1. Modeling large-scale atmospheric and oceanic motions: Primitive equations in Geophysical Fluid Dynamics

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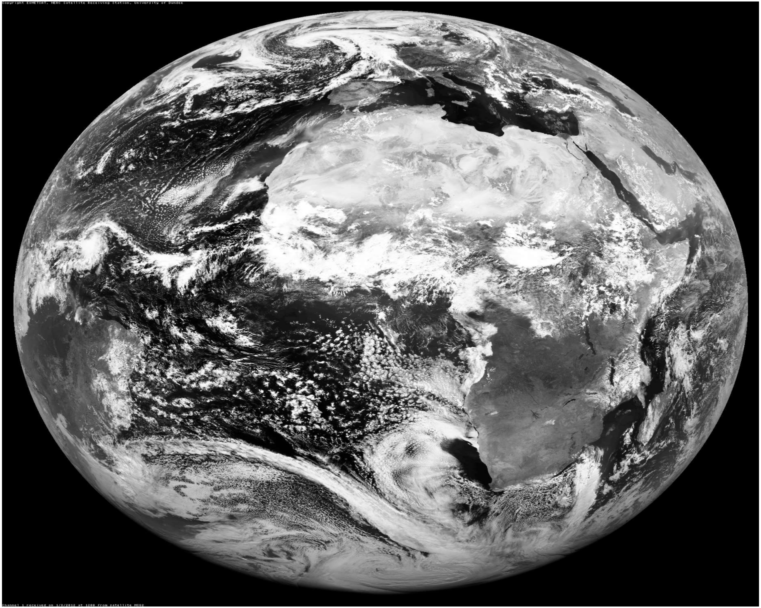
Mathematics of the atmosphere and oceans,
SUSTECH, 2023

Plan

Geophysical Fluid Dynamics : space view



GFD : what's that ?

Hydrodynamics in all its complexity plus :

- Rotating frame
- Variable temperature/density effects
- Spherical geometry (large- and meso-scales)
- Fluid in the complex domains (coasts, topography/bathymetry)
- Multi-phase fluid (water vapor, ice)

But !

Some of these additional effects, like fast rotation, allow to simplify the analysis

Scales :

Horizontal scales

- Large : planetary 10^4 km
- Medium : atmosphere - synoptic, 10^3 km ; ocean - meso-scale $10 - 10^2$ km
- Small : atmosphere - meso-scale $1 - 10$ km ; ocean - sub-meso scale 1 km
- Very small : meters

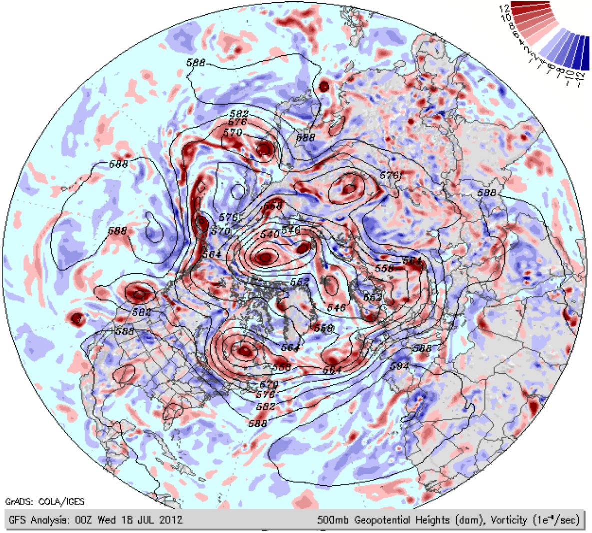
Scales of interest : large and medium.

Vertical scales

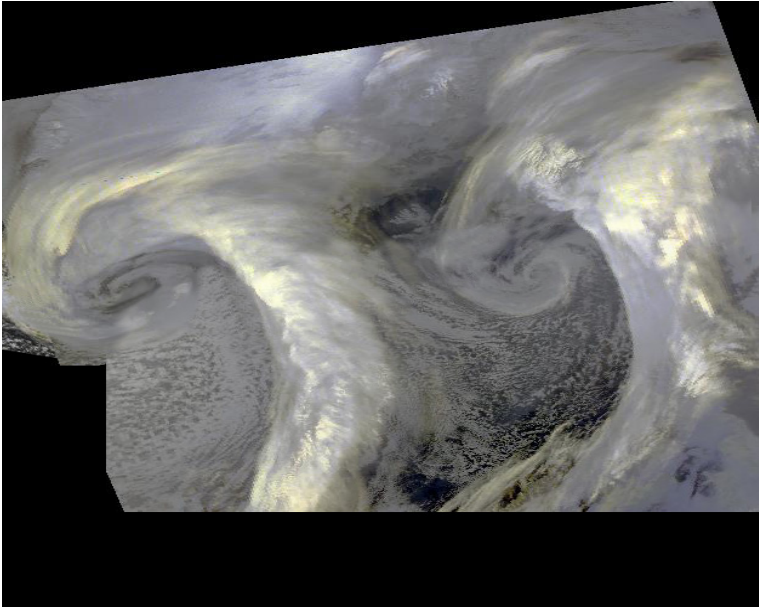
Synoptic motions in the atmosphere : whole troposphere, or its significant part, $\mathcal{O}(10km)$.
Meso-scale motions in the ocean : whole oceanic depth, or its significant part, $\mathcal{O}(1km)$.

$\Rightarrow$ Strong disparity between horizontal and vertical scales.

