4. Getting rid of fast waves: Slow dynamics

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Plan

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Horizontal motion in Hydrostatic Primitive Equations

$$\frac{\partial \boldsymbol{v}_h}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}_h + f \hat{z} \wedge \boldsymbol{v}_h = -\boldsymbol{\nabla}_h \Phi. \quad (1)$$

$$f = f_0(1 + \beta y), \quad \Phi = \Phi_0 + \phi = g(H_0 + h) \quad (2)$$

h - geopotential (perturbation) height.

Scaling for vortex-like motions

- Velocity $\boldsymbol{v}_h = (u, v)$, $u, v \sim U$, $w \sim W << U$
- Unique horizontal spatial scale L,
- Vertical scale $H << L$,
- Time-scale : turn-over time $T \sim L/U$.

Characteristic parameters

Intrinsic scale of the system : deformation (Rossby) radius :

$$R_d = \frac{\sqrt{gH_0}}{f_0} \quad (3)$$

- Rossby number : $Ro = \frac{U}{f_0 L}$,
- Burger number : $Bu = \frac{R_d^2}{L^2}$,
- Characteristic non-linearity : $\lambda = \Delta H / H_0$, where ΔH is the typical value of h,
- Dimensionless gradient of f : $\tilde{\beta} \sim \beta L$

Non-dimensional equations of horizontal motion

$$Ro\,(\partial_t \mathbf{v}_h + \mathbf{v}\cdot\nabla\mathbf{v}_h) + (1+\tilde{\beta})\hat{\mathbf{z}}\wedge\mathbf{v}_h = -\frac{\lambda Bu}{Ro}\nabla_h h\,, \qquad (4)$$

Geostrophic equilibrium

Equilibrium between the Coriolis force and the pressure force $\rightarrow$ geostrophic wind :

$$\hat{\mathbf{z}}\wedge\mathbf{v}_g = -\boldsymbol{\nabla} h \qquad (5)$$

Conditions of realization :

- $Ro \rightarrow 0$,
- $\lambda\ Bu \sim Ro$,
- $\tilde{\beta} \rightarrow 0$.

Non-dimensional RSW equations

$$Ro\,(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) + (1+\tilde{\beta}y)\hat{\mathbf{z}}\wedge\mathbf{v} = -\frac{\lambda Bu}{Ro}\nabla\eta\,, \quad (6)$$

$$\lambda\partial_t\eta + \nabla\cdot(\mathbf{v}(1+\lambda\eta)) = 0\,. \quad (7)$$

Large-scale regimes close to geostrophy : $Ro \equiv \epsilon \ll 1$

- Quasi-geostrophic(QG) : weak non-linearity :

$$\lambda \sim Ro, \Rightarrow Bu \sim 1, \Rightarrow L \sim R_d,\ \tilde{\beta} \sim Ro \quad (8)$$

- Frontal geostrophic (FG) : strong non-linearity :

$$\lambda \sim 1, \Rightarrow Bu \sim Ro, \Rightarrow L \gg R_d,\ \tilde{\beta} \sim Ro \quad (9)$$

Equations of motion of RSW in QG regime

$$\epsilon\left(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}\right) + (1+\epsilon y)\hat{\mathbf{z}}\wedge\mathbf{v} = -\nabla\eta\,, \qquad (10)$$

$$\epsilon\partial_t\eta + \nabla\cdot\left(\mathbf{v}(1+\epsilon\eta)\right) = 0\,. \qquad (11)$$

Asymptotic expansions :

$$\mathbf{v} = \mathbf{v}^{(0)} + \epsilon\mathbf{v}^{(1)} + \epsilon^2\mathbf{v}^{(2)} + \dots \qquad (12)$$

Not necessary to expand η, the velocity is slaved to pressure.
Geostrophic wind :

$$u^{(0)} = -\partial_y\eta, \quad v^{(0)} = \partial_x\eta \quad \Rightarrow \quad \partial_x u^{(0)} + \partial_y v^{(0)} = 0, \quad (13)$$

