3. Waveguides and waveguide modes

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Plan

Coastal waveguide
- Kelvin waves in RSW with an idealized coast
- Waves in RSW with an idealized coast and a shelf
- Waves on the coastal current

Topographic waveguides
- Topographic waves

Equatorial waveguide
- Equatorial waves in RSW model
- Barotropic vs baroclinic equatorial waves

Conclusions

RSW in a right half f-plane

Linearized RSW equations with a straight "coast" at $x = 0$:

$$\begin{cases} u_t - fv + g\eta_x = 0, \\ v_t + fu + g\eta_y = 0, \\ \eta_t + H_0(u_x + v_y) = 0, \\ u|_{x=0} = 0. \end{cases} \tag{1}$$

Substituting $(u, v, \eta) = \int dld\omega\, (\bar{u}_0, \bar{v}_0, \bar{\eta}_0)e^{i(ly-\omega t)} + c.c.$ in (1) and eliminating $\bar{u}_0$, $\bar{v}_0 \to$ ODE for $\bar{\eta}_0$:

$$\bar{\eta}_0'' + (\omega^2 - f^2 - gH_0 l^2)\bar{\eta}_0 = 0. \tag{2}$$

As

$$\bar{u}_0 = i\frac{l\bar{\eta}_0 - \omega\bar{\eta}_0'}{\omega^2 - f^2}, \tag{3}$$

boundary condition for $\bar{\eta}_0(x)$ is

$$fl\bar{\eta}_0 - \omega\bar{\eta}_0'\big|_{x=0} = 0. \tag{4}$$

RSW waves in the half-plane

Solutions of two types :

- Free inertia-gravity waves :

$$\omega^2 - f^2 - gH_0 l^2 \equiv k^2 > 0, \tag{5}$$

$$\bar{\eta}_0 \propto e^{\pm ikx}, \Rightarrow \omega^2 = f^2 + gH_0(k^2 + l^2). \tag{6}$$

- Trapped boundary Kelvin waves :

$$\omega^2 - f^2 - gH_0 l^2 \equiv -\kappa^2 < 0, \tag{7}$$

$$\bar{\eta}_0 \propto e^{-\kappa x}, \Rightarrow \kappa > 0. \tag{8}$$

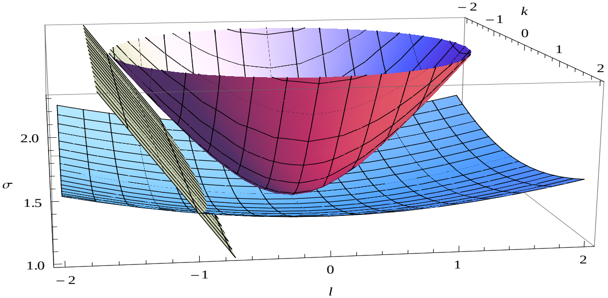
Boundary condition :

$$fl\bar{\eta}_0 - \omega\bar{\eta}_0'\big|_{x=0} = 0 \quad \Rightarrow \quad \kappa = -\frac{fl}{\omega},$$

$$\omega^2 - f^2 - gH_0 l^2 + gH_0 \frac{f^2 l^2}{\omega^2} = 0 \quad \omega^2 = gH_0 l^2,$$ no dispersion.

$$\kappa > 0 \quad \Rightarrow \quad \omega = -\sqrt{gH_0}\, l$$ – unidirectional propagation.

Dispersion diagram of the 2-layer RSW with a lateral boundary



Dispersion relation for internal-gravity and Kelvin waves in the 2-layer RSW model. Baroclinic Kelvin waves are not shown. Upper surface : barotropic inertia-gravity waves, lower surface : baroclinic inertia-gravity waves, plane : barotropic Kelvin waves.

Reflexion of inertia-gravity waves

Scaling with time in units of f^{-1}, and distances - in units of $R_d = \frac{\sqrt{gH_0}}{f}$, the deformation radius.
Any "free" wave is a sum of incident and reflected waves :

$$(u, v, \eta) = (u_i, v_i, \eta_i) + (u_r, v_r, \eta_r)$$

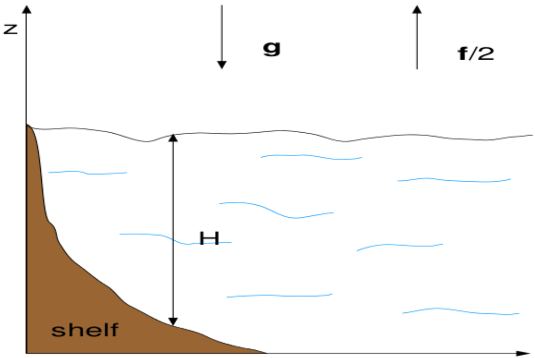
$$\begin{aligned}(u_i, v_i, \eta_i) &= A_i \left(\frac{k\omega + il}{\omega^2 - 1}, \frac{l\omega - ik}{\omega^2 - 1}, 1 \right) e^{i(kx+ly-\omega t)} + \text{c.c.}, \\ (u_r, v_r, \eta_r) &= A_r \left(\frac{-k\omega + il}{\omega^2 - 1}, \frac{l\omega + ik}{\omega^2 - 1}, 1 \right) e^{i(-kx+ly-\omega t)} + \text{c.c.}.\end{aligned}$$

Boundary condition :

$$u_i + u_r|_{x=0} = 0, \Rightarrow A_r = A_i \frac{k\omega + il}{k\omega - il}, \ \omega^2 = 1 + k^2 + l^2. \quad (9)$$

Snell's law.

