{1}{g} dxdydP \tag{74}$$

Mass conservation $\Rightarrow$ Volume conservation in P.

Adiabatic primitive equations

Equations of motion

$$div(\boldsymbol{v}) = \boldsymbol{\nabla}_h \cdot \boldsymbol{v}_h + \partial_p \omega = 0, \quad \omega = \frac{dP}{dt}. \qquad (75)$$

$$\frac{\partial \boldsymbol{v}_h}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}_h + f \hat{\boldsymbol{z}} \wedge \boldsymbol{v}_h = -\boldsymbol{\nabla}_h \phi. \qquad (76)$$

$$\partial_t \theta + \boldsymbol{v} \cdot \boldsymbol{\nabla} \theta = 0. \qquad (77)$$

$$\frac{\partial \phi}{\partial P} = -\frac{RT}{P} = -\frac{R}{P} \left(\frac{P}{P_s} \right)^{\frac{R}{c_p}} \theta. \qquad (78)$$

Boundary conditions

Bottom: ground $\equiv$ free surface in terms of pressure, geopotential fixed.
Top: rigid lid $\equiv$ fixed value of pressure, e.g. tropopause.

Boussinesq approximation for atmosphere

Varying background density in atmosphere: $\rho_0 = \rho_0(z)$.
Boussinesq approximation in x, y, z coordinates, with $\rho = \rho_0(z) + \tilde{\rho}$, $P = P_0(z) + \tilde{p}$, $\theta = \theta_0(z) + \tilde{\theta}$, $(\tilde{...})$ omitted below:

$$\frac{\partial \boldsymbol{v}_h}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}_h + f \hat{\boldsymbol{z}} \wedge \boldsymbol{v}_h = -\boldsymbol{\nabla}_h \phi, \tag{79}$$

with geopotential $\phi = \frac{p}{\rho_0}$. Hydrostatics:

$$-\frac{\partial \phi}{\partial z} - \frac{p}{\rho_0^2}\frac{\partial \rho_0}{\partial z} - g\frac{\rho}{\rho_0} = 0. \tag{80}$$

Equation of state (ideal gas) + (71) $\rightarrow$

$$-\frac{\partial \phi}{\partial z} + b = 0, \tag{81}$$

$b = g\frac{\theta}{\theta_0}$ - buoyancy, $\frac{\partial b}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} b = 0$ for adiabatic motions.
Continuity equation $\rightarrow$ anelastic equation:

$$\boldsymbol{\nabla} \cdot (\rho_0(z)\boldsymbol{v}) = 0.$$