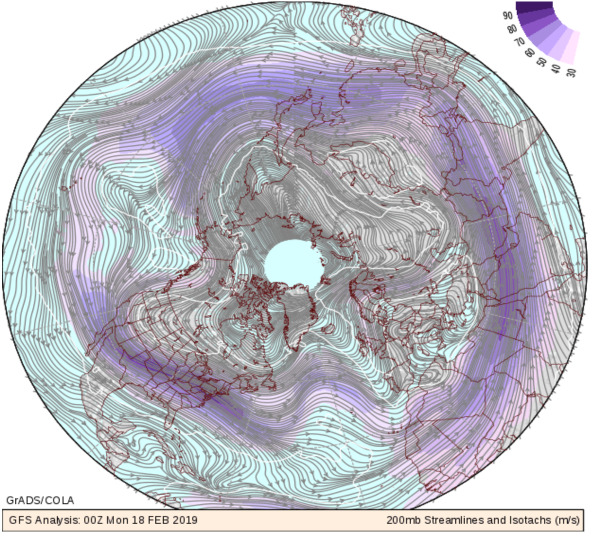
Atmospheric vortices in data



Atmospheric vortices : satellite view

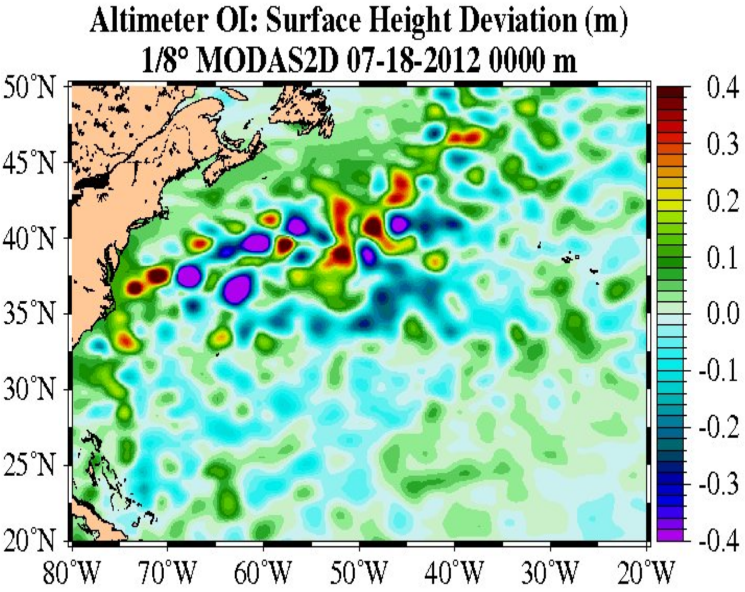


Atmospheric jet in data

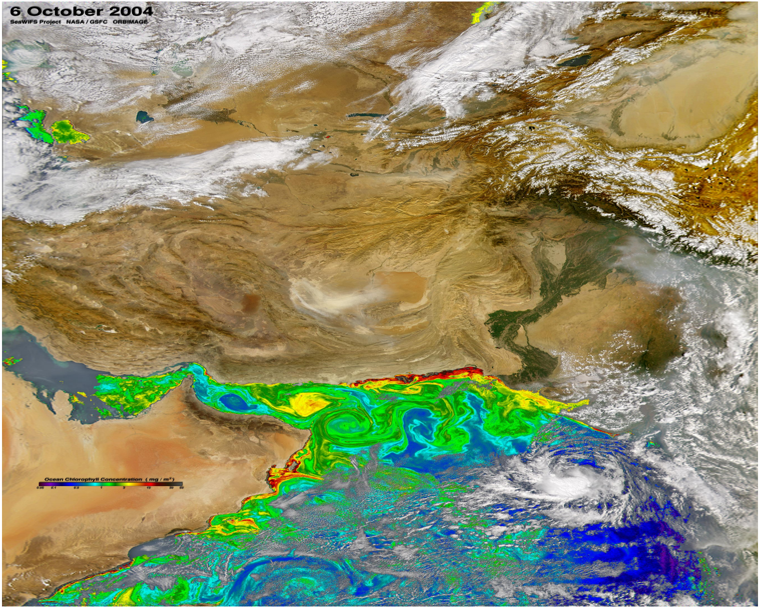


Atmospheric jet : satellite view

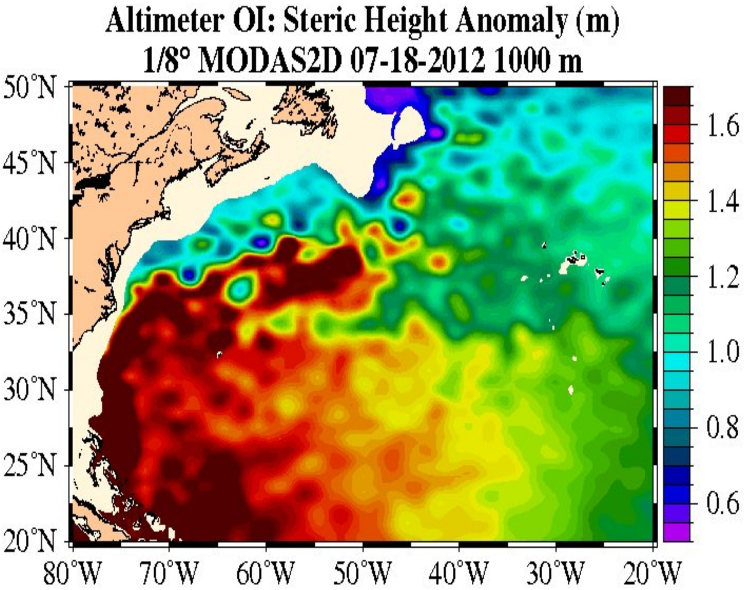
Oceanic vortices in satellite data



Oceanic vortices : satellite view of plankton



Oceanic jet in satellite data



Atmospheric waves

Oceanic waves

Where the governing equations come from :

- Mechanical system $\Rightarrow$ Newton's 2nd law $\leftrightarrow$ momentum conservation.
- Continuous medium $\Rightarrow$ local mass conservation
- Thermodynamical system $\Rightarrow$ 1st and 2nd laws of thermodynamics, equation of state
- Dissipative effects $\Rightarrow$ flux-gradient relations.

Dynamics of the perfect fluid : Lagrange's view

Description in terms of positions of fluid parcels $\boldsymbol{X}(\boldsymbol{a}, t)$, along their trajectories, where $\boldsymbol{a} = (a, b, c)$ are initial positions (Lagrangian labels) and $\boldsymbol{v}(\boldsymbol{X}, t) = \frac{d\boldsymbol{X}}{dt} \equiv \dot{\boldsymbol{X}}$ is their velocity.