Geostrophic advection :

$$\frac{d^{(0)}}{dt}\dots = \partial_t\dots + u^{(0)}\partial_x\dots + v^{(0)}\partial_y\dots \equiv \partial_t\dots + \mathcal{J}(\eta,\dots). \qquad (14)$$

Order ϵ^1 : obtaining the QG equation

$$u^{(1)} = -\frac{d^{(0)}}{dt} v^{(0)} - y u^{(0)}, \quad v^{(1)} = \frac{d^{(0)}}{dt} u^{(0)} - y v^{(0)}, \Rightarrow \quad (15)$$

$$\partial_x u^{(1)} + \partial_y v^{(1)} = -\frac{d^{(0)}}{dt} \nabla^2 \eta - v^{(0)}, \Rightarrow \quad (16)$$

QG equation :

$$\frac{d^{(0)}}{dt} \left(\eta - \nabla^2 \eta\right) - \partial_x \eta = 0 \Leftrightarrow \frac{d^{(0)}}{dt} \left(\eta - \nabla^2 \eta - y\right) = 0. \quad (17)$$

QG equation with restored dimensions

$$\frac{d^{(0)}}{dt} \left(\frac{1}{R_d^2} h - \nabla^2 h - \frac{1}{R_d} (1 + \beta y) \right) = 0. \quad (18)$$

f- plane :

$$\frac{d^{(0)}}{dt} \left(\frac{1}{R_d^2} h - \nabla^2 h \right) \Rightarrow \text{no waves} \quad (19)$$

QG versus 2D Euler equations

From 2D Euler to vorticity equation

2D Euler equations for constant-density incompressible fluid :

$$\frac{d_h}{dt}\boldsymbol{v}_h = \partial_t \boldsymbol{v}_h + \boldsymbol{v}_h \cdot \boldsymbol{\nabla} \boldsymbol{v}_h = -\boldsymbol{\nabla}_h \Phi, \quad \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0, \qquad (20)$$

with $\Phi = \rho^{-1} P$. Introducing streamfunction : $\boldsymbol{v}_h = \hat{\boldsymbol{z}} \wedge \boldsymbol{\nabla}_h \psi$ and cross-differentiating $\rightarrow$

$$\frac{d_h}{dt}\boldsymbol{\nabla}_h^2 \psi = \partial_t \boldsymbol{\nabla}_h^2 \psi + \mathcal{J}\left(\psi, \boldsymbol{\nabla}_h^2 \psi\right) = 0. \qquad (21)$$

QG vs 2D Euler

2D Euler $\Leftrightarrow$ QG with

modified vorticity-streamunction relation :

$$\zeta = \boldsymbol{\nabla}_h^2 \psi \Longrightarrow \boldsymbol{\nabla}_h^2 \psi - \frac{1}{R_d^2}\psi$$

Barotropic Rossby waves

Non-dimensional QG equation on the β- plane :

$$\partial_t \eta - \nabla^2 \partial_t \eta - \mathcal{J}(\eta, \nabla \eta) - \partial_x \eta = 0. \qquad (22)$$

Physical meaning : conservation of quasi-geostrophic PV.
Formal linearization :

$$\partial_t \eta - \nabla^2 \partial_t \eta - \partial_x \eta = 0. \qquad (23)$$

Wave solutions $\eta \propto exp^{i(kx+ly-\omega t)}$ - dispersion relation :

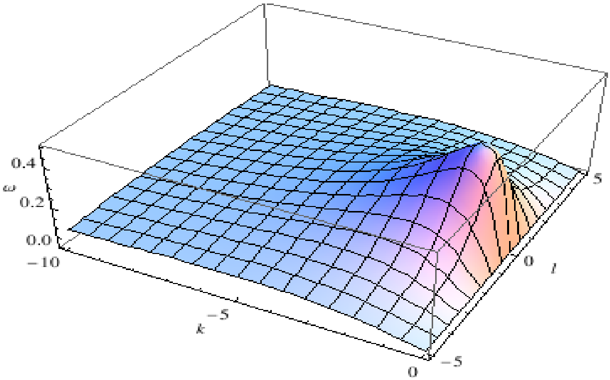
$$\omega = -\frac{k}{k^2 + l^2 + 1}. \qquad (24)$$

With restored dimensions :

$$\omega = -\beta \frac{k}{k^2 + l^2 + R_d^{-2}}. \qquad (25)$$

Rossby waves : strongly dispersive, anisotropic dispersion.