Shallow-water model with a shelf.



Linearized non-dimensional equations in the presence of topography :

$$\begin{aligned} u_t - v + \eta_x &= 0, \\ v_t + u + \eta_y &= 0, \\ \eta_t + (Hu)_x + (Hv)_y &= 0. \end{aligned} \quad (10)$$

H - unperturbed depth of the fluid.

- Abrupt shelf : typical scale $L << R_d \leftrightarrow \frac{L}{R_d} = \epsilon$.
- Gentle-slope shelf : typical scale $L \sim R_d$

Fourier-transform and reduction to a single equation :
$(u, v, \eta) = (\bar{u}_0(x), \bar{v}_0(x), \bar{\eta}_0(x))e^{i(ly-\omega t)} + \text{c.c.} \rightarrow$

$$(H\bar{\eta}_0')' + (\omega^2 - 1 - l^2 H - \frac{l}{\omega}H')\bar{\eta}_0 = 0. \quad (11)$$

Abrupt shelf

Asymptotic analysis in ϵ :

- "Open-sea" domain :

$$\bar{\eta}_0^{(h)\prime\prime} + (\omega^2 - 1 - l^2)\bar{\eta}_0^{(h)} = 0. \qquad (12)$$

Solution - trapped wave : $\bar{\eta}_0^{(h)} = Ae^{-\kappa x}$, $\kappa > 0$

$$\kappa^2 = l^2 + 1 - \omega^2. \qquad (13)$$

Suppose : $\kappa = \kappa_0 + \epsilon\kappa_1 + \ldots,\ \omega = \omega_0 + \epsilon\omega_1 + \ldots.$

- "Coastal" domain :

$$\frac{1}{\epsilon^2}\left(H(\xi)\bar{\eta}_0^{(c)}(\xi)'\right)' + \left(\omega^2 - 1 - l^2 H(\xi) - \frac{1}{\epsilon}\frac{l}{\omega}H'(\xi)\right)\bar{\eta}_0^{(c)} = 0. \qquad (14)$$

$$\bar{\eta}_0^{(c)}(\xi) = \bar{\eta}^{(0)}(\xi) + \epsilon\bar{\eta}^{(1)}(\xi) + \ldots, \quad \xi = \frac{x}{\epsilon} \qquad (15)$$

Hierarchy of equations for $\bar{\eta}^{(n)}$, $n = 0, 1, \ldots$:

$$\begin{aligned} \left(H(\xi)\bar{\eta}^{(0)}(\xi)'\right)' &= 0, \\ \left(H(\xi)\bar{\eta}^{(1)}(\xi)'\right)' - \frac{l}{\omega_0} H'(\xi))\bar{\eta}^{(0)}(\xi) &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{16}$$

Order zero

$$H(\xi)\bar{\eta}^{(0)}(\xi)' = C = \text{const.} \tag{17}$$

$C \neq 0, \Rightarrow$ singularity at $x = 0$, $\quad \Rightarrow \bar{\eta}^{(0)} = \text{const.}$
Matching with the domain (h) à $x = \epsilon\xi$:

$$\bar{h}_0^{(h)} = A\left(1 - \kappa_0\epsilon\xi + \frac{1}{2}\kappa_0^2(\epsilon\xi)^2 - \epsilon^2\kappa_1\xi + \ldots\right), \Rightarrow \tag{18}$$

$\bar{\eta}^{(0)} = A.$

Order 1

$$\left(H(\xi)\bar{\eta}^{(1)}(\xi)'\right)' - \frac{l}{\omega_0}H'(\xi))A = C_1 = \text{const.} \tag{19}$$

Solution regular for $\bar{u}_0, \bar{v}_0 \quad C_1 = 0 \Rightarrow$

$$\bar{\eta}^{(1)} = \frac{l}{\omega_0}A\xi + \text{const.} \tag{20}$$

Matching of $\bar{\eta}^{(0)} + \epsilon\bar{\eta}^{(1)}$ with $\bar{h}_0^{(h)}$ à $x = \epsilon\xi$
$\Rightarrow \frac{l}{\omega_0} = -\kappa_0, \;$ const $= 0$.
Since $\kappa^2 = l^2 + 1 - \omega^2, \;\; \omega^2 \neq 1 \;\; \Rightarrow \kappa_0 = 1. \rightarrow$
Kelvin wave in the leading order.
Higher orders $\rightarrow$ corrections to the dispersion relation.

Shelf with a gentle slope : Ball's model

Ball's model : $H(x) = H_0(1 - e^{-ax})$.
Change of variables(trapped solutions) $x \to s = e^{-ax}$,
$\bar{h}_0 \to s^p \tilde{h}_0$, where p is defined by

$$\omega^2 - 1 - l^2 = -p^2 < 0, \Rightarrow \tag{21}$$

Hypergeometric equation :

$$s(1-s)\tilde{h}_0'' + [\gamma - (\alpha + \beta + 1)]\,\tilde{h}_0' - \alpha\beta\tilde{h}_0 = 0, \tag{22}$$

solutions $F(\alpha, \beta, \gamma, s)$ - hypergeometric functions,

$$\gamma = 2p+1, \quad \alpha = p + \frac{1}{2} - \sqrt{l^2 - \frac{l}{\omega} + \frac{1}{4}}, \quad \beta = p + \frac{1}{2} + \sqrt{l^2 - \frac{l}{\omega} + \frac{1}{4}}. \tag{23}$$

Trapped wave solutions

A regular at $x = 0$ and decaying at $x \to \infty$ solution $\Rightarrow$ $\alpha = -n, \; n = 0, 1, \ldots$. In this case

$$\bar{h}_0 = s^p F(-n, \beta, \gamma, s), \quad n = 0, 1, \ldots, \tag{24}$$

where

$$F(-n, \beta, \gamma, s) = \sum_{m=0}^{n} \frac{(-n)_m (\beta)_m}{(\gamma)_m m!} s^m, \quad (a)_m := a(a+1)\ldots(a+m-1) \tag{25}$$

$\alpha = -n \to$ dispersion relation :

$$p + \frac{1}{2} + n = \sqrt{l^2 - \frac{l}{\omega} + \frac{1}{4}}, n = 0, 1, \ldots. \tag{26}$$

Free wave solutions

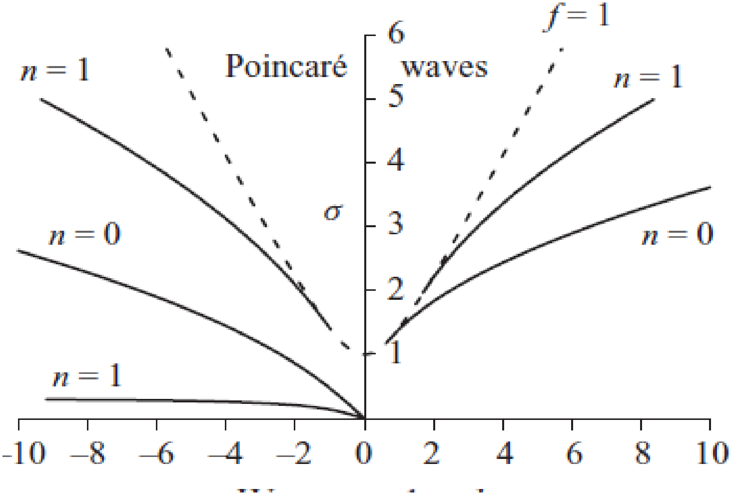
Solution for propagating (incident and reflected) Poincaré waves : $p \to ik$ in the above-displayed formulas. Solution is then given in terms of hypergeomeric functions :

$$\bar{h}_0 = A\left[e^{-ikx}F(\alpha^*, \beta^*, \gamma^*, s) - re^{ikx}F(\alpha, \beta, \gamma^*, s)\right], \quad (27)$$

A is the amplitude of the wave, * means complex conjugation, r is reflection coefficient :

$$r = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha^* + \beta^*)}{\Gamma(\alpha^*)\Gamma(\beta^*)\Gamma(\alpha + \beta)}, \; \Gamma - \text{gamma-function}. \quad (28)$$

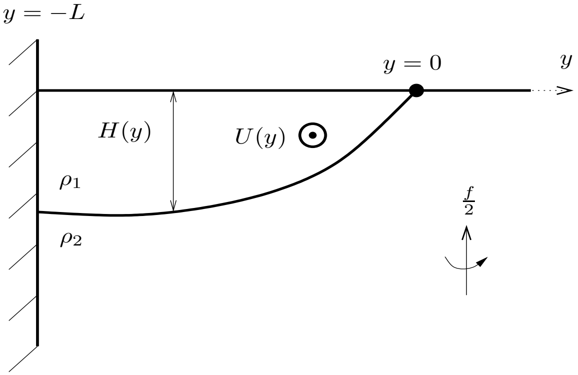
Dispersion relation for the coastal waves (n - number of nodes in the cross-coast direction



General properties of the coastal waves :

- Unique Kelvin wave,
- Discrete spectrum of sub-inertial trapped waves with $\omega < f$ (shelf waves) with unique sense of propagation (coast at their right)
- Discrete spectrum of supra-inertial trapped waves with $\omega > f$ (edge waves) with double sense of propagation
- Continuous spectrum of incident/reflected supra-inertial inertia-gravity (Poincaré) waves

Outcropping coastal density current



Outcropping ⇒ non-trivial profile of the layer thickness *H* in a steady state ⇒ non-zero mean velocity via the geostrophic balance

$$U(y) = -\frac{g}{f} H_y(y) \tag{29}$$

LInearization and boundary conditions

$$\begin{cases} u_t + Uu_x + vU_y - v = -\, h_x, \\ v_t + Uv_x + u = -\, h_y, \\ h_t + Uh_x = -(Hu_x + (Hv)_y). \end{cases} \tag{30}$$