Newton's 2nd law :

$$\rho(\boldsymbol{X}, t)\frac{d^2\boldsymbol{X}}{dt^2} = -\frac{\partial P(\boldsymbol{X}, t)}{\partial \boldsymbol{X}} \equiv -\boldsymbol{\nabla}_{\boldsymbol{X}} P(\boldsymbol{X}, t), \qquad (1)$$

where ρ and P are density and pressure in the fluid. Mass conservation :

$$\rho_i(\boldsymbol{a})d^3\boldsymbol{a} = \rho(\boldsymbol{X}, t)d^3\boldsymbol{X}, \leftrightarrow \rho_i(x) = \rho(\boldsymbol{X}, t)\mathcal{J}(\boldsymbol{X}, \boldsymbol{a}) \qquad (2)$$

where ρ_i is initial distribution of density, $\mathcal{J}(\boldsymbol{X}, \boldsymbol{a}) = \frac{\partial(X,Y,Z)}{\partial(a,b,c)}$ is the Jacobi determinant (Jacobian). $\frac{d\mathcal{J}}{dt} = \frac{\rho_i}{\rho}\boldsymbol{\nabla}_{\boldsymbol{X}} \cdot \dot{\boldsymbol{X}}$

No heat exchange between parcels :

$$\frac{ds}{dt} = 0, \qquad (3)$$

where $s(\boldsymbol{X}, t)$ is the entropy of the parcel.

Dynamics of the perfect fluid : Euler's view

Description in terms of velocity, density and pressure fields at a fixed point of space : $\boldsymbol{v}(\boldsymbol{x},t)$, $\rho(\boldsymbol{x},t)$, $P(\boldsymbol{x},t)$.
Euler-Lagrange duality : $\forall\ \boldsymbol{x}, t\ \exists\ \boldsymbol{X}(t) = \boldsymbol{x}$. Chain differentiation $\rightarrow$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}. \tag{4}$$

Newton's 2nd law :

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}\right) = -\boldsymbol{\nabla}P. \tag{5}$$

Continuity (mass conservation) equation :

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0 \leftrightarrow \frac{d}{dt}(\rho\mathcal{J}) = \frac{d\rho}{dt}\mathcal{J} + \rho\frac{d\mathcal{J}}{dt} = \frac{d\rho_i}{dt} = 0 \tag{6}$$

Momentum conservation :

$$\frac{\partial \rho\boldsymbol{v}}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}\otimes\boldsymbol{v} + P\mathbb{1}) = 0 \tag{7}$$

Adding thermodynamics : equation of state

General equation of state

$$P = P(\rho, s), \quad (8)$$

where s - entropy per unit mass ;

- Barotropic (isentropic) fluid :

$$s = \text{const} \;\Rightarrow P = P(\rho), \quad (9)$$

- Baroclinic fluid :

$$P = P(\rho, s), \quad (10)$$

- Incompressible - particular case of barotropic- fluid :

$$\mathcal{J} = 1 \leftrightarrow \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \Rightarrow \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\rho = 0. \quad (11)$$

$\Rightarrow$ pressure is not independent variable.

Thermodynamics : reminder

1st principle, "dry" thermodynamics

$$d\epsilon = Tds - Pdv, \qquad (12)$$

where ϵ - internal energy and $v = \frac{1}{\rho}$ - volume per unit mass.
Enthalpy per unit mass : $h = \epsilon + Pv$:

$$dh = Tds + vdP. \qquad (13)$$

Energy density of the fluid :

$$e = \frac{\rho \boldsymbol{v}^2}{2} + \rho\epsilon. \qquad (14)$$

Local conservation of energy :

$$\frac{\partial e}{\partial t} + \boldsymbol{\nabla} \cdot \left[\rho \boldsymbol{v} \left(\frac{\boldsymbol{v}^2}{2} + h\right)\right] = 0. \qquad (15)$$

Volume-preserving diffeomorphisms

Flow of the incompressible fluid in the domain $\mathcal{D}$ in the absence of singularities $\rightarrow$ bijective mapping $\boldsymbol{a} \mapsto \boldsymbol{X}(\boldsymbol{a}, t)$, $\boldsymbol{a}, \boldsymbol{X} \in \mathcal{D}$ of $\mathcal{D}$ onto itself : volume-preserving diffeomorphism belonging to infinite-dimensional Lie group. Lagrangian equations of motion : Euler-Lagrange equations

$$\frac{d}{dt}\frac{\delta \mathcal{L}}{\delta \dot{\boldsymbol{X}}} - \frac{\delta \mathcal{L}}{\delta \boldsymbol{X}} = 0 \tag{17}$$

for variational (Hamilton's) principle with action

$$\mathcal{S} = \int dt \int d^3\mathbf{a}\mathcal{L}; \; \mathcal{L} = \rho_i(\boldsymbol{a})\frac{\dot{\boldsymbol{X}}^2(\boldsymbol{a}, t)}{2} + P(\boldsymbol{a}, t)\left(\mathcal{J}(\boldsymbol{X}, \boldsymbol{a}) - 1\right), \rightarrow \tag{18}$$

geodesic equations on the group manifold, after relabeling $\boldsymbol{a} \rightarrow \tilde{\boldsymbol{a}} : \; \rho_i(\boldsymbol{a})d^3\boldsymbol{a} = d^3\tilde{\boldsymbol{a}}$ (mass-weighted labels). P - Lagrange multiplier to the incompressibility constraint.

Dissipative phenomena as molecular fluxes

Perfect fluid : local conservation laws with macroscopic fluxes of related quantities. Dissipation : correction of the macroscopic fluxes of

- momentum
- mass
- internal energy (heat)

by corresponding molecular fluxes, calculated from the flux - gradient relations (LeChatelier principle) :

$$\boldsymbol{f}_A = -k_A \boldsymbol{\nabla} A, \qquad (19)$$

A - a thermodynamical variable, $\boldsymbol{f}_A$ - corresponding molecular flux, k_A - transport coefficient.

Introducing viscosity

Stress tensor in momentum conservation corrected by viscous stresses - (density of) molecular momentum flux :

$$\rho \boldsymbol{v} \otimes \boldsymbol{v} \to \rho \boldsymbol{v} \otimes \boldsymbol{v} - \mu \boldsymbol{\nabla} \otimes \boldsymbol{v} \tag{20}$$

$\mu = \rho\nu$, μ - dynamic, ν - kinematic viscosities

Incompressible case : Navier -Stokes (NS) equation

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = -\frac{\boldsymbol{\nabla} P}{\rho} + \nu \boldsymbol{\nabla}^2 \boldsymbol{v}, \ \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0. \tag{21}$$

Reynolds' number

Dimensionless form of the NS equation :

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = -\frac{\boldsymbol{\nabla} P}{\rho} + \frac{1}{Re} \boldsymbol{\nabla}^2 \boldsymbol{v}, \tag{22}$$

where $Re = UL/\nu$, U, L -typical scales of velocity and length.

Synoptic motions far from the boundaries : $Re \to \infty$.

Diffusivity, thermal conductivity

Molecular fluxes of mass and heat :

$$-D\boldsymbol{\nabla}\rho, \qquad -\kappa\boldsymbol{\nabla}T \tag{23}$$

Corrected continuity equation :

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = D\boldsymbol{\nabla}^2\rho. \tag{24}$$

Equation of heat/temperature

$$\frac{\partial T}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}T = \chi\boldsymbol{\nabla}^2 T. \tag{25}$$

Non-dimensional form $\rightarrow$ Péclet numbers
$Pe = UL/D$, $Pe = UL/\chi$, small for synoptic motions far from the boundaries.

Motion in the rotating frame

Material point in the rotating frame :

$$m\frac{d\boldsymbol{v}}{dt} + 2m\boldsymbol{\Omega} \wedge \boldsymbol{v} + m\boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \boldsymbol{x}) = \boldsymbol{F}, \qquad (26)$$

where $\boldsymbol{v} = \frac{d\boldsymbol{x}}{dt}$, and $\boldsymbol{F}$ is a sum of external forces.

Euler equations with rotation

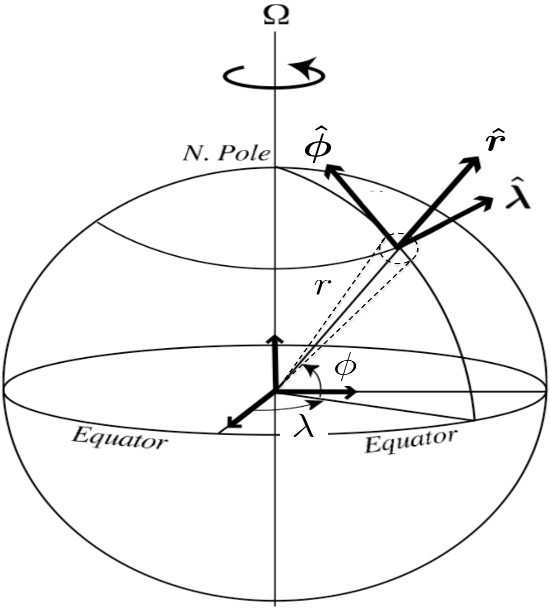
Replacements $m \to \rho, \frac{d}{dt} \to \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla},\ \boldsymbol{F} \to -\boldsymbol{\nabla} P$:

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} + 2\boldsymbol{\Omega} \wedge \boldsymbol{v} + \boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \boldsymbol{x}) = -\frac{\boldsymbol{\nabla} P}{\rho}, \qquad (27)$$

Including rotation in the variational principle

$$\mathcal{L} \to \mathcal{L} + \boldsymbol{\mathcal{R}} \cdot \dot{\boldsymbol{X}}; \quad \boldsymbol{\mathcal{R}} : \boldsymbol{\nabla}_{\boldsymbol{X}} \wedge \boldsymbol{\mathcal{R}} = 2\boldsymbol{\Omega}. \qquad (28)$$

Spherical coordinates



Euler and continuity equations

$$\frac{dv_r}{dt} - \frac{v_\lambda^2 + v_\phi^2}{r} - 2\Omega \cos\phi v_\lambda + g = -\frac{1}{\rho}\partial_r P,$$

$$\frac{dv_\lambda}{dt} + \frac{v_r v_\lambda - v_\phi v_\lambda \tan\phi}{r} + 2\Omega(-\sin\phi v_\phi + \cos\phi v_r) = -\frac{1}{\rho r\cos\phi}\partial_\lambda P,$$

$$\frac{dv_\phi}{dt} + \frac{v_r v_\phi + v_\lambda^2 \tan\phi}{r} + 2\Omega \sin\phi v_\lambda = -\frac{1}{\rho r}\partial_\phi P,$$

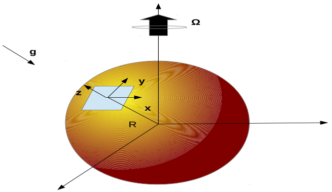
$$\frac{d\rho}{dt} + \rho\left[\frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\cos\phi}\left(\frac{\partial(\cos\phi v_\phi)}{\partial\phi} + \frac{\partial v_\lambda}{\partial\lambda}\right)\right],$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_r\partial_r + \frac{v_\phi}{r}\partial_\phi + \frac{v_\lambda}{r\cos\phi}\partial_\phi$$

Traditional approximation : green + red $\to$ out, $r \to R =$ const, centrifugal acceleration neglected.

Non-traditional approximation : green $\to$ out.

Tangent plane approximation (traditional)



$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} + f \hat{\boldsymbol{z}} \wedge \boldsymbol{v} = -\frac{\boldsymbol{\nabla} P}{\rho} + \boldsymbol{g}^* \qquad (29)$$

Coriolis parameter $f = 2\Omega \sin\phi$, $\hat{\boldsymbol{z}}$ - unit **z**, effective gravity :

$$\boldsymbol{g}^* = \boldsymbol{g} + \boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \boldsymbol{x}) \qquad (30)$$

"Traditional" : $\boldsymbol{g}^* \approx \boldsymbol{g}$ (correction several %). f - plane : $f = const$; β - plane : $f = f + \beta y$;

Hydrostatics. Stratification

State of rest $\boldsymbol{v} = (u, v, w) \equiv 0$ solution of (29) if hydrostatic equilibrium holds :

$$0 = -\frac{\boldsymbol{\nabla} P}{\rho} + \boldsymbol{g}$$

Continuity equation : satisfied by time-independent ρ in a state of rest.

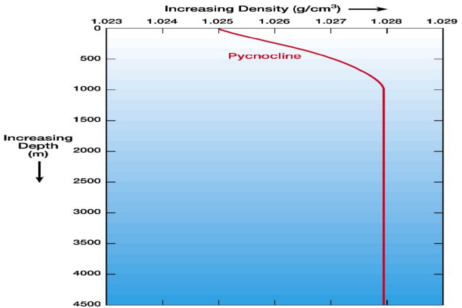
$$\frac{d\rho}{dt} + \rho \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$$

Statically stable states : $\rho = \rho_0(z)$, $\rho_0'(z) \leq 0$.
Dependence of ρ_0 on z : stratification.
Hydrostatic approximation : vertical accelerations small - valid for large scales $\Leftrightarrow$ large masses.

$$\frac{dw}{dt} \ll g. \qquad (31)$$

Ocean : observations and approximations

Typical averaged density profile :



$$\rho(\boldsymbol{x}, t) = \rho_0 + \rho_s(z) + \sigma(x, y, z; t), \;\; \rho_0 \gg ||\rho_s|| \gg ||\sigma||. \quad (32)$$

Temperature (T) and salinity (S) : density variations $\delta\rho \propto (\delta T, \delta S)$.

Hydrostatics :

$$P = P_0 + P_s(z) + \pi(x, y, z; t), \quad (33)$$

Equations of motion

Incompressibility and hydrostatics :

$$\nabla \cdot \boldsymbol{v} = 0, \quad g\rho + \partial_z P = 0, \tag{34}$$

Euler equations in Boussinesq approximation :

$$\frac{\partial \boldsymbol{v}_h}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}_h + f\hat{\boldsymbol{z}} \wedge \boldsymbol{v}_h = -\frac{\nabla_h \pi}{\rho} \approx -\nabla_h \phi. \tag{35}$$

$\phi = \frac{\pi}{\rho_0}$ - geopotential, $\boldsymbol{v} = \boldsymbol{v}_h + \hat{\boldsymbol{z}} w \equiv u\hat{\boldsymbol{x}} + v\hat{\boldsymbol{y}} + w\hat{\boldsymbol{z}}$.

Mass conservation :

$$\frac{d\rho}{dt} \equiv \partial_t \rho + \boldsymbol{v} \cdot \nabla \rho = 0. \tag{36}$$

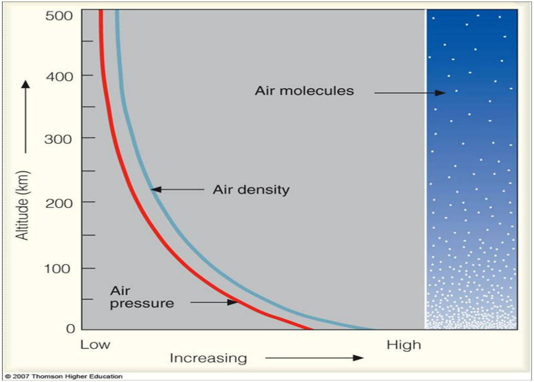
Boundary conditions :

Rigid lid and flat bottom : $w|_{z=0} = w|_{z=H} = 0$.

Non-trivial bathymetry $b(x, y)$ and/or free-surface : $h(x, y, t)$:

$$w|_{z=b} = \frac{db}{dt}, \quad w|_{z=h} = \frac{dh}{dt}$$

Atmosphere : pressure coordinatees



One-to-one correspondence between altitude and pressure $\rightarrow$ pressure as vertical coordinate.

Thermodynamics of the dry atmosphere

Equation of state - ideal gas :

$$P = \rho R T, \quad c_{p,v} = T \left(\frac{\partial s}{\partial T} \right)_{P,V} = const, \quad c_p - c_v = R. \tag{37}$$

Entropy :

$$s = c_p \ln T - R \ln P + const. \tag{38}$$

Adiabatic process :

$$s = \text{const} \Rightarrow c_p \frac{dT}{T} - R \frac{dP}{P} = 0, \Rightarrow T = T_s \left(\frac{P}{P_s} \right)^{\frac{R}{c_p}}. \tag{39}$$

Potential temperature :

$$\theta = T \left(\frac{P_s}{P} \right)^{\frac{R}{c_p}}, \, s = c_p \ln \theta + \text{const.} \tag{40}$$

Geopotential

$d\phi = gdz$, where $z = z(P)$ via hydrostatics, and thus becomes a thermodynamical variable.

Hydrostatics

$$d\phi = -\frac{RT}{P} dP \rightarrow$$

$$\frac{\partial \phi}{\partial P} = -\frac{RT}{P} = -\frac{1}{\rho} \rightarrow \quad (41)$$

$$\frac{\partial \phi}{\partial P} = -\frac{R}{P} \left(\frac{P}{P_s} \right)^{\frac{R}{c_p}} \theta \quad (42)$$

Elimination of ρ in Euler equations

"Triangular" relation :

$$\left(\frac{\partial P}{\partial x}\right)_z \left(\frac{\partial x}{\partial z}\right)_P \left(\frac{\partial z}{\partial P}\right)_x = -1 \Rightarrow \tag{43}$$

$$\left(\frac{\partial P}{\partial x}\right)_z = -\left(\frac{\partial P}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_P = \rho \left(\frac{\partial \phi}{\partial x}\right)_P . \tag{44}$$

Incompressibility in pressure coordinates

Lagrangian view : mass element $\rightarrow$ volume element in pressure coordinates in hydrostatic approximation :

$$\rho dxdydz = -\frac{1}{g} dxdydP \tag{45}$$

Mass conservation $\Rightarrow$ volume conservation with pressure as vertical coordinate $\Rightarrow$ divergence-less velocity.

Adiabatic equations of motion

$$div(\boldsymbol{v}) = \boldsymbol{\nabla}_h \cdot \boldsymbol{v}_h + \partial_p \omega = 0, \quad \omega \equiv \frac{dP}{dt}. \tag{46}$$

$$\frac{\partial \boldsymbol{v}_h}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}_h + f \hat{z} \wedge \boldsymbol{v}_h = -\boldsymbol{\nabla}_h \phi. \tag{47}$$

$$\partial_t \theta + \boldsymbol{v} \cdot \boldsymbol{\nabla} \theta = 0. \tag{48}$$

$$\frac{\partial \phi}{\partial P} = -\frac{RT}{P} = -\frac{R}{P} \left(\frac{P}{P_s} \right)^{\frac{R}{c_p}} \theta. \tag{49}$$

Remark : Coordinates (x, y, P) curvilinear, but curvature terms are neglected in divergence and material derivative ↔ weak curvature of the isobars, mostly valid for synoptic motions.

"Pseudo-height" coordinate

New vertical coordinate :

$$\bar{z} = z_0 \left(1 - \left(\frac{P}{P_s} \right)^{\frac{R}{c_p}} \right) \equiv z_0 \left(1 - \left(\frac{P}{P_s} \right)^{\frac{\gamma - 1}{\gamma}} \right), \qquad (50)$$

$$z_0 = \frac{\gamma}{\gamma - 1} \frac{P_s}{g \rho_s} \approx 28\text{km}. \qquad (51)$$

Pseudo- density :

$$r : \quad r d\bar{z} = \rho dz = -\frac{1}{g} dP. \qquad (52)$$

Mass conservation :

$$dxdydP = -gr(\bar{z})dxdyd\bar{z} \Rightarrow \tag{53}$$

$$r\left(\nabla_h \cdot \boldsymbol{v}_h + \frac{\partial \bar{w}}{\partial \bar{z}}\right) + \bar{w}\frac{\partial r}{\partial \bar{z}} = 0, \quad \boldsymbol{v} = (\boldsymbol{v}_h, \bar{w} = \frac{d\bar{z}}{dt}). \tag{54}$$

Approximation $\bar{z} \ll z_0$:

$$\nabla_h \cdot \boldsymbol{v}_h + \frac{\partial \bar{w}}{\partial \bar{z}} = -\frac{\bar{w}}{r}\frac{\partial r}{\partial \bar{z}} = \frac{\bar{w}}{(\gamma - 1)z_0\left(1 - \frac{\bar{z}}{z_0}\right)} \approx 0. \tag{55}$$

Equations of motion

$$\frac{\partial \boldsymbol{v}_h}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}_h + f \hat{z} \wedge \boldsymbol{v}_h = -\boldsymbol{\nabla}_h \phi, \tag{56}$$

$$-g \frac{\theta}{\theta_0} + \frac{\partial \phi}{\partial \bar{z}} = 0, \tag{57}$$

$$\frac{\partial \theta}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \theta = 0; \quad \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0. \tag{58}$$

Identical to oceanic PE with the exchange $\sigma \to -\theta$.

Conservation of PV in the PE model

Absolute vorticity :

$$\boldsymbol{\zeta}_a = \boldsymbol{\zeta} + \hat{\mathbf{z}} f, \, \boldsymbol{\zeta} = \boldsymbol{\nabla} \wedge \boldsymbol{v}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{\zeta}_a = 0, \tag{59}$$

Application of $\boldsymbol{\nabla}\wedge$ to PE + "hydrodynamic identity" :

$$\boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = \frac{1}{2} \boldsymbol{\nabla} \boldsymbol{v}^2 - \boldsymbol{v} \wedge (\boldsymbol{\nabla} \wedge \boldsymbol{v}) \tag{60}$$

$\rightarrow$ equation for $\boldsymbol{\zeta}_a$:

$$\frac{d\boldsymbol{\zeta}_a}{dt} = \boldsymbol{\zeta}_a \cdot \boldsymbol{\nabla} \boldsymbol{v} + \frac{g}{\rho_0} \hat{\mathbf{z}} \wedge \boldsymbol{\nabla} \rho. \tag{61}$$

Baroclinic torque

Last term in the r.h.s. : creation of horizontal vorticity by density gradients.

Conservation of potential vorticity (PV)

$$PV \equiv q = \zeta_a \cdot \nabla \rho, \quad \frac{dq}{dt} = 0. \tag{62}$$

PV anomaly (zero at the rest state) : $PVA = q - f\rho_s'(z)$.

Proof by direct calculation :

$$\begin{aligned}
\partial_t (\zeta_a \cdot \nabla \rho) &= (\partial_t \zeta_a) \cdot \nabla \rho + \zeta_a \cdot \nabla (\partial_t \rho) \\
&= \nabla \rho \cdot (\nabla \wedge (\boldsymbol{v} \wedge \zeta_a)) - \zeta_a \cdot \nabla (\boldsymbol{v} \cdot \nabla \rho) \\
&= -\nabla \cdot (\nabla \rho \wedge (\boldsymbol{v} \wedge \zeta_a)) - \zeta_a \cdot \nabla (\boldsymbol{v} \cdot \nabla \rho) \\
&= -\nabla \cdot (\boldsymbol{v} (\zeta_a \cdot \nabla \rho)) + \nabla \cdot (\zeta_a (\boldsymbol{v} \cdot \nabla \rho)) \\
&- \zeta_a \cdot \nabla (\boldsymbol{v} \cdot \nabla \rho) = -\boldsymbol{v} \cdot \nabla (\zeta_a \cdot \nabla \rho) . \qquad (63)
\end{aligned}$$

using

$$\nabla \cdot \boldsymbol{v} = \nabla \cdot \zeta_a = 0, \quad \nabla A \cdot (\nabla \wedge \boldsymbol{B}) = -\nabla \cdot (\nabla A \wedge \boldsymbol{B})$$

$$\boldsymbol{A} \wedge (\boldsymbol{B} \wedge \boldsymbol{C}) = \boldsymbol{B} (\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C} (\boldsymbol{A} \cdot \boldsymbol{B})$$

Symmetry with respect to Lagrangian relabeling

Volume -preserving parcel relabeling in the action principle :

$$\boldsymbol{a} \to \boldsymbol{a'} = \boldsymbol{a} + \delta \boldsymbol{a}(\boldsymbol{a}, t), \ \frac{\partial(a', b', c')}{\partial(a, b, c)} = 1 \Rightarrow$$

$$\boldsymbol{\nabla_a} \cdot \delta \boldsymbol{a} = 0 \Rightarrow \delta \boldsymbol{a} = \boldsymbol{\nabla_a} \wedge \delta \boldsymbol{\alpha}(\boldsymbol{a}, t). \tag{64}$$

Chain differentiation :

$$\dot{\boldsymbol{X}}\Big|_{\boldsymbol{a'}} = \dot{\boldsymbol{X}}\Big|_{\boldsymbol{a}} + (\boldsymbol{\nabla_a} \otimes \boldsymbol{X}) \cdot \dot{\boldsymbol{a}}|_{\boldsymbol{a'}} \Rightarrow \delta \dot{\boldsymbol{X}} = -(\boldsymbol{\nabla_a} \otimes \boldsymbol{X}) \cdot \delta \dot{\boldsymbol{a}}.$$

Invariance of action (mass-weighted labels) :

$$\delta S = \int dt \int d^3\mathbf{a}\, \dot{\boldsymbol{X}} \cdot \delta \dot{\boldsymbol{X}} = -\int dt \int d^3\mathbf{a}\, \delta \dot{\boldsymbol{a}} \cdot \boldsymbol{\nu} = 0, \tag{65}$$

where $\boldsymbol{\nu} = (\boldsymbol{\nabla_a} \otimes \boldsymbol{X}) \cdot \dot{\boldsymbol{X}} \equiv (\boldsymbol{\nabla_a} \otimes \boldsymbol{X}) \cdot \boldsymbol{v}$, with duality :

$$\boldsymbol{\nu} = (\boldsymbol{\nabla_a} \otimes \boldsymbol{X}) \cdot \boldsymbol{v} \ \Leftrightarrow \ \boldsymbol{v} = (\boldsymbol{\nabla_X} \otimes \boldsymbol{a}) \cdot \boldsymbol{\nu}. \tag{66}$$

Integration by parts in space and time $\rightarrow$

$$\int dt \int d^3\mathbf{a} \frac{d}{dt} (\boldsymbol{\nabla_a} \wedge \boldsymbol{\nu}) \cdot \delta \boldsymbol{\alpha} = 0 \Rightarrow \frac{d}{dt} (\boldsymbol{\nabla_a} \wedge \boldsymbol{\nu}) = 0. \tag{67}$$

Lagrangian meaning of PV conservation

PV in terms of ν and Lagrangian variables, using (66) :

$$PV := (\nabla_{\boldsymbol{X}} \wedge \boldsymbol{v}) \cdot \nabla \rho = (\nabla_{\boldsymbol{X}} \wedge [\boldsymbol{\nu} \cdot (\nabla_{\boldsymbol{X}} \otimes \boldsymbol{a})]) \cdot \nabla \rho$$

Right-hand side is equivalent to $\sum_i \frac{\partial(\nu_i, a_i, \rho)}{\partial(X,Y,Z)} \equiv \frac{\partial(\boldsymbol{\nu}, \boldsymbol{a}, \rho)}{\partial(X,Y,Z)}$, where $i = 1, 2, 3$ indicates the x, y, z- components of a vector. Using $\frac{\partial(a,b,c)}{\partial(X,Y,Z)} = \rho_i(\boldsymbol{a})$:

$$PV = \rho_i(\boldsymbol{a}) \frac{\partial(\boldsymbol{\nu}, \boldsymbol{a}, \rho)}{(a, b, c)} = \rho_i(\boldsymbol{a}) \left[(\nabla_{\boldsymbol{a}} \wedge \boldsymbol{\nu}) \cdot \nabla_{\boldsymbol{a}} \rho \right],$$

and from (67) and $\dot{\rho} = 0$, PV is Lagrangian invariant :

$$\frac{d}{dt} \left[(\nabla \wedge \boldsymbol{v}) \cdot \rho \right] = 0 \qquad (68)$$

Stationary solutions, jets

- Trivial stationary solution : a state of rest in hydrostatic equilibrium.
- Non-trivial stationary solutions :
  - β- plane : zonal flows $u = U(y,z)$ in geostrophic and hydrostatic equilibrium

$$-f(y)U(y,z) = -\partial_y \Phi(y,z), \; v = w = \sigma = 0, \quad (69)$$

$$g\frac{\rho_s(y,z)}{\rho_0} = -\partial_z \Phi(y,z) \quad (70)$$

  - f- plane : arbitrary oriented (zonal or meridional) flows.

Jets : localized $U(y,z) \rightarrow$ nonzero PV anomaly $(-\partial_y U)(\partial_z \rho_s) \neq 0$.

Polar coordinates

$$(x, y, z) \to (r, \theta, z)$$

Horizontal velocity :

$$\boldsymbol{v}_h = (u\hat{\boldsymbol{r}} + v\hat{\boldsymbol{\theta}})$$

Horizontal divergence :

$$\boldsymbol{\nabla}_h \cdot \boldsymbol{v}_h = \frac{1}{r}\left(\partial_r(ru) + \partial_\theta v\right)$$

Lagrangian derivative :

$$\frac{d}{dt} = \partial_t + u\partial_r + \frac{v}{r}\partial_\theta + w\partial_z$$

Stationary solutions : vortices

Primitive equations for horizontal motion

$$\frac{d\boldsymbol{v}_h}{dt} + \left(f + \frac{v}{r}\right)\hat{z} \wedge \boldsymbol{v}_h + \boldsymbol{\nabla}_h \phi = 0, \tag{71}$$

(72)

Vortex solutions

Stationary axisymmetric solution on the f- plane :
cyclo-geostrophic + hydrostatic equilibria :
$u = w = \sigma = 0, v = V(r, z)$

$$\left(f + \frac{V(r,z)}{r}\right) V(r,z) = \partial_r \Phi(r,z) \tag{73}$$

$$g\frac{\rho_s(r,z)}{\rho_0} = -\partial_z \Phi(r,z) \tag{74}$$

Small perturbations over the state of rest on the f- plane

Linearized equations :

Perturbations about the state of rest : $\boldsymbol{v} = 0$, with linear stratification on the f- plane, $f = \text{const}$. Linearized equations :

$$\begin{aligned} u_t - fv + \phi_x &= 0, \\ v_t + fu + \phi_y &= 0, \\ \phi_z + \frac{g}{\rho_0}\sigma = 0, \;\; \sigma_t + w\rho_s' &= 0, \end{aligned} \tag{75}$$

$$u_x + v_y + w_z = 0, \tag{76}$$

where u, v, w - three components of velocity perturbation, ϕ - geopotential perturbation, σ - perturbation of density, ρ_s - background density $\rho_s' = const$.
Notation : $(...)_x := \partial_x(...)$ etc.

Dispersion relation for Fourier-mode solutions

Elimination of σ and w :

$$\begin{aligned} u_t - fv + \phi_x &= 0, \\ v_t + fu + \phi_y &= 0, \\ u_x + v_y - N^{-2}\phi_{zzt} &= 0, \end{aligned} \tag{77}$$

a system of PDEs with constant coefficients. $N^2 = -\frac{g\rho_s'}{\rho_0}$ - Brunt - Väisälä frequency.

Fourier-transform$\leftrightarrow$ harmonic waves :

$$(u, v, \phi) = (u_0, v_0, \phi_0)e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} + \text{c.c.}, \Rightarrow \tag{78}$$

System of homogeneous algebraic equations for (u_0, v_0, ϕ_0), with solvability condition giving dispersion equation

$$\omega\left(\omega^2 - \left(N^2\frac{k_x^2 + k_y^2}{k_z^2} + f^2\right)\right) = 0. \tag{79}$$

Physical meaning of solutions

Three roots of this equation provide :

- Stationary solutions $\omega = 0$;
  correspond to linearized PV conservation equation :

$$\partial_t q_L = 0, \quad q_L = -(v_x - u_y)\frac{\rho_0}{g}N^2 + f\sigma_z, \tag{80}$$

  and hence to vortex motions.

- Propagative waves with dispersion relation :

$$\omega = \pm\sqrt{N^2\frac{k_x^2 + k_y^2}{k_z^2} + f^2} \tag{81}$$

  and two senses of propagation, internal inertia-gravity waves (IGW). They are supra-inertial : $|\omega| \geq f$.

Waves on the background of a jet

Linearized equations :

Perturbations about a stationary jet :
$u = U(y,z)$, $v = w = 0$, with stratification $\rho_s(z)$. Linearized equations :

$$\begin{aligned} u_t + Uu_x + vU_y + wU_z - fv + \phi_x &= 0, \\ v_t + Uv_x + fu + \phi_y &= 0, \qquad (82) \\ \phi_z + \frac{g}{\rho_0}\sigma = 0, \;\; \sigma_t + U\sigma_x + w\rho_s' &= 0, \\ u_x + v_y + w_z &= 0, \qquad (83) \end{aligned}$$

Fourier-transformation → waves propagating along the jet :

$$(u, v, \phi) = (u_0, v_0, \phi_0)(y, z)e^{i(\omega t - kx)} + \text{c.c.}, \qquad (84)$$

In general, complex eigen-frequencies $\omega \Rightarrow$ growing eigen-modes → instability of the jet.

Full linearized non-hydrostatic equations

$$g\frac{\sigma}{\rho_0} = -\phi_z \to w_t + g\frac{\sigma}{\rho_0} = -\phi_z. \quad (85)$$

Elimination of buoyancy $b = g\frac{\sigma}{\rho_0}$ and vertical velocity w :

$$b = -\phi_z - w_t, \quad -\left(\partial_{tt} + N^2\right)(u_x + v_y) + \phi_{zzt} = 0 \Rightarrow \quad (86)$$

$$u_t - fv = -\phi_x, \quad (87)$$

$$v_t + fu = -\phi_y, \quad (88)$$

$$\left(\partial_{tt} + N^2\right)(u_x + v_y) - \phi_{zzt} = 0, \quad (89)$$

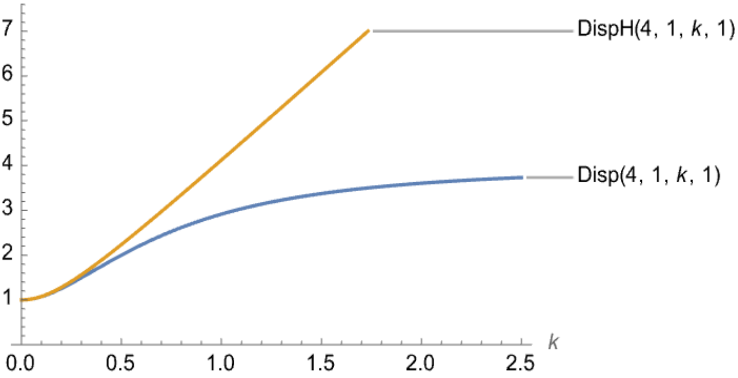
Dispersion relation :

$$\omega\left[\omega^2 - \left(N^2\frac{k^2+l^2}{k^2+l^2+m^2} + f^2\frac{m^2}{k^2+l^2+m^2}\right)\right] = 0 \quad (90)$$

Typically in the atmosphere and ocean

$$N^2 > f^2 \Rightarrow f^2 \le \omega^2 \le N^2 \quad (91)$$

Hydrostatic vs non-hydrostatic dispersion curves



Frequency as a function of the modulus of horizontal wavenumber k at fixed vertical wavenumber $m = 1$, and $N = 4$, $f = 1$ as given by hydrostatic (DispH) vs non-hydrostatic (Disp) dispersion relations.

- Large-scale atmospheric and oceanic motions : same set of PE, modulo changes of variables
- Crucial property of PE : Lagrangian conservation of PV.
- Main dynamical entities in PE : vortices, jets, and waves
- Vortices : slow motions carrying non-zero PV anomaly, have zero frequency in linear approximation,
- Jets : particular case of vortex motions.
- Inertia-gravity waves : fast motions bearing no PV anomaly
- Frequencies of wave and vortices are separated by a spectral gap $[0, f]$, frequencies of the waves bounded from below by f.
- Non-hydrostatic effects, if included, impose an upper boundary to the wave spectrum.