Dispersion diagram for barotropic Rossby waves



Phase velocity negative (westward propagation), group velocity negative for long and positive for short waves

Scaling and non-dimensional TRSW equations

Same scaling and hypotheses as in QG RSW + buoyancy :
$(u, v) \sim U, \; h \sim H_0 \, (1 + (Ro/Bu)\eta)$,
$b \sim B_0 \, (1 + 2(Ro/Bu)\mathcal{B})$.
$Bu = \mathcal{O}(1)$ and $Ro = \epsilon = \ll 1$.
Non-dimensional TRSW equations :

$$\begin{cases} \epsilon \left(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}\right) + (1 + \epsilon y)\hat{\mathbf{z}} \wedge \mathbf{v} = \\ -(1 + 2\epsilon\mathcal{B})\nabla\eta - (1 + \epsilon\eta)\nabla\mathcal{B} \, , \\ \epsilon\partial_t\eta + \nabla \cdot (\mathbf{v}(1 + \epsilon\eta)) = 0 \, , \\ \partial_t\mathcal{B} + \mathbf{v} \cdot \nabla\mathcal{B} = 0 \, . \end{cases} \tag{26}$$

Asymptotic expansion in ϵ, leading order :

$$\hat{\mathbf{z}} \wedge \mathbf{v}^{(0)} = -\nabla(\eta + \mathcal{B}), \;\; \Rightarrow \nabla \cdot \mathbf{v}^{(0)} = 0, \tag{27}$$

thermo-geostrophic equilibrium

TQG model

First-order : as RSW, but with thermal corrections :

$$\begin{cases} u^{(1)} = -\frac{d^{(0)}}{dt} v^{(0)} - y u^{(0)} - 2\mathcal{B}\,\partial_y \eta - \eta\,\partial_y \mathcal{B} \\ v^{(1)} = \frac{d^{(0)}}{dt} u^{(0)} - y v^{(0)} + 2\mathcal{B}\,\partial_x \eta + \eta\,\partial_x \mathcal{B}, \end{cases} \tag{28}$$

where $\frac{d^{(0)}}{dt} \cdots = \partial_t \cdots + \mathcal{J}(\psi, \ldots),\ \psi = \eta + \mathcal{B} \rightarrow$ TQG equations

$$\begin{cases} \partial_t \left(\nabla^2 \psi - \psi + \mathcal{B} + y \right) + \mathcal{J}(\psi, \nabla^2 \psi) = 0, \\ \partial_t \mathcal{B} + \mathcal{J}(\psi, \mathcal{B}) = 0. \end{cases} \tag{29}$$

Can be rewritten in terms of η instead of ψ.

RSW model with 2 layers with a rigid lid.

Equations of horizontal motion layerwise

$$\partial_t \mathbf{v}_i + \mathbf{v}_i \cdot \nabla \mathbf{v}_i + f\hat{\mathbf{z}} \wedge \mathbf{v}_i + \frac{1}{\rho_i} \nabla \pi_i = 0\,, i = 1, 2; \qquad (30)$$

Conservation of mass layer-wise

$$\partial_t (H_i - (-1)^{i+1}\eta) + \nabla \cdot \left(\mathbf{v}_i (H_i - (-1)^{i+1}\eta)\right) = 0\,, i = 1, 2; \qquad (31)$$

$H_i, i = 1, 2$ - non-perturbed thicknesses of the layers, $H_1 + H_2 = H$, η - position of the interface, $i + 1$ - modulo 2.