Free-slip boundary condition at the coast : $v(-1) = 0$. The outcropping line is a material line $\Rightarrow$:

$$H(y) + h(x,y,t)|_{y=Y_0} = 0, \quad \frac{dY_0}{dt} = v\Big|_{y=Y_0} . \tag{31}$$

$y = 0$ - location of the free streamline of the mean flow, $Y_0(x,t)$ - position of the perturbed free streamline, $\frac{d}{dt}$ - Lagrangian derivative. Linearised boundary conditions :

$$Y_0 = -\frac{h}{H_y}\Big|_{y=0}, \tag{32}$$

and continuity equation evaluated at $y = 0 \Rightarrow$ the only constraint to impose on the solutions of (30) is regularity at $y = 0$.

Constant PV flows

PV of the mean flow in non-dimensional terms :

$$Q(y) = \frac{1 - U_y}{H(y)}, \quad U(y) = -H_y(y), \Rightarrow \tag{33}$$

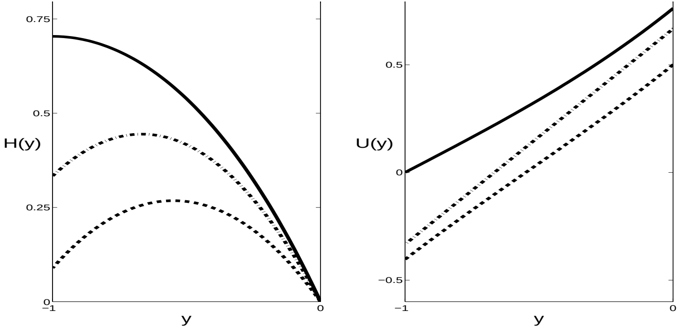
$$H_{yy}(y) - Q(y)H(y) + 1 = 0, \quad H(0) = 0, \; H_y(0) = -U_0, \tag{34}$$

$U(0) = U_0$ is the mean flow velocity at the outcropping.
Flows with constant : $Q(y) = Q_0 \neq 0$:

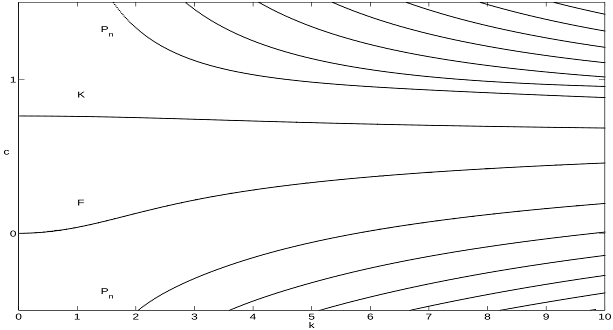
$$\begin{cases} H(y) = \frac{1}{Q_0}[1 - U_0\sqrt{Q_0}\, sinh(\sqrt{Q_0}y) - \; cosh(\sqrt{Q_0}y)], \\ U(y) = U_0\, cosh(\sqrt{Q_0}y) + \frac{1}{\sqrt{Q_0}}\, sinh(\sqrt{Q_0}y). \end{cases} \tag{35}$$

Advantage : for $(u, v, h) = (\bar{u}(y), \bar{v}(y), \bar{h}(y))e^{ik(x-ct)} + c.c.$, the wave equation does not have singularity, which is otherwise the case, at critical levels y_c : $U(y_c) - c = 0$.

Examples of constant PV flows

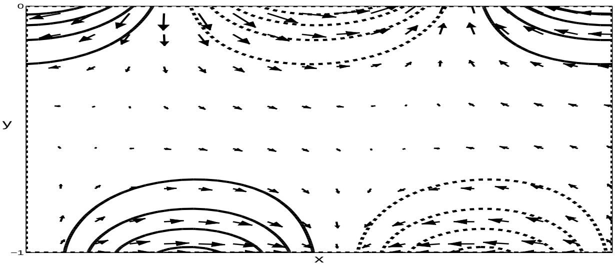


Dispersion diagram



Dispersion diagram for waves in the flow with $Q_0 = 1$. K - coastal Kelvin wave, F - frontal wave, P_n - Poincaré (inertia-gravity) wave, n - number of nodes of the mode in the span-wise direction.

Phase portraits of Kelvin and Frontal waves

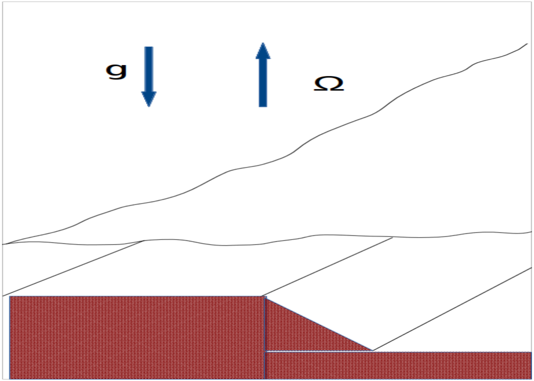


Pressure (contours) and velocity (arrows) anomalies of Kelvin (bottom) and frontal (top) waves propagating over a uniform PV flow flow with $Q_0 = 1$.

Remark :

At small enough current velocities Kelvin and Frontal wave can couple and form an unstable mode.

Escarpment topography



Wave spectrum over escarpment

Wave equation :

$$(gH\bar{h}_0')' + (\omega^2 - f^2 - l^2 gH - \frac{l}{\omega} gH')\bar{h}_0 = 0. \qquad (36)$$

At $x \to \pm\infty$ depth is constant, albeit different : $H = H_\pm =$ const. Asymptotics of $\bar{h}_{0\pm}$:

$$gH_\pm \bar{h}_{0\pm}'' + (\omega^2 - f^2 - l^2 gH_\pm)\bar{h}_{0\pm} = 0. \qquad (37)$$

Two kinds of solutions, depending on the signs of $p_\pm^2 = \omega^2 - f^2 - l^2 gH_\pm$.

- $p_\pm^2 > 0 \to$ a wave propagating to or out of escarpment,
- $p_\pm^2 < 0 \to$ trapped at the escarpment wave.

Linear escarpment

Dimensionless wave equation at the escarpment :

$$((H_m + x)\bar{h}_0')' + (\omega^2 - f^2 - l^2(H_m + x) - \frac{fl}{\omega})\bar{h}_0 = 0, \quad (38)$$

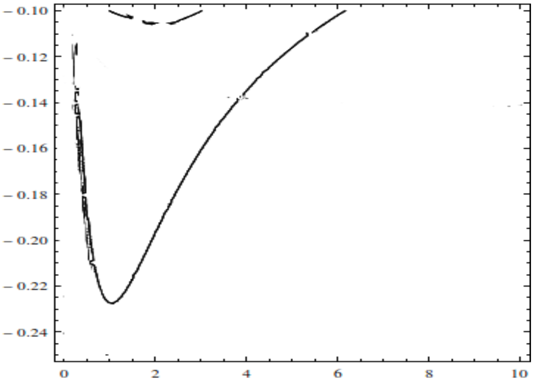
where H_m - mean depth. May be explicitly solved in terms of confluent hypergeometric functions M and U :

$$\begin{aligned} \bar{h}_0[x) &= C_1 U\left(-\frac{-fl - f^2\omega - l\omega + \omega^3}{2l\omega}, 1, 4l + 2lx\right) \\ &+ C_2 M\left(\frac{-fl - f^2\omega - l\omega + \omega^3}{2l\omega}, 1, 4l + 2lx\right), (39) \end{aligned}$$

where $C_{1,2} = \text{const}$.

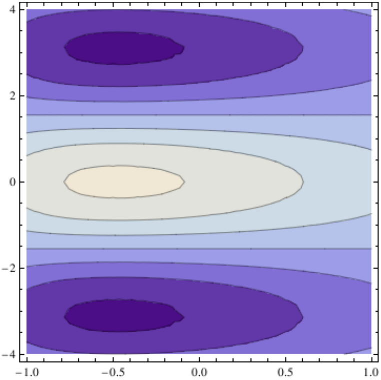
To be matched to the asymptotics $\bar{h}_0(x) = C_\pm e^{\mp\sqrt{-p_\pm^2}}$. Continuity of $\bar{h}_0$ and $\bar{h}_0'$ at $x = \pm 1$ - four homogeneous linear algebraic equations for the constants $C_\pm$, $C_{1,2}$, solvability condition → dispersion relation $\omega = \omega(l)$.

Dispersion curves for topographic waves trapped by the linear escarpment



Two lowest modes shown (respectively, zero- and one-node. Resemblance with Rossby waves ↔ same origin : gradient of background PV.

Phase portrait of the $n = 0$ mode



Isolines of h for the gravest topographic wave with maximal frequency over the escarpment region ($x \in (-1, 1)$. Trapped waves can propagate only in the negative direction along the escarpment, i.e. leaving the shallower region on their right.

RSW model on the equatorial tangent plane

Equator $\Rightarrow$ rotation of the planet is parallel to the tangent plane $\Rightarrow$ no f_0 :

$$\begin{cases} \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \beta y \, \hat{\mathbf{z}} \wedge \mathbf{v} + g \nabla h & = 0 \, , \\ \partial_t h + \nabla \cdot (\mathbf{v} h) & = 0 \, . \end{cases} \tag{40}$$

Decay boundary conditions in y (confinement in the equatorial region).

Scaling

Spatial scale - equatorial deformation radius : $L \sim \left(\frac{\sqrt{gH}}{\beta} \right)^{\frac{1}{2}}$,

Time-scale - $T \sim (\beta L)^{-1}$, Velocity scale - $U \sim \frac{g' \Delta H}{\beta L^2}$, $\Rightarrow$ - Rossby number $\epsilon = \frac{\Delta H}{H}$.

Linearized non-dimensional equations - explicit y-dependence :

$$\begin{cases} u_t - y \, v + h_x & = 0, \\ v_t + y \, u + h_y & = 0, \\ h_t + u_x + v_y & = 0. \end{cases} \tag{41}$$

Gauss - Hermite basis

Change of dependent variables

$$f = \frac{1}{2}(u+h); \quad g = \frac{1}{2}(u-h). \tag{42}$$

$$\begin{cases} f_t + f_x + \frac{1}{2}(v_y - yv) = & 0, \\ g_t - g_x - \frac{1}{2}(v_y + yv) = & 0, \\ v_t + y(f+g) + (f-g)_y = & 0, \end{cases} \tag{43}$$

appearance of operators $\partial_y \pm y$. $\exists$ a set of orthonormal functions with decay boundary conditions such that :

$$\phi_n' + y\phi_n = \sqrt{2n}\phi_{n-1}, \quad \phi_n' - y\phi_n = -\sqrt{(2n+1)}\phi_{n+1}. \tag{44}$$

Gauss-Hermite functions, H_n - Hermite polynomials

$$\phi_n(y) = \frac{H_n(y)e^{-\frac{y^2}{2}}}{\sqrt{2^n n! \sqrt{\pi}}}, \tag{45}$$

$$\phi_n''(y) + (2n+1-y^2)\phi_n(y) = 0. \tag{46}$$

Special solutions : Kelvin wave

Particular solution with $v \equiv 0$:

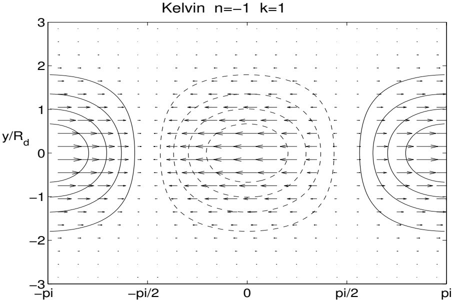
$$f_t + f_x = 0,\ g_t - g_x = 0,\ \Rightarrow f = F(x-t, y),\ g = G(x+t, y),$$

$$y(f+g) + (f-g)_y = 0,\ \Rightarrow F \propto e^{-\frac{y^2}{2}},\ G \propto e^{+\frac{y^2}{2}}.$$

Decay boundary conditions impose $G \equiv 0 \Rightarrow$

$$u = F_0(x-t)e^{-\frac{y^2}{2}};\ \ h = F_0(x-t)e^{-\frac{y^2}{2}};\ \ v = 0. \qquad (47)$$

Equatorial Kelvin wave with unique sense of propagation, eastwards, and no dispersion.



Special solutions : Yanai waves

Another particular solution with $g = 0$, $f \neq 0, v \neq 0 \Rightarrow$

$$\begin{cases} f_t + f_x + \frac{1}{2}(v_y - yv) = 0, \\ v_y + yv \quad = 0, \\ v_t + yf + f_y \quad = 0, \end{cases} \tag{48}$$

Separation of variables :

$$v = v_0(x,t)\,\phi_0(y), \quad f = F_1(x,t)\,\phi_1(y) \Rightarrow \tag{49}$$

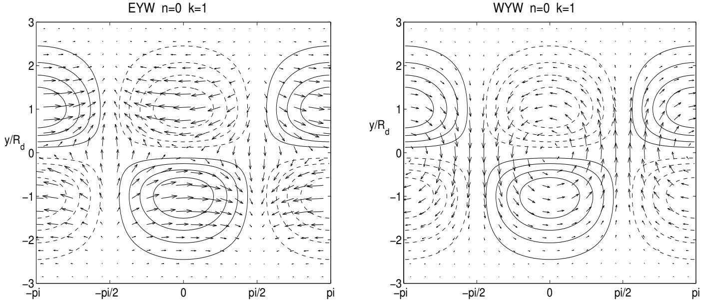
equations with constant coefficients for $F_1(x,t)$, $v_0(x,t)$:

$$F_{1_t} + F_{1_x} - \frac{1}{\sqrt{2}} v_0 = 0, \quad v_{0_t} + \sqrt{2} F_1 = 0. \tag{50}$$

Looking for wave solutions $\propto e^{i(\omega t - kx)}$ we get the dispersion relation :

$$\omega = \frac{k}{2} \pm \sqrt{\frac{k^2}{4} + 1}. \tag{51}$$

Phase portraits of Yanai waves



Pressure (contours) and velocity (arrows) distribution in the equatorial eastward- (left panel) and westward- (right panel) propagating Yanai waves with zonal wavenumber $k = 1$.

General solution : inertia-gravity and Rossby waves

Elimination of u and h (or f and g) in favour of v :

$$\partial_t \left(\nabla^2 v - y^2 v - \partial_{tt} v\right) + \partial_x v = 0. \qquad (52)$$

Expansion of v in ϕ_n : $v = \sum_n v_n(x,t)\phi_n(y)$ gives :

$$\partial_t \left[\partial^2_{xx} v_n - (2n+1)v_n - \partial^2_{tt} v_n\right] + \partial_x v_n = 0. \qquad (53)$$

After Fourier-transformation
$\tilde{v}_n(k,t) = \int dx e^{-ikx} v_n(x,t) + c.c.$ we get

$$\partial^3_{ttt}\tilde{v}_n + (k^2 + 2n + 1)\partial_t \tilde{v}_n - ik\tilde{v}_n = 0. \qquad (54)$$

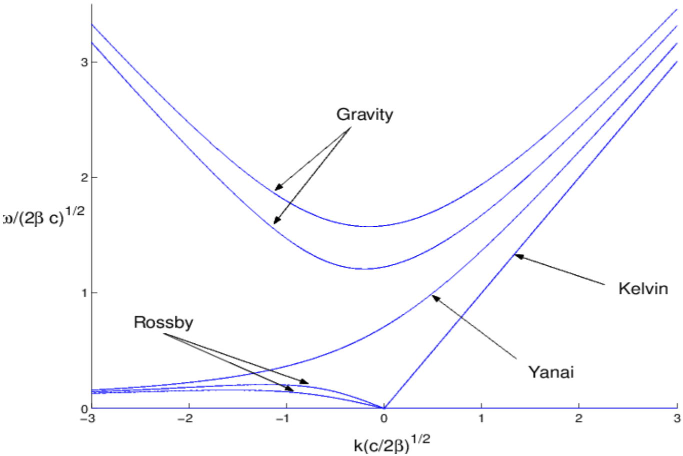
General solution

$$\tilde{v}_n = v_{n_1}(k)e^{-i\omega_{n_1} t} + v_{n_2}(k)e^{-i\omega_{n_2} t} + v_{n_3}(k)e^{-i\omega_{n_3} t}, \qquad (55)$$

where $\omega_{n_\alpha},\ \alpha = 1,2,3$ are roots of the dispersion relation :

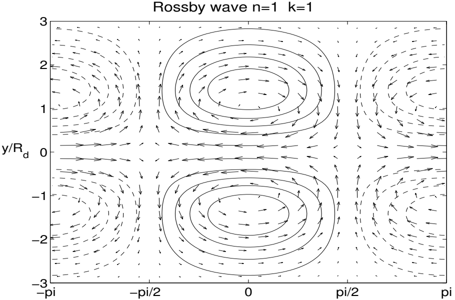
$$\omega^3_{n_\alpha} - (k^2 + 2n + 1)\omega_{n_\alpha} - k = 0. \qquad (56)$$

Dispersion diagram



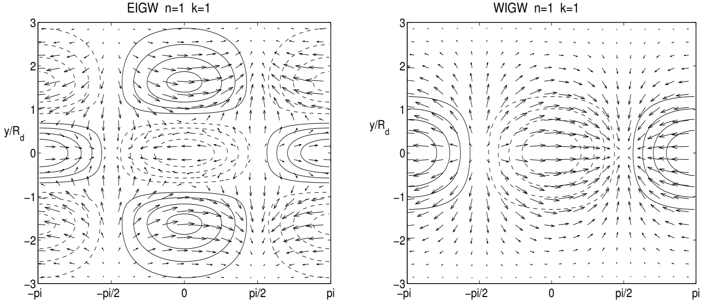
Dispersion diagram for equatorial waves in the 1-layer RSW. Only two lowest meridional modes for Rossby and inertia-gravity waves are shown.

Phase portrait of a Rossby wave



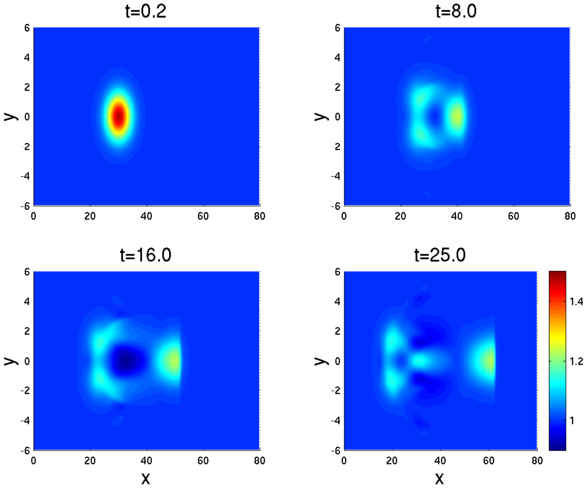
Pressure (contours) and velocity (arrows) distribution in the equatorial Rossby wave with zonal wavenumber $k = 1$.

Phase portraits of inertia-gravity waves



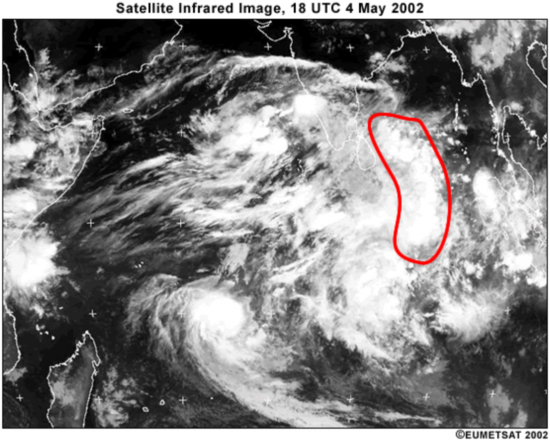
Pressure (contours) and velocity (arrows) distribution in the equatorial eastward- (left panel) and westward- (right panel) propagating inertia-gravity waves with zonal wavenumber $k = 1$.

Generation of Kelvin and Rossy waves by pressure anomaly : numerical simuations



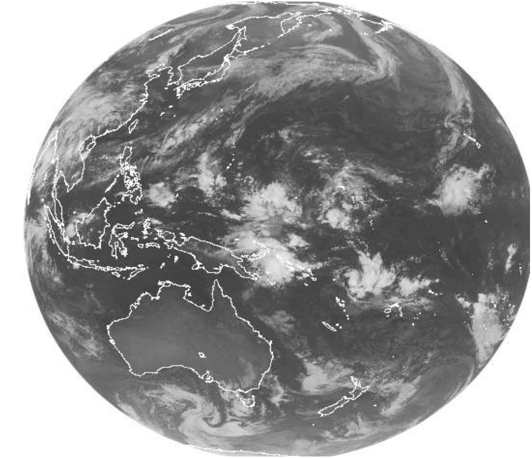
Relaxation of a pressure anomaly of large zonal scale at the equator, with formation of Rossby and Kelvin waves

Equatorial Kelvin wave in satellite observations



Kelvin front at the equator.

Equatorial Rossby wave in satellite observations



Symmetric with respect to equator twin depression visible in the cloud cover in a satellite image and associated with an equatorial Rossby wave.

2-layer RSW with a rigid lid on the equatorial β- plane

$$\partial_t \mathbf{v}_i + \mathbf{v}_i \cdot \nabla \mathbf{v}_i + \beta y \hat{\mathbf{z}} \wedge \mathbf{v}_i + \frac{1}{\rho_i} \nabla \pi_i = 0 \, , i = 1, 2; \tag{57}$$

$$\partial_t h_i + \nabla \cdot (h_i \mathbf{v}_i) = 0 \tag{58}$$

$$(\rho_2 - \rho_1) g \eta = \pi_2 - \pi_1, \;\; h_1 + h_2 = H. \tag{59}$$

Simplifying hypotheses :

- $\rho_1 \to \rho_2, \; \pi_2 = \pi_1 + \rho_1 g' h_1, \; g' = g \frac{\rho_2 - \rho_1}{\rho_1}$
- $H_1 = H_2$

Barotropic/baroclinic decomposition :

$$\mathbf{v}_{bt} = \frac{h_1 \mathbf{v}_1 + h_2 \mathbf{v}_2}{H}, \quad \mathbf{v}_{bc} = \mathbf{v}_1 - \mathbf{v}_2 \tag{60}$$

Barotropic streamfunction :

$$h_1 + h_2 = \text{const} \Rightarrow$$

$$\nabla \cdot (h_1 \mathbf{v}_1 + h_2 \mathbf{v}_2) = H \nabla \cdot \mathbf{v}_{bt} = 0 \Rightarrow \mathbf{v}_{bt} = \hat{\mathbf{z}} \wedge \nabla \psi$$

Equatorial scaling with $g \to g' \Rightarrow$

Non-dimensional linearized equations for ψ, $\mathbf{v}_{bc} = (u, v)$, η

$$\begin{aligned} \nabla^2 \psi_t + \psi_x &= 0, \\ \mathbf{v}_t + \nabla h + y \hat{\mathbf{z}} \times \mathbf{v} &= 0, \\ h_t + \nabla \cdot \mathbf{v} &= 0. \end{aligned} \tag{61}$$

Wave solutions

- "Free" barotropic Rossby waves

$$\psi_0 = A_\psi e^{i(\theta + ly)} + c.c.;\ \theta = kx - \omega t, \qquad (62)$$

with dispersion relation

$$\omega = -k/(k^2 + l^2), \qquad (63)$$

- "Trapped" baroclinic waves :

$$(u, v, \eta) = (iU_n, V_n, iH_n)\, Ae^{i\theta_n} + c.c.;\ \theta_n = kx - \omega_n t \qquad (64)$$

with dispersion relation

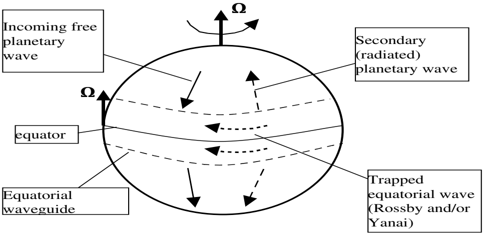
$$\omega_n^3 - (k^2 + 2n + 1)\omega_n - k = 0;\ n = -1, 0, 1, 2, \dots, \qquad (65)$$

- Kelvin, Yanai, Rossby, Inertia-Gravity

Waveguide transparent for barotropic waves

Equatorial waveguide and planetary waves

Interaction free planetary waves -trapped equatorial waves



First conclusions

- Variety of large-scale waveguides : coastal/topographic, mean current, equatorial with corresponding waveguide modes
- Waveguide modes include : weakly dispersive (non-dispersive in hydrostatic approximation) Kelvin waves, strongly dispersive Rossby modes
- Wavegude modes in coastal and equatorial waveguides partially fill in spectral gap $\Rightarrow$ care needed in identifying slow motions as vortex ones.
- Linear waveguide modes coexist with free-waves - possibility of interactions at nonlinear level (semi-transparent wavegudes).
- Breaking and front formation expected for non-dispersive Kelvin waves at nonlinear level.