2. Vertically averiging primitive equations: rotating shallow water models and their properties

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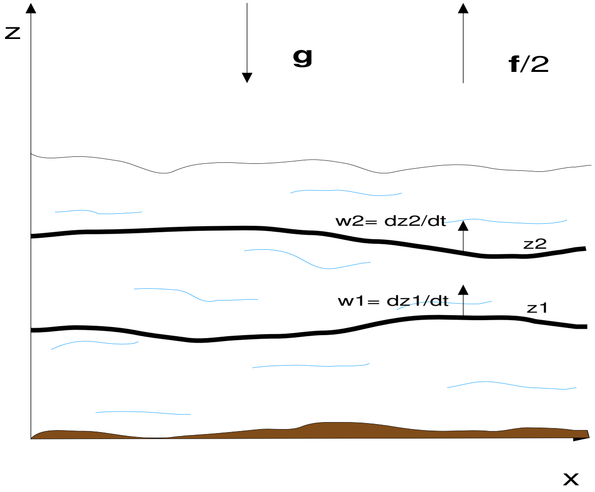
Plan

Motivation

Disparity of typical horizontal (L) and vertical (H) scales of synoptic motions in the atmosphere and meso-scale motions in the ocean : *Example* : typical weather system with $L \sim 1000$km, $H \sim 10$km $\rightarrow$ rough description can be provided by vertical averaging of PE.

Key element : averaging between material (Lagrangian) surfaces moving with the fluid..

Material surfaces



Vertical averaging

- Take horizontal momentum equation in conservative form :

$$\partial_t (\rho \boldsymbol{v}_h) + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v}_h) + f \hat{\boldsymbol{z}} \wedge (\rho \boldsymbol{v}_h) + \boldsymbol{\nabla}_h \pi = 0, \quad (1)$$

and integrate between a pair of material surfaces $z_{1,2}$:

$$w|_{z_i} = \frac{dz_i}{dt} = \partial_t z_i + \boldsymbol{v}_h \cdot \boldsymbol{\nabla}_h z_i \quad i = 1, 2. \qquad (2)$$

- Use Leibniz formula and get :

$$\partial_t \int_{z_1}^{z_2} dz \rho \boldsymbol{v}_h + \boldsymbol{\nabla}_h \cdot \int_{z_1}^{z_2} dz \left(\rho \boldsymbol{v}_h \otimes \boldsymbol{v}_h\right) + f \hat{\boldsymbol{z}} \wedge \int_{z_1}^{z_2} dz \left(\rho \boldsymbol{v}_h\right) = -\boldsymbol{\nabla}_h \int_{z_1}^{z_2} dz \pi - \boldsymbol{\nabla}_h z_1 \, \pi|_{z_1} + \boldsymbol{\nabla}_h z_2 \, \pi|_{z_2}, \qquad (3)$$

- Use density advection and incompressibility equations and similarly get

$$\partial_t \int_{z_1}^{z_2} dz \rho + \boldsymbol{\nabla}_h \cdot \int_{z_1}^{z_2} dz \rho \boldsymbol{v}_h = 0, \tag{4}$$

$$\partial_t (z_2 - z_1) + \boldsymbol{\nabla}_h \cdot \int_{z_1}^{z_2} dz \boldsymbol{v}_h = 0 \tag{5}$$

- Introduce vertical averages : $\langle F \rangle = \frac{\int_{z_1}^{z_2} dz\, F}{z_2 - z_1}$ and obtain averaged equations :

$$\begin{aligned} \partial_t \left((z_2 - z_1) \langle \boldsymbol{v}_h \rangle \right) &+ \boldsymbol{\nabla}_h \cdot \left((z_2 - z_1) \langle \boldsymbol{v}_h \otimes \boldsymbol{v}_h \rangle \right) + \\ f(z_2 - z_1) f \hat{\boldsymbol{z}} \wedge \langle \boldsymbol{v}_h \rangle &= -\boldsymbol{\nabla}_h \int_{z_1}^{z_2} dz \pi - \boldsymbol{\nabla}_h z_1 \, \pi|_{z_1} \\ &+ \boldsymbol{\nabla}_h z_2 \, \pi|_{z_2}, \end{aligned} \tag{6}$$

$$\partial_t \left((z_2 - z_1) \rho \right) + \boldsymbol{\nabla}_h \cdot \left((z_2 - z_1) \langle \rho \boldsymbol{v}_h \rangle \right) = 0. \tag{7}$$

$$\partial_t (z_2 - z_1) + \boldsymbol{\nabla}_h \cdot \left((z_2 - z_1) \langle \boldsymbol{v}_h \rangle \right) = 0 \tag{8}$$

- Introduce mean density $\bar{\rho} := \langle \rho \rangle$, and use hydrostatics to get,

$$\pi(x, y, z, t) \approx -g\bar{\rho}(z - z_1) + \pi|_{z_1}(x, y, t). \qquad (9)$$

- Use the mean-field (= columnar motion) approximation :

$$\langle \boldsymbol{v}_h \otimes \boldsymbol{v}_h \rangle \approx \langle \boldsymbol{v}_h \rangle \otimes \langle \boldsymbol{v}_h \rangle, \ \langle \rho \boldsymbol{v}_h \rangle \approx \langle \rho \rangle \langle \boldsymbol{v}_h \rangle. \qquad (10)$$

and get Master Equation for the layer :

$$\begin{aligned} &\bar{\rho}(z_2 - z_1)(\partial_t \boldsymbol{v}_h + \boldsymbol{v}_h \cdot \nabla \boldsymbol{v}_h + f\hat{\mathbf{z}} \wedge \mathbf{v}_h) = \\ - \quad &\nabla_h \left(-g\bar{\rho}\frac{(z_2 - z_1)^2}{2} + (z_2 - z_1)\, \pi|_{z_1} \right) \\ - \quad &\nabla_h z_1 \, \pi|_{z_1} + \nabla_h z_2 \, \pi|_{z_2} . \end{aligned} \qquad (11)$$

- Pile up layers, with lowermost boundary fixed by topography, and uppermost free or fixed.

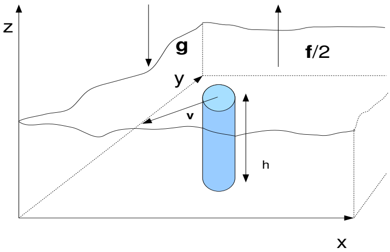
Standard shallow-water models : $\bar{\rho} = \text{const}$.

1-layer RSW : $z_1 = 0,\ z_2 = h$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + f \hat{\mathbf{z}} \wedge \mathbf{v} + g \nabla h = 0\,, \qquad (12)$$

$$\partial_t h + \nabla \cdot (\mathbf{v} h) = 0\,. \qquad (13)$$

$\Rightarrow$ 2d barotropic gas dynamics, with h playing the role of density, and the equation of state $P = \frac{gh^2}{2}$, in the presence of the Coriolis force.
In the presence of non-trivial topography $b(x, y) : h \to h - b$ in the second equation.



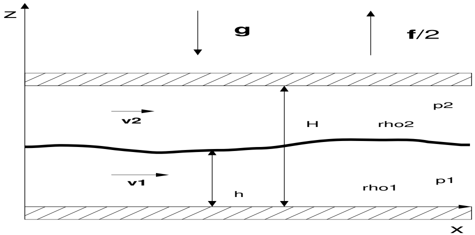
2-layer RSW, flat bottom, rigid lid : $z_1 = 0$, $z_2 = h$, $z_3 = H = \text{const}$

$$\partial_t \mathbf{v}_i + \mathbf{v}_i \cdot \nabla \mathbf{v}_i + f\hat{\mathbf{z}} \wedge \mathbf{v}_i + \frac{1}{\rho_i} \nabla \pi_i = 0\,, i = 1, 2; \tag{14}$$

$$\partial_t h + \nabla \cdot (\mathbf{v}_1 h) = 0\,, \tag{15}$$

$$\partial_t (H - h) + \nabla \cdot (\mathbf{v}_2 (H - h)) = 0\,, \tag{16}$$

$$\pi_1 = (\rho_1 - \rho_2) g h + \pi_2\,. \tag{17}$$



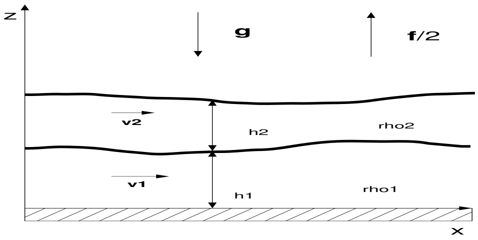
2-layer RSW with a free surface and flat bottom : $z_1 = 0$, $z_2 = h_1$, $z_3 = h_1 + h_2$

$$\partial_t \mathbf{v}_2 + \mathbf{v}_2 \cdot \nabla \mathbf{v}_2 + f\hat{\mathbf{z}} \wedge \mathbf{v}_2 = -g\nabla(h_1 + h_2) \quad (18)$$

$$\partial_t \mathbf{v}_1 + \mathbf{v}_1 \cdot \nabla \mathbf{v}_1 + f\hat{\mathbf{z}} \wedge \mathbf{v}_1 = -g\nabla(rh_1 + h_2), \quad (19)$$

$$\partial_t h_{1,2} + \nabla \cdot (\mathbf{v}_{1,2} h_{1,2}) = 0\,, \quad (20)$$