Dynamical boundary condition at the interface

$$(\rho_2 - \rho_1) g \eta = \pi_2 - \pi_1\,. \qquad (32)$$

Conservation laws

Conservation of PV layer-wise

$$(\partial_t + \mathbf{v}_i \cdot \nabla)\, q_i = 0, \quad q_i = \frac{\zeta_i + f}{H_i - (-1)^{i+1}\eta}, \tag{33}$$

where $\zeta_i = \hat{\mathbf{z}} \cdot \nabla \wedge \mathbf{v}_i$ relative vorticity in the layer i.

Conservation of energy

$$E = \int dxdy \left(\sum_{i=1,2} \rho_i (H_i - (-1)^{i+1}\eta) \frac{\mathbf{v}_i^2}{2} + (\rho_2 - \rho_1) g \frac{\eta^2}{2} \right) \tag{34}$$

First term - kinetic, second - available potential energy.

Scales and parameters

Characteristic scales

- Typical horizontal velocity : U
- Typical horizontal scale : L
- Time-scale : $T \sim L/U$ - turn-over time
- Pressure scale layerwise : $P_i \sim \rho_i U L f_0$
- Typical vertical scale : H; $D_i = \frac{H_i}{H}$

Parameters

- Rossby number : $Ro = \frac{U}{f_0 L}$
- Typical dimensionless deviation of the interface : λ
- Dimensionless gradient of the Coriolis parameter : $\tilde{\beta}$
- Aspect ratio : $d = \frac{H_1}{H_2}$
- Stratification parameter : $N = \frac{\rho_2 - \rho_1}{\rho_2} = 1 - r$
- Burger number : $Bu = \frac{R_d^2}{L^2}$, $R_d^2 = \frac{NgH}{f_0^2}$

Baroclinic deformation radius : $R_d^2 = \frac{g' H}{f_0}$, $g' = gN$ - reduced gravity.

Non-dimensional equations

$$\epsilon \frac{d_i}{dt}\mathbf{v}_i + (1+\tilde{\beta} y)\hat{z} \wedge \mathbf{v}_i = -\boldsymbol{\nabla}\pi_i, \quad i = 1, 2. \tag{35}$$

$$\begin{aligned} -\lambda \frac{d_1}{dt}\eta + (D_1 - \lambda\eta)\boldsymbol{\nabla}\cdot\mathbf{v}_1 &= 0 \\ \lambda \frac{d_2}{dt}\eta + (D_2 + \lambda\eta)\boldsymbol{\nabla}\cdot\mathbf{v}_2 &= 0 \end{aligned} \tag{36}$$

$$\pi_2 - r\pi_1 = \frac{\lambda Bu}{\epsilon}\eta. \tag{37}$$

$$\frac{d_i}{dt} = \partial_t + \mathbf{v}_i \cdot \boldsymbol{\nabla} \tag{38}$$

QG regime

$$\lambda \sim \tilde{\beta} \sim \epsilon \ll 1, \quad \Rightarrow \quad L \sim R_d \tag{39}$$

Asymptotic expansion in $\epsilon \Rightarrow$

$$\begin{aligned} u_i &= u_i^{(0)} - \epsilon \left[\partial_t v_i^{(0)} + \mathcal{J}(\pi_i, v_i^{(0)}) + y u_i^{(0)} \right] + \dots \\ v_i &= v_i^{(0)} + \epsilon \left[\partial_t u_i^{(0)} + \mathcal{J}(\pi_i, u_i^{(0)}) - y v_i^{(0)} \right] + \dots \end{aligned} \tag{40}$$

Geostrophic wind :

$$u_i^{(0)} = -\partial_y \pi_i, \quad v_i^{(0)} = \partial_x \pi_i \Rightarrow \partial_x u_i^{(0)} + \partial_y v_i^{(0)} \equiv 0. \tag{41}$$

Divergence of first-order velocity

$$\partial_x u_i^{(1)} + \partial_y v_i^{(1)} = - \left[\partial_t \nabla^2 \pi_i + \mathcal{J}(\pi_i, \nabla^2 \pi_i) + \partial_x \pi \right] \tag{42}$$

2-layer QG equations

Mass conservation layer-wise :

$$\partial_t \eta + \mathcal{J}(\pi_i, \eta) - (-1)^i D_i \left[\partial_t \boldsymbol{\nabla}^