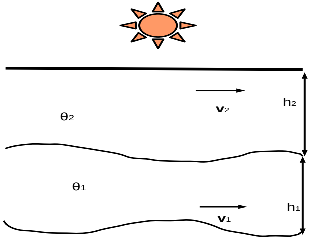
where $r = \frac{\rho_1}{\rho_2} \leq 1$ - density ratio, and $h_{1,2}$ - thicknesses of the layers.



Atmospheric vs oceanic models

Standard atmospheric shallow-water models : vertical averaging of Primitive Equations in pseudo-hight pressure coordinates, supposing constant mean potential temperature θ layer-wise. Layers upside-down : static stability $\Rightarrow \theta$ increasing with height. Constant geopotential at the ground, *and pressure is not !* $\Rightarrow$ ground = "free surface". Upper layer bounded by an isobaric surface.

Two-layer atmospheric RSW model



$$\begin{cases} \partial_t \boldsymbol{v}_1 + (\boldsymbol{v}_1 \cdot \nabla)\boldsymbol{v}_1 + f\hat{\boldsymbol{z}} \wedge \boldsymbol{v}_1 = -g\nabla(h_1 + h_2), \\ \partial_t \boldsymbol{v}_2 + (\boldsymbol{v}_2 \cdot \nabla)\boldsymbol{v}_2 + f\hat{\boldsymbol{k}} \wedge \boldsymbol{v}_2 = -g\nabla(h_1 + \alpha h_2), \\ \partial_t h_1 + \nabla \cdot (h_1 \boldsymbol{v}_1) = 0, \\ \partial_t h_2 + \nabla \cdot (h_2 \boldsymbol{v}_2) = 0, \end{cases} \quad (21)$$

$\alpha = \frac{\theta_2}{\theta_1} \geq 1$ - stratification parameter.

Lagrangian view of 1-layer RSW

RSW : ensemble of fluid columns of variable depth moving in the plane. Trajectories of columns :

$$(x, y) \to (X(x, y; t), Y(x, y; t))$$

(x, y) - initial positions.
Velocity of the column : $\left(\dot{X}, \dot{Y}\right) = (u(X, Y; t), v(X, Y; t))$.
Volume conservation :

$$h(X, Y; t)\, dX\, dY = h_I(x, y)\, dx\, dy, \;\; \Rightarrow \tag{22}$$

h is not an independent variable :

$$h(X, Y; t) = \frac{h_I(x, y)}{\mathcal{J}_2}, \tag{23}$$

where $\mathcal{J}_2 = \frac{\partial(X,Y)}{\partial(x,y)}$ - Jacobian in 2 dimensions.

Lagrangian equations of motion

Momentum equations :

$$\begin{cases} \ddot{X} - f\dot{Y} = -g\partial_X h = -\dfrac{g}{\mathcal{J}_2}\dfrac{\partial(h,Y)}{(x,y)} , \\ \ddot{Y} + f\dot{X} = -g\partial_Y h = -\dfrac{g}{\mathcal{J}_2}\dfrac{\partial(X,h)}{(x,y)} , \end{cases} \tag{24}$$

with $h(X,Y;t) = \frac{h_I(x,y)}{\mathcal{J}_2}$. To be solved with initial conditions : $\dot{X}(x,y,0) = u_I(x,y)$, $\dot{Y}(x,y,0) = v_I(x,y)$.
"Straightening" of h by additional change of labels :
$h_I \to H =$ const in mass-weighted labels (a,b).
Euler-Lagrange equations for the action $S = \int dt\, L_{RSW}$ in mass-weighted variables :

$$L_{RSW} = \int dadb \left[\frac{\dot{X} + \dot{Y}}{2} + \frac{f}{2}\left(X\dot{Y} - Y\dot{X} \right) - \frac{gh(X,Y)}{2} \right], \tag{25}$$

with variations calculated with (23) give (24).

Hydrostatics and columnar motion approximations in variational principle

Lagrangian for a layer of incompressible fluid with unit density on the tangent plane to a rotating planet, with gravity :

$$
\begin{aligned}
L &= \int d^3\mathbf{a} \left[\frac{\dot{\boldsymbol{X}}^2(\boldsymbol{a},t)}{2} + \frac{f}{2} (\hat{\boldsymbol{z}} \wedge \boldsymbol{X}(\boldsymbol{a},t)) \cdot \dot{\boldsymbol{X}}(\boldsymbol{a},t) \right. \\
&- \left. g \hat{\boldsymbol{z}} \cdot \boldsymbol{X}(\boldsymbol{a},t) + P(\boldsymbol{a},t) \left(\mathcal{J}(\boldsymbol{X},\boldsymbol{a}) - 1 \right) \right], \qquad (26)
\end{aligned}
$$

with $\boldsymbol{a} = (a,b,c)$, $\boldsymbol{X} = (X,Y,Z)$.

- Mass-weighted labels : $\int d^3\mathbf{a} = \int_{-\infty}^{\infty} dadb \int_0^H dc$
- Hydrostatics : $\dot{Z} \ll (\dot{X}, \dot{Y}) \to \dot{Z}$ out.
- Columnar motion : $X = X(a,b,t)$, $Y = Y(a,b,t) \Rightarrow$

$$
\mathcal{J}(\boldsymbol{X},\boldsymbol{a}) = \frac{\partial(X,Y)}{\partial(a,b)} \frac{\partial Z}{\partial c} \Rightarrow Z = \mathcal{J}_2^{-1} c = h(X,Y;t)c/H. \qquad (27)
$$

After (explicit) integration by c, $L = L_{RSW}$.

Lagrangian form of 2-layer RSW equations

$$\ddot{\boldsymbol{X}}_i + f\hat{\boldsymbol{z}} \wedge \dot{\boldsymbol{X}}_i = -g\nabla_{\boldsymbol{X}_i}\left(r^{i-1}h_1 + h_2\right),\; i = 1, 2. \qquad (28)$$

$\boldsymbol{X}_i = (X_i, Y_i)$ functions of Lagrangian labels (x, y), and

$$h(X_i, Y_i; t) = h_{I_i}(x, y)\frac{\partial (x, y)}{\partial (X_i, Y_i)},\; i = 1, 2. \qquad (29)$$

"Foreign" h in (28) is considered as a function of "native" $\boldsymbol{X}$. Lagrangian has three entries $L = L_1 + L_2 + L_{12}$, with

$$L_i = \rho_i \int dxdy \left(\frac{\dot{X}_i^2 + \dot{Y}_i^2}{2} - \frac{gh_i(X, Y)}{2}\right),\; i = 1, 2, \qquad (30)$$

$$L_{12} = \rho_1 \int dx_1 dy_1 \int dx_2 dy_2\, \delta\left(\boldsymbol{X}_1(x_1, y_1) - \boldsymbol{X}_2(x_2, y_2)\right). \qquad (31)$$

To vary with respect to e.g. $\boldsymbol{X}_1$ $dx_2dy_2 \to dX_2dY_2$. Jacobian, and hence h_2, emerge via (29). Integration over dX_2dY_2 is lifted by delta-function, which makes h_2 a function of X_1, Y_1.

Thermal rotating shallow water (TRSW) model

Replacing in (26) $g \to b = g\frac{\rho}{\rho_0}$, where

$$\dot{b}(\boldsymbol{X}, t) = 0, \Rightarrow b(\boldsymbol{X}, t) = b_i(a, b, c)$$

transforms L into Lagrangian for non-hydrostatic PE. Hydrostatic and columnar motion approximations, $b_i = b_i(a, b) \to$

$$L_{TRSW} = \int dadb \left[\frac{\dot{X} + \dot{Y}}{2} + \frac{f}{2}\left(X\dot{Y} - Y\dot{X} \right) - \frac{b(Y, Y)\, h(X, Y)}{2} \right], \tag{32}$$

Corresponding Eulerian equations can be as well obtained by $\bar{\rho} \to \bar{\rho}(x, y, t)$ in the master equation (11) :

$$\begin{cases} \partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} + f\hat{z} \wedge \boldsymbol{v} = -b\boldsymbol{\nabla} h - \frac{h}{2}\boldsymbol{\nabla} b \equiv \frac{1}{h}\boldsymbol{\nabla}\frac{bh^2}{2}, \\ \partial_t h + \boldsymbol{\nabla} \cdot (\boldsymbol{v} h) = 0, \\ \partial_t b + \boldsymbol{v} \cdot \nabla b = 0. \end{cases} \tag{33}$$

$\Rightarrow$ dynamics of an ideal gas with density h, entropy b, and pressure $P = \dfrac{bh^2}{2}$ in a rotating frame.

(Serre-)Green-Naghdi equation for rotating fluid

Inserting results of columnar motion approximation :

$$Z = \frac{\partial(a,\ b)}{\partial(X,\ Y)}\, c = h(X, Y; t)c/H \tag{34}$$

in the Lagrangian of rotating Euler equations (26), and not imposing hydrostatics gives :

$$L_{GN} = \int dadb \left[\frac{\dot{X} + \dot{Y}}{2} + \frac{\dot{h}(X, Y)}{6} + \frac{f}{2}\left(X\dot{Y} - Y\dot{X}\right) - \frac{gh(X, Y)}{2} \right], \tag{35}$$

where the expression (34) of h in terms of $\boldsymbol{X}_h = (X,\ Y)$ to be used while calculating variations, which gives :

$$\ddot{\boldsymbol{X}}_h + f\hat{\boldsymbol{z}} \wedge \dot{\boldsymbol{X}}_h = -g\boldsymbol{\nabla}_h h(\boldsymbol{X}_h) - \frac{1}{3\,h(\boldsymbol{X}_h)}\boldsymbol{\nabla}_h \cdot \left(h^2(\boldsymbol{X}_h)\ddot{h}(\boldsymbol{X}_h)\right), \tag{36}$$

$\Leftrightarrow$ Green-Naghdi equation with rotation (subscript omitted)

$$\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} + f\hat{\boldsymbol{z}}\wedge\boldsymbol{v} + g\boldsymbol{\nabla}h + \frac{1}{3h}\boldsymbol{\nabla}\left(h^2(\partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla})^2 h\right) = 0. \tag{37}$$

Miyata-Choi-Camassa equations for rotating fluid

Relaxing hydrostatic approximation in the two-layer model with a rigid lid :

$$\begin{aligned} \partial_t \boldsymbol{v}_i + \boldsymbol{v}_i \cdot \boldsymbol{\nabla} \mathbf{v}_i + f \hat{\mathbf{z}} \wedge \boldsymbol{v}_i \quad & + \\ \frac{1}{\rho_i} \boldsymbol{\nabla} \pi_i + \frac{1}{3h_i} \boldsymbol{\nabla} \left(h_i^2 (\partial_t + \boldsymbol{v}_i \cdot \boldsymbol{\nabla})^2 h_i \right) \quad & = \quad 0 \,, \end{aligned} \tag{38}$$

$$\partial_t h_i + \nabla \cdot (\mathbf{v}_i h_i) = 0 \,, i = 1, 2, \tag{39}$$

$$\pi_1 = (\rho_1 - \rho_2) g h_1 + \pi_2 \,, \tag{40}$$

with no summation over repeated i.

Action symmetries ⇒ conservation laws in RSW

- Local mass conservation (embedded) :

$$\partial_t h + \nabla \cdot (h\mathbf{v}) = 0 \rightarrow \int dxdy\, h = \text{const} \tag{41}$$

- Local momentum conservation (symmetry to translations in the absence of rotation) :

$$\partial_t (h\mathbf{v}) + \nabla \cdot \left(h\mathbf{v} \otimes \mathbf{v} + g\frac{h^2}{2} \right) = -f\hat{\mathbf{z}} \wedge (h\mathbf{v}) \rightarrow \tag{42}$$

$$\int dxdy\, h\mathbf{v} = \text{const}, \text{if } f \equiv 0 \tag{43}$$

- Local energy conservation (symmetry to time-shifts) :

$$\partial_t e + \nabla \cdot \left(\mathbf{v} h \left(\frac{\mathbf{v}^2}{2} + gh \right) \right) = 0 \rightarrow \int dxdy\, e = \text{const} \tag{44}$$

$$e = h\frac{\mathbf{v}^2}{2} + g\frac{h^2}{2} \tag{45}$$

Lagrangian conservation law in RSW : PV

Potential vorticity :

$$q = \frac{\zeta + f}{h}, \tag{46}$$

$\zeta = v_x - u_y$ - relative vorticity, $\zeta + f$ - absolute vorticity.
Lagrangian conservation of PV :

$$\frac{dq}{dt} = (\partial_t + \mathbf{v} \cdot \nabla)\, q = 0, \tag{47}$$

Follows by combining vorticity equation obtained by cross-differentiation of the equations for two components of velocity :

$$\frac{d(\zeta + f)}{dt} + (\zeta + f)\nabla \cdot \mathbf{v} = 0, \tag{48}$$

and mass conservation in the form :

$$\frac{dh}{dt} + h\,\nabla \cdot \mathbf{v} = 0. \tag{49}$$

Eulerian expression : time-independence of any integral

$$C_{\mathcal{F}} = \int_D dxdy\; h\mathcal{F}(q), \tag{50}$$

Lagrangian vorticity and PV

Relative vorticity ζ in Lagrangian form :

$$\zeta = \frac{\partial \dot{Y}}{\partial X} - \frac{\partial \dot{X}}{\partial Y} = \frac{\partial(\dot{Y}, Y)}{\partial(X, Y)} - \frac{\partial(X, \dot{X})}{\partial(X, Y)} = \frac{1}{\mathcal{J}_2}\left[\frac{\partial(\dot{Y}, Y)}{\partial(x, y)} - \frac{\partial(X, \dot{X})}{\partial(x, y)}\right], \tag{51}$$

Lagrangian expression for PV :

$$q = \frac{\zeta + f}{h} = \frac{1}{h_I}\left[\frac{\partial(\dot{Y}, Y)}{\partial(x, y)} - \frac{\partial(X, \dot{X})}{\partial(x, y)} + f\mathcal{J}_2\right]. \tag{52}$$

Symmetry of L_{RSW} with respect to volume- ($\Rightarrow \mathcal{J}_2 - \Rightarrow h-$) preserving relabeling

$$\delta x = -\partial_y \delta\chi, \quad \delta y = \partial_x \delta\chi \;\Rightarrow$$

Lagrangian conservation $\dot{q} = 0$.

Eulerian conservation laws in 2-layer RSW

- Overall momentum (in the absence of the Coriolis force)

$$\mathbf{M} = \int dxdy\, \left(\rho_1 h_1 \mathbf{v}_1 + \rho_2 h_2 \mathbf{v}_2\right),$$

- Mass layer-wise

$$\rho_i \int dxdy\, h_i, \quad i = 1, 2,$$

- Energy :

$$\begin{aligned} E &= \int dxdy \left[\left(\frac{\rho_1}{2} h_1 \mathbf{v_1}^2 + \frac{\rho_2}{2} h_2 \mathbf{v_2}^2 \right) \right. \\ &+ \left. \left(\frac{\rho_1}{2} g h_1^2 + \frac{\rho_2}{2} g h_2^2 \right) + \rho_1 g h_1 h_2 \right] \end{aligned}$$

Lagrangian conservation laws in 2-layer RSW : PV

Potential vorticities layer-wise :

$$q_1 = \frac{f + \partial_x v_1 - \partial_y u_1}{h_1} \quad , \quad q_2 = \frac{f + \partial_x v_2 - \partial_y u_2}{h_2},$$

are Lagrangian invariants :

$$\frac{d_i q_i}{dt} = 0, \quad i = 1, 2 \tag{53}$$

where

$$\frac{d_i}{dt} = \partial_t = u_i \partial_x + v_i \partial_y$$

Energy and PV conservation in Serre - Green - Naghdi equations

Energy

$$E = \int dxdy \left(h\frac{\mathbf{v}^2}{2} + g\frac{h^2}{2} + \frac{h^2(\boldsymbol{\nabla}\cdot\boldsymbol{v})^2}{6} \right) = \text{const} \quad (54)$$

Potential vorticity

Volume-preserving parcel relabeling in the action principle ⇒ PV conservation :

$$q = \frac{\zeta + f + \frac{1}{3}\mathcal{J}(\boldsymbol{\nabla}\cdot\boldsymbol{v}), h)}{h}, \quad (\partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla})q = 0, \quad (55)$$

with relative vorticity $\zeta = \hat{\boldsymbol{z}}\cdot\boldsymbol{\nabla}\wedge\boldsymbol{v}$.

Linearized equations on the f- plane, and their solutions

Small perturbations about the state of rest with $\boldsymbol{v}_h = 0,\ h = H_0 = \text{const},\ f = \text{const}$:

$$\begin{cases} \partial_t \boldsymbol{v}_h + f\hat{\boldsymbol{z}} \wedge \boldsymbol{v}_h + g\boldsymbol{\nabla}_h \eta = 0, \\ \partial\eta + H_0 \boldsymbol{\nabla}_h \cdot \boldsymbol{v}_h = 0, \end{cases} \tag{56}$$

$\boldsymbol{v}_h = (u, v)$ - velocity perturbation, η - free-surface perturbation. Fourier-transformation (solutions : harmonic waves) $(u, v, \eta) = (u_0, v_0, \eta_0)e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})} + c.c.$, Solvability condition of the resulting algebraic system $\to$ dispersion relation between the frequency and wave-number :

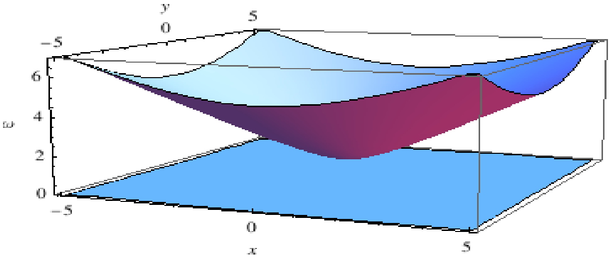
$$\omega\left(\omega^2 - gH_0\mathbf{k}^2 - f^2\right) = 0. \tag{57}$$

Roots : 1) stationary solutions $\omega = 0 \leftrightarrow$ linearized PV conservation : $\partial_t q_L = 0,\ q_L = v_x - u_y - \frac{f}{H_0}\eta \Rightarrow$ vortex motions.

2) propagative waves

$$\omega = \sqrt{gH_0\mathbf{k}^2 + f^2} \geq f. \tag{58}$$

Dispersion diagram in 1-layer RSW



Dispersion relation for inertia-gravity waves. $c = \sqrt{gH_0} = 1$, $f = 1$, $\omega < 0$ not shown. $\omega = 0$ also displayed to illustrate the spectral gap between wave and vortex motions.

Linear waves in Green-Naghdi equations on the f-plane

Linearized GN system

$$\begin{cases} \partial_t \boldsymbol{v}_h + f\hat{\boldsymbol{z}} \wedge \boldsymbol{v}_h + \boldsymbol{\nabla}_h \left(g\eta + \frac{H_0}{3}\partial^2_{tt}\eta \right) = 0, \\ \partial\eta + H_0 \boldsymbol{\nabla}_h \cdot \boldsymbol{v}_h = 0, \end{cases} \tag{59}$$

Dispersion relation

$$\omega \left(\omega^2 - \left(g - \frac{H_0}{3}\omega^2 \right) H_0 \mathbf{k}^2 - f^2 \right) = 0 \Rightarrow \tag{60}$$

Vortex motions with $\omega = 0$ and dispersive waves with frequency bounded from above and below, like in PE :

$$\omega = \pm \sqrt{\frac{gH_0\mathbf{k}^2 + f^2}{1 + \frac{H_0}{3}\mathbf{k}^2}}, \quad f \leq |\omega| \leq \sqrt{\frac{3g}{H_0}}. \tag{61}$$

Vorticity waves on the β- plane

If $\boldsymbol{v} = (u, v)$ is non-divergent in the RSW model, it is given by a streamfunction :

$$u = -\partial_y \psi, \quad v = \partial_x \psi$$

and vorticity equation (48) on the β- plane becomes :

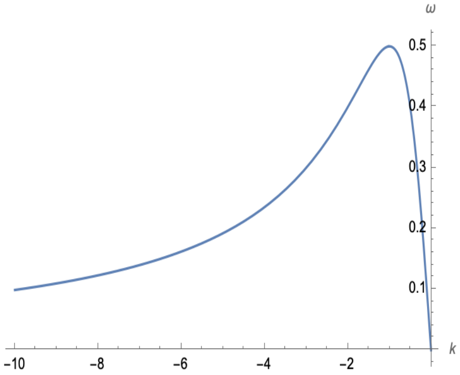
$$(\partial_t + \boldsymbol{v} \cdot \boldsymbol{\nabla})(\boldsymbol{\nabla}^2 \psi + f_0 + \beta y) = \partial_t \boldsymbol{\nabla}^2 \psi + \mathcal{J}\left(\psi, \boldsymbol{\nabla}^2 \psi\right) + \beta \partial_x \psi = 0. \tag{62}$$

Linearization $\rightarrow \partial_t \boldsymbol{\nabla}^2 \psi + \beta \partial_x \psi = 0$. S Harmonic wave solution $\psi \propto e^{i(\omega t - kx - ly)}$ exists, if

$$\omega = -\beta \frac{k}{k^2 + l^2} \Rightarrow \tag{63}$$

Eastward- (leftward) propagating Rossby waves

Dispersion relation for Rossby waves



Frequency ω as a function of zonal wavenumber k at a fixed meridional wavenumber $l \neq 0$.

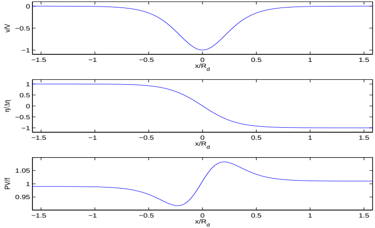
Stationary zonal jets

Stationary solution of the full nonlinear system : 1-dimensional zonal flow $u = U(y)$, $h = H(y)$, $v = 0$ in geostrophic equilibrium :

$$-f(y)U(y) = gH'(y)\,. \tag{64}$$

Jet : localized $U(y) \Rightarrow$ localized $H'(y) \Rightarrow$ pressure front.
Stationary jet orientation is arbitrary on the f - plane.
Bickley jet on the f- plane :

$$V = -\frac{g\Delta\eta}{fL}\mathrm{sech}^2\,(x/L)\,, \quad H = H_0 - \Delta\eta \tanh\,(x/L) \tag{65}$$



Axisymmetric vortices in polar coordinates

$(x, y) \to (r, \theta) \Rightarrow \mathbf{v} = u\hat{\boldsymbol{r}} + v\hat{\boldsymbol{\theta}}, \ \frac{d}{dt} = \partial_t + u\partial_r + (v/r)\partial_\theta$

RSW equations :

$$\frac{du}{dt} - \frac{v^2}{r} - fv = -g\,\partial_r h\,, \tag{66a}$$

$$\frac{dv}{dt} + \frac{uv}{r} + fu = -\frac{g}{r}\partial_\theta h\,, \tag{66b}$$

$$\partial_t h + \frac{1}{r}\partial_r(hru) + \frac{1}{r}\partial_\theta(hv) = 0\,. \tag{66c}$$

Exact solution : axisymmetric vortex with

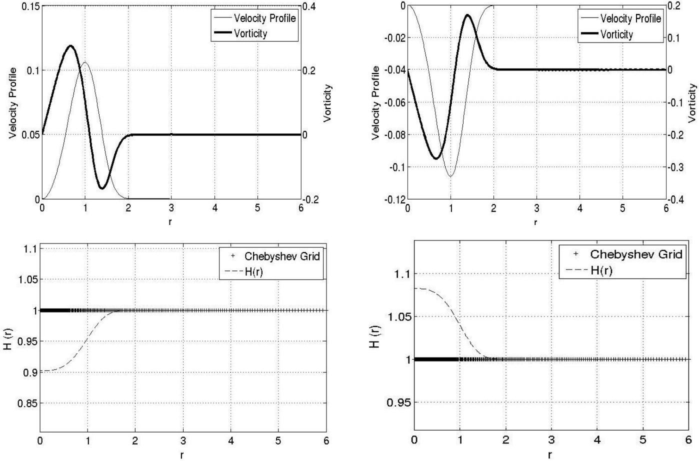
$u = 0$, $v = V(r)$, $h = H(r)$ in cyclo-geostrophic equilibrium

$$\left(\frac{V(r)}{r} + f\right)V(r) = g\,H'(r) \tag{67}$$

Analogous solution in TRSW with axisymmetric buoyancy $B(r)$ - vortex in thermo - cyclo-geostrophic equilibrium :

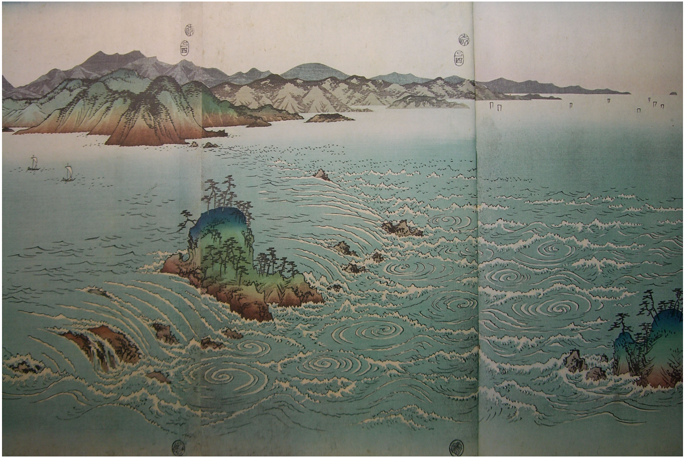
$$\left(\frac{V(r)}{r} + f\right)V(r) = B(r)\,H'(r) + \frac{H(r)}{2}\,B'(r). \tag{68}$$

Example of localized vortex



Velocity and relative vorticity (upper row), and thickness (lower row) of localized cyclonic (left column) and anticyclonic (right column) vortices. Stretched Chebyshev grid is superimposed in the lower row.

Artist's view of shallow vortex dynamics : mean flow, vortices and waves (and topography)



Linearized equation of motion

Linearisation about state of rest with thicknesses $H_{1,2}$:

$$\begin{cases} \partial_t \mathbf{v}_1 + f\hat{\mathbf{z}} \wedge \mathbf{v}_1 + \nabla(\eta_1 + \eta_2) = 0, \\ \partial_t \eta_1 + H_1 \nabla \cdot \mathbf{v}_1 = 0, \\ \partial_t \mathbf{v}_2 + f\hat{\mathbf{z}} \wedge \mathbf{v}_2 + \nabla(r\eta_1 + \eta_2) = 0, \\ \partial_t \eta_2 + H_2 \nabla \cdot \mathbf{v}_2 = 0. \end{cases} \tag{69}$$

Below $H_1 = H_2 = \frac{H}{2}$, for simplicity. Barotropic-baroclinic decomposition $\mathbf{v}^{\pm} = \sqrt{r}\mathbf{v}_1 \pm \mathbf{v}_2, \quad \eta^{\pm} = 2\left(\sqrt{r}\eta_1 \pm \eta_2\right), \rightarrow$

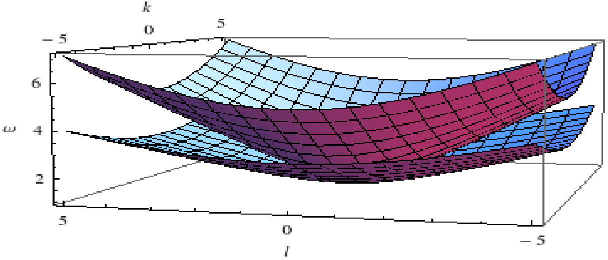
$$\begin{cases} \partial_t \mathbf{v}^+ + f\hat{z} \wedge \mathbf{v}^+ + g\frac{1+\sqrt{r}}{2}\nabla\eta^+ &= 0, \\ \partial_t \eta^+ + H\nabla \cdot \mathbf{v}^+ &= 0, \\ \partial_t \mathbf{v}^- + f\hat{z} \wedge \mathbf{v}^- + g\frac{1-\sqrt{r}}{2}\nabla\eta^- &= 0, \\ \partial_t \eta^- + H\nabla \cdot \mathbf{v}^- &= 0. \end{cases} \tag{70}$$

Two RSW subsystems with effective gravities $\frac{1\pm\sqrt{r}}{2}g \rightarrow$

$$\omega_\pm^2 = c_\pm^2 \mathbf{k}^2 + f^2, \quad c_\pm = \sqrt{gH\frac{1 \pm \sqrt{r}}{2}}, \tag{71}$$

$+$ two roots $\omega_\pm = 0$ corresponding to PV's conservation.

Inertia-gravity waves dispersion in 2-layer RSW



Dispersion relation for inertia-gravity waves in the 2-layer RSW model. $c_+ = 1$, $c_- = .3$, $f = 1$ (non-positive values $\omega \le 0$ not shown).

Stationary 2-layer zonal jets

Stationary solutions : geostrophic equilibria layer-wise for either u_i or v_i, $i = 1, 2$:

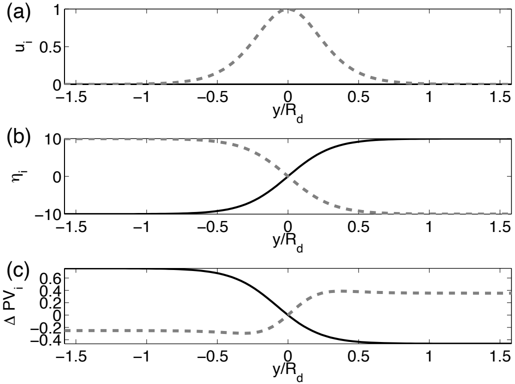
$$-fU_i(y) = g\left(r^{i-1}(H_1 + \eta_1'(y)) + H_2 + \eta_2'(y)\right), \quad i = 1, 2, \tag{72}$$

r - density ratio.

Example : upper-layer zonal atmospheric Bickley jet on the f- plane (in non-dimensional terms, α - potential temperatures ratio) :

$$U_1 = 0, \quad \eta_1 = \frac{1}{\alpha - 1}\tanh(y),$$
$$U_2 = \mathrm{sech}^2(y), \quad \eta_2 = \frac{-1}{\alpha - 1}\tanh(y). \tag{73}$$

Upper atmospheric jet



Baroclinic upper-layer Bickley jet : Span-wise profiles of : normalised zonal velocity (upper panel) U_i, thickness anomaly $\bar{\eta}_i$ (middle panel), PV anomaly (lower layer) of the baroclinic Bickley jet. Lower (upper) layer : continuos (dashed).

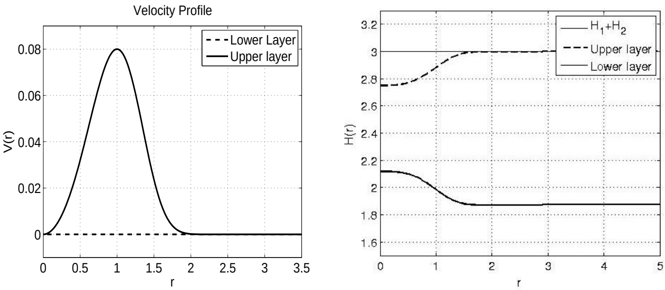
Stationary 2-layer vortices

$$\begin{cases} \frac{\partial \boldsymbol{v}_i}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}_i + \left(f + \frac{v_i}{r}\right) \hat{\boldsymbol{z}} \wedge \boldsymbol{v}_i + g\nabla(r^{i-1}h_1 + h_2) = 0, \\ \frac{\partial h_i}{\partial t} + \nabla \cdot (h_i \boldsymbol{v}_i) = 0, \quad i = 1, 2. \end{cases} \tag{74}$$

$\boldsymbol{v}_i = (u_i, v_i)$ - velocity in layer i counted from the top, h_i - thickness of layer i, and density ratio is $r = \rho_1/\rho_2 < 1$. Stationary solutions on the f-plane- cyclo-geostrophic equilibria layer-wise : $u_i = 0$, $v_i = V_i(r)$, $h_i = H_i(r)$

$$V_i \left(\frac{V_i}{r} + f \right) = -g\partial_r (r^{i-1} H_1 + H_2) \ , \quad i = 1, 2. \tag{75}$$

Upper-layer atmospheric vortex



Left : radial profiles of azimuthal velocity of the upper-layer vortex with at $s = 1.37$, *Right :* corresponding radial profiles of thickness.

Barotropic instability

Localized jet in RSW (see above) :

$\bar{u} = 0, \quad \bar{v} = -V_0 \operatorname{sech}^2(x/L), \quad h = H_0 + \bar{\eta} = H_0 - \Delta\eta \tanh(x/L),$

$V_0 = \frac{g\Delta\eta}{fL}$ - peak velocity, L -jet width, H_0 - mean height.
Small perturbations u, v, η : $||u||, ||v||, ||\eta|| \ll 1$

$$u \to u, \; v \to \bar{v} + v, \; \eta \to \bar{\eta} + \eta$$

Non-dimensional linearized equations :

$$\begin{cases} Ro\,(\partial_t u + \bar{v}\partial_y u) - v + \partial_x \eta = 0, \\ Ro\,(\partial_t v + u\partial_x \bar{v} + \bar{v}\partial_y v) + u + \partial_y \eta = 0, \\ Ro\,(\partial_t \eta + \partial_x(u\bar{\eta}) + \bar{v}\partial_y \eta + \bar{\eta}\partial_y v) + Bu\,(\partial_x u + \partial_y v) = 0. \end{cases} \tag{76}$$

$Ro = \frac{V_0}{fL}, \; Bu = \frac{R_d^2}{L^2} = \frac{gH_0}{f^2 L^2}.$

Reduction to a system of ODEs

Fourier-transform in stream-wise direction

$$(u, v, \eta) = (ik\hat{u}, \hat{v}, \hat{\eta}) \exp\{i(ky - \omega t)\} + \text{c.c.} \rightarrow$$

Eigenproblem for eigenvalues ω and eigenvectors $\boldsymbol{a} = (\hat{u}, \hat{v}, \hat{\eta})$ at fixed k :

$$\mathcal{M}\boldsymbol{a} = c\,\boldsymbol{a}$$

with

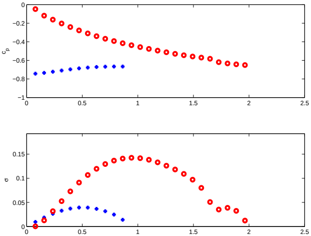
$$\mathcal{M} = \begin{pmatrix} \bar{v} & \frac{1}{Ro\,k^2} & -\frac{1}{Ro\,k^2}\partial_x \\ \frac{1}{Ro} + \partial_x \bar{v} & \bar{v} & 1 \\ (\partial_x \bar{\eta} + \bar{\eta}\partial_x) + \frac{Bu}{Ro}\partial_x & \bar{\eta} + \frac{Bu}{Ro} & \bar{v} \end{pmatrix}. \tag{77}$$

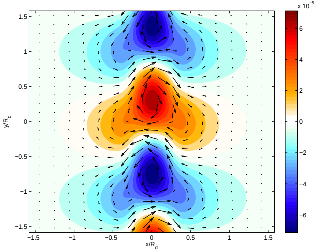
Solution : reduction to an algebraic eigenproblem by spatial discretization using Chebyshev collocation method. Complex eigenvalues $\omega = \omega_r + i\omega_i \Rightarrow$ instability with growth rate $|\omega_i|$.

Long-wave stability diagram and most unstable mode for a jet with $Ro = 0.1$, $Bu = 10$

Left : phase velocity (top) and growth rate (bottom) of long-wave unstable modes.

Right : Phase portrait (velocity and thickness anomalies) of the most unstable mode.



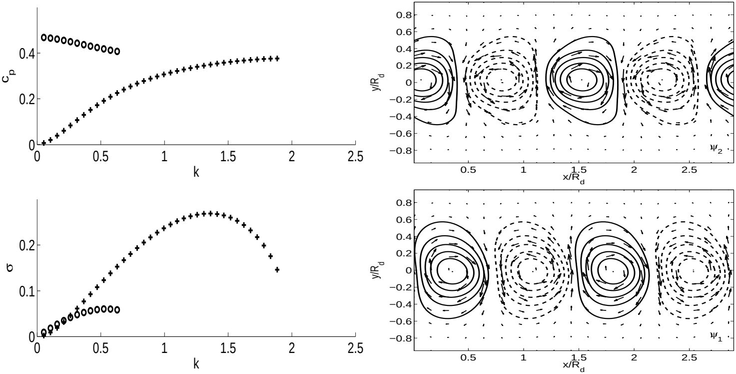


Baroclinic instability

Localized upper-layer jet in 2-layer RSW (see above)

$$\bar{u}_1 = 0, \quad \bar{\eta}_1 = \frac{1}{\alpha - 1} \tanh(y),$$
$$\bar{u}_2 = \operatorname{sech}^2(y), \quad \bar{\eta}_2 = \frac{-1}{\alpha - 1} \tanh(y).$$

Same-approach $\rightarrow$ long-wave stability diagram and most unstable mode (upper & lower layers) at $Ro = 0.1$, $Bu = 10$:



Kelvin-Helmholtz instability in 2-layer RSW

1D (no rotation) 2-layer RSW under a rigid lid :

$$\partial_t u_1 + u_1 \partial_x u_1 + \rho_1^{-1} \partial_x \pi = 0 \,, \tag{78a}$$

$$\partial_t u_2 + u_2 \partial_x u_2 + \rho_2^{-1} \partial_x \pi + g' \partial_x \eta = 0 \,, \tag{78b}$$

$$\partial_t (H_1 - \eta) + \partial_x ((H_1 - \eta) u_1) = 0 \,, \tag{78c}$$

$$\partial_t (H_2 + \eta) + \partial_x ((H_2 + \eta) u_2) = 0 \,, \tag{78d}$$

$H = H_1 + H_2 = const$, $g' = g(\rho_2 - \rho_1)/\rho_2 \equiv g \Delta\rho_2$.
Steady state $u_{1,2} = U_{1,2}$, $\eta = 0$, $\pi =$ const. Linearization :

$$D_1 u_1 + \rho_1^{-1} \partial_x \pi = 0 \,, \tag{79a}$$

$$D_2 u_2 + \rho_2^{-1} \partial_x \pi + g' \partial_x \eta = 0 \,, \tag{79b}$$

$$- D_1 \eta + H_1 \partial_x u_1 = 0 \,, \tag{79c}$$

$$D_2 \eta + H_2 \partial_x u_2 = 0 \,, \tag{79d}$$

where $D_{1,2} = \partial_t + U_{1,2} \partial_x$.

Instability criterion

Elimination of variables $\rightarrow$

$$\frac{\rho_1}{H_1} D_1^2 \eta + \frac{\rho_2}{H_2} D_2^2 \eta - g \Delta\rho \partial_{xx}^2 \eta = 0. \tag{80}$$

Fourier-transform $\eta = \bar{\eta} e^{ik(x-ct)} + \text{c.c.} \Rightarrow$ quadratic equation for phase velocity c with discriminant

$$\mathcal{D} = g\Delta\rho \left(\frac{\rho_1}{H_1} + \frac{\rho_2}{H_2} \right) - (U_1 - U_2)^2 \frac{\rho_1}{H_1} \frac{\rho_2}{H_2}, \tag{81}$$

which is negative ($\rightarrow$ imaginary eigenfrequencies $\rightarrow$ linear instability) for strong velocity shears

$$|U_1 - U_2| > \sqrt{g\Delta\rho \left(\frac{H_1}{\rho_1} + \frac{H_2}{\rho_2} \right)}.$$

1D quasi-linear systems

Quasi-linear system :

$$\partial_t V_i(x,t)+\sum_{j=1}^{N} A_{ij}\left(\vec{V}\right)\partial_x V_j(x,t) = B_i\left(\vec{V}\right),\ i=1,2,...,N. \tag{82}$$

Let $\vec{l}^{(a)}$ - left eigenvectors and $\xi^{(a)}$ - related eigenvalues, $a = 1,2,...$: $\vec{l}^{(a)} \cdot A = \xi^{(a)}\vec{l}^{(a)}$, $\Rightarrow$

$$\vec{l}^{(a)}\cdot\left(\partial_t\vec{V} + A\circ\partial_x\vec{V}\right) = \vec{l}^{(a)}\cdot\left(\partial_t\vec{V} + \xi^{(a)}\partial_x\vec{V}\right) := \vec{l}^{(a)}\cdot\dot{\vec{V}} \rightarrow \tag{83}$$

Advection along a characteristic :

$$\frac{dx}{dt} = \xi^{(a)} \Rightarrow \tag{84}$$

$$\vec{l}^{(a)}\cdot\dot{\vec{V}} = \vec{l}^{(a)}\cdot\vec{B} \tag{85}$$

- ordinary differential equations.

Hyperbolic systems :

Quasi-linear system is hyperbolic if all of N eigenvalues $\xi^{(a)}$ real and different .

Riemann invariants :

If $\vec{l}^{(a)} = const$ (or integrating factor exists) - Riemann variables (invariants if $\vec{B} = 0$) :

$$r^{(a)} = \vec{l}^{(a)} \cdot \vec{V} :, \quad \frac{dr^{(a)}}{dt} = \vec{l}^{(a)} \cdot \vec{B} \tag{86}$$

Shocks :

Intersection of characteristics $\leftrightarrow$ derivatives of Riemann invariants become infinite in finite time.

1D SW model as a hyperbolic system

Quasi - linear form of the SW equations :

$$\partial_t \begin{pmatrix} u \\ h \end{pmatrix} + \begin{pmatrix} u & 1 \\ h & u \end{pmatrix} \partial_x \begin{pmatrix} u \\ h \end{pmatrix} = 0, \qquad (87)$$

Eigenvectors and eigenvalues :

$$\vec{l}^{\pm} = (\pm\sqrt{h}, 1), \quad \xi^{\pm} = u \pm \sqrt{h}. \qquad (88)$$

Riemann invariants :

$$r^{\pm} = u \pm 2\sqrt{h}, \quad \frac{dr^{\pm}}{dt^{\pm}} = 0, \ \frac{d}{dt^{\pm}} \equiv \partial_t + \xi^{\pm}\partial_x. \qquad (89)$$

Breaking in SW model

Equation for derivatives of Riemann invariants :
$D^{\pm} \equiv \partial_x r^{\pm}$

$$\frac{dD^{\pm}}{dt^{\pm}} + \partial_x \xi^{\pm} D^{\pm} = 0, \; \xi^{\pm} = \frac{3}{4} r^{\pm} + \frac{1}{4} r^{\mp}, \Rightarrow \qquad (90)$$

$$\frac{dD^{\pm}}{dt^{\pm}} + \frac{3}{4} \left(D^{\pm}\right)^2 + \frac{1}{4} D^{\pm} D^{\mp}. \qquad (91)$$

Suppose one invariant is zero identically $\Rightarrow$ Riccati equation for remaining D along the characteristic :

$$\frac{dD}{dt} + \frac{3}{4} (D)^2 = 0, \; \rightarrow D \propto (D_I + t)^{-1} \qquad (92)$$

$\Rightarrow$ singularity in finite time.

1.5D RSW in Lagrangian variables

Change of Lagrangian labels $x = x(a)$, "straightening" initial $h_I(x) \to H = \text{const}$. Hence $J = \partial X/\partial a = H/h(X,t)$, and $g\,\partial_X h = \partial_a P$, where $P = gH/(2J^2)$ - Lagrangian pressure. Non-dimensional Lagrangian equations :

$$\begin{cases} \dot{u} - v + \partial_a P = 0, \\ \dot{v} + u = 0, \\ \dot{J} - \partial_a u = 0, \end{cases} \tag{93}$$

Quasilinear form v is not an independent variable :

$$\partial_a v = Q(a) - J \tag{94}$$

$\Rightarrow$ quasi-linear system :

$$\begin{pmatrix} \dot{u} \\ \dot{J} \end{pmatrix} + \begin{pmatrix} 0 & -J^{-3} \\ -1 & 0 \end{pmatrix} \partial_a \begin{pmatrix} u \\ J \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix}. \tag{95}$$

Eigenvectors and eigenvalues

Left eigenvectors of the advection matrix :

$$\left(1\,,\pm J^{-\frac{3}{2}}\right)$$

Left eigenvalues :

$$\mu_{\pm} = \pm J^{-\frac{3}{2}}$$

Riemann invariants : $r_{\pm} = u \pm 2J^{-\frac{1}{2}}$, obey the following equations :

$$\dot{r}_{\pm} + \mu_{\pm}\partial_a r_{\pm} = v\,. \tag{96}$$

Expressions of original variables in terms of $r_{\pm}$:

$$u = \frac{1}{2}(r_{+}+r_{-}), \quad J = \frac{16}{(r_{+}-r_{-})^2} > 0, \quad \mu_{\pm} = \pm\left(\frac{r_{+}-r_{-}}{4}\right)^3. \tag{97}$$

Wave-breaking

Equations for $D_\pm = \partial_a r_\pm$

$$\dot{D}_\pm + \mu_\pm \partial_a D_\pm + \frac{\partial \mu_\pm}{\partial r_+} D_+ D_\pm + \frac{\partial \mu_\pm}{\partial r_-} D_- D_\pm = \partial_a v = Q(\alpha) - J \,. \tag{98}$$

In terms of derivatives along the characteristics :
$\frac{dD}{dt_\pm} = \dot{D} + \mu_\pm \partial_a D$, as

$$\frac{dD_\pm}{dt_\pm} + \frac{\partial \mu_\pm}{\partial r_+} D_+ D_\pm + \frac{\partial \mu_\pm}{\partial r_-} D_- D_\pm = Q(\alpha) - J \,. \tag{99}$$

Generalized Riccati equations. Qualitative analysis of Riccati equations :

1. If initial relative vorticity $Q - J = \partial_a v$ is sufficiently negative, breaking always takes place whatever initial conditions,
2. If relative vorticity is positive, as well as the derivatives of the Riemann invariants at the initial moment, breaking never takes place.

Conservation laws and Rankine-Hugoniot conditions

Conservative form of the one-layer 1.5D RSW equations :

$$\begin{cases} (hu)_t + (hu^2 + \frac{1}{2}gh^2)_x - fhv &= 0, \\ (hv)_t + (huv)_x + fhu &= 0, \\ h_t + (hu)_x &= 0. \end{cases} \quad (100)$$

Weak solutions : conservation of mass and momentum across a discontinuity$\rightarrow$ Rankine-Hugoniot (RH) conditions

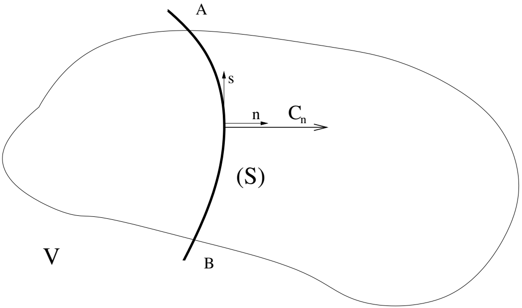
$$\begin{cases} -U\,[hu] + \left[hu^2 + gh^2/2\right] &= 0, \\ -U\,[hv] + [huv] &= 0, \\ -U\,[h] + [hu] &= 0, \end{cases} \quad (101)$$

U - speed of the discontinuity, $[A]$- jump of any quantity A across the discontinuity, following a fluid parcel : $[A] = A_{front} - A_{rear}$. Physically relevant solutions : dissipation of total energy E across the discontinuity :

$$-\,U\,[E] + \left[(E + gh^2/2)u\right] \leq 0 \text{ with } E = \frac{h}{2}(gh + u^2 + v^2), \quad (102)$$

Curved shocks

Shock in a reference frame formed by normal and tangential unit vectors $(\mathbf{n}, \mathbf{s})$, and moving at the local speed of the shock $\mathbf{n}\,\mathcal{C}_n$.



Velocity and Bernoulli function in the moving frame :

$$\bar{\mathbf{u}} = \begin{pmatrix} \bar{u}^{(n)} \\ \bar{u}^{(s)} \end{pmatrix}_{loc} = \begin{pmatrix} u^{(n)} - \mathcal{C}_n \\ u^{(s)} \end{pmatrix}_{loc} \qquad \bar{B} = gh + \bar{\mathbf{u}}^2/2 \tag{103}$$

Rankine-Hugoniot conditions in the moving frame

$$\begin{cases} -\mathcal{C}_n [h] + \left[hu^{(n)}\right] = 0, \\ -\mathcal{C}_n \left[hu^{(n)}\right] + \left[hu^{(n)2} + \frac{1}{2}gh^2\right] = 0, \\ -\mathcal{C}_n \left[hu^{(s)}\right] + \left[hu^{(n)}u^{(s)}\right] = 0. \end{cases} \tag{104}$$

Jump in $\bar{B}$ follows from combining the first and the second RH conditions (104) :

$$\left[\bar{B}\right] = -\frac{g}{4}\frac{[h]^3}{h_f h_r}. \tag{105}$$

Subscripts : f - front, r - rear.
Relation to PV jump :

$$h\bar{u}^{(n)} [q] = -\partial_s \left[\bar{B}\right]. \tag{106}$$

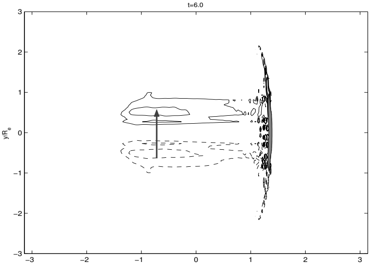
PV generation by shocks

For moving shocks of any shape :

$$[q] = \frac{g}{4h\bar{u}^{(n)}}\partial_s \frac{[h]^3}{h_f h_r}. \qquad (107)$$

Rate of vorticity change in volume V :

$$\frac{d}{dt}\int_V hq\, dV = -\int_{(S)} h\bar{u}\,[q]\, dS = \left[\bar{B}\right]_{front} - \left[\bar{B}\right]_{rear}. \qquad (108)$$



PV production behind a breaking equatorial Kelvin wave. Positive (negative) PV : continuous (dashed). Arrow : PV flux.

Atmospheric 2-layer model in one dimension without rotation

$$\begin{cases} \partial_t u_1 + u_1 \partial_x u_1 - f v_1 + g \partial_x (h_1 + h_2) = 0 \, , \\ \partial_t v_1 + u_1 (f + \partial_x v_1) = 0 \, , \\ \partial_t u_2 + u_2 \partial_x u_2 - f v_2 + g \partial_x (h_1 + \alpha h_2) = 0 \, , \\ \partial_t v_2 + u_2 (f + \partial_x v_2) = 0 \, , \\ \partial_t h_1 + \partial_x (h_1 u_1) = 0 \, , \\ \partial_t h_2 + \partial_x (h_2 u_2) = 0 \, , \end{cases} \tag{109}$$

$u_{1,2}, \; v_{1,2}$ - components of velocity in lower (1) and upper (2) layers, $h_{1,2}$ - thicknesses, $\alpha = \frac{\theta_2}{\theta_1} > 1$ - stratification parameter.

Quasi-linear form, hyperbolicity, characteristic equation

System (109) has standard quasi-linear form :

$$\partial_t \boldsymbol{f} + \mathbf{A}(\boldsymbol{f})\partial_x \boldsymbol{f} = \boldsymbol{b}(\boldsymbol{f}). \quad (110)$$

where $\boldsymbol{f} = (u_1, u_2, v_1, v_2, h_1, h_2)$,
$\boldsymbol{b} = (fv_1, -fu_1, fv_2, -fu_2, 0, 0)$, and 6×6 matrix $\mathbf{A}$ easily recovered from (109). Hyperbolicity : left eigenvalues of $\mathbf{A}$ real and different. $\Rightarrow$ propagation velocities along the characteristics $c(x, t)$.
Characteristic equation $\det(\mathbf{A} - c\mathbf{I}) = 0$, if "advective" characteristics $c = u_1$ and $c = u_2$ are discarded :

$$\left[(u_1 - c)^2 - gh_1\right]\left[(u_2 - c)^2 - \alpha g h_2\right] - g^2 h_1 h_2 = 0, \quad (111)$$

Complex solutions $\Rightarrow$ hyperbolicity loss

Linearized model

Solutions of the system linearized over the state of rest $h_{1(2)} = H_{1(2)}$ - speeds of linear gravity waves in the system :

$$c_{\pm}^2 = g(H_1 + \alpha H_2)\frac{1 \pm \sqrt{\Delta}}{2}, \qquad (112)$$

where

$$\Delta = 1 - \frac{4H_1 H_2(\alpha - 1)}{(H_1 + \alpha H_2)^2} = \frac{(H_1 - \alpha H_2)^2 + 4H_1 H_2}{(H_1 + \alpha H_2)^2}. \qquad (113)$$

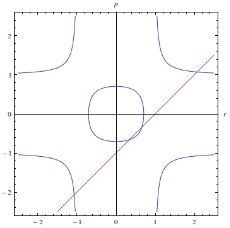
Stable stratification $\alpha > 1$, $\rightarrow 0 < \Delta < 1$ and $c_{\pm}^2 > 0$, $\Rightarrow c_k$ $(k = 1, \dots, 4)$ are real and different $\Rightarrow$ linearised system is hyperbolic. Slower solutions c_{-}^2 - baroclinic, faster solutions c_{+}^2 - barotropic waves.

Geometric proof of possible hyperbolicity loss

Define $p = \frac{u_1 - c}{\sqrt{g h_1}}$, $r = \frac{u_2 - c}{\sqrt{\alpha g h_2}}$, and rewrite characteristic equation as

$$(p^2 - 1)(r^2 - 1) = 1/\alpha \tag{114}$$

Solutions : intersections of the curve (114) and straight line $r = \sqrt{\frac{d}{\alpha}} p - \frac{u_2 - u_1}{\sqrt{\alpha g h_2}}$, where $d = \frac{h_1}{h_2}$.



Hyperbolicity criterion :

$$\frac{|u_2 - u_1|}{\sqrt{g h_1}} \leq \begin{cases} \sqrt{\frac{\alpha - 1}{d}} & \text{if} \quad d \leq \alpha, \\ \sqrt{\frac{\alpha - 1}{\alpha}} & \text{if} \quad d \geq \alpha, \end{cases} \tag{115}$$

Problem with RH conditions in 2 layers

Exchange terms with "foreign" h_i in momentum equations $\Rightarrow$ impossible to write 2-layer RSW equations (109) as pure conservation laws :

$$\partial_t f_j + \nabla \cdot \boldsymbol{F}_j(\boldsymbol{f}) = d_j(\boldsymbol{f}).$$

Only $h_{1,2}$ and the total momentum density $u_1 h_1 + u_2 h_2$ are conserved $\rightarrow$ the system of available RH conditions is incomplete $\rightarrow$ some extra *ad hoc* condition needed.
Usually : condition of energy loss in one of the layers, but no universal rule.

$$\begin{cases} -U[u_1 h_1 + u_2 h_2] + \left[u_1^2 h_1 + u_2^2 h_2 + g\frac{h_1^2}{2} + g\alpha\frac{h_2^2}{2} + g h_1 h_2\right] = 0, \\ -U[v_1 h_1] + [u_1 v_1 h_1] = 0, \\ -U[v_2 h_2] + [u_2 v_2 h_2] = 0, \\ -U[h_1] + [h_1 u_1] = 0, \\ -U[h_2] + [h_2 u_2] = 0, \end{cases} \qquad (116)$$

RSW models :

- obtained either by vertical averaging between material surfaces or from the Hamilton's principle, under approximations of columnar motion and hydrostatics
- both methods allow for "improving" RSW by relaxing the constraints of
  - horizontally uniform density/potential temperature $\rightarrow$ TRSW models
  - hydrostatics $\rightarrow$ Green-Naghdi models
- contain a specific Lagrangian invariant, PV
- display both linear waves and vortex solutions, separated by a spectral gap in the f- plane approximation
- hyperbolic, capture wave-breaking on nonlinear level
- hyperbolicity loss happens in multi-layer models when velocity shear is too strong $\rightarrow$ relation to the Kelvin-Helmholtz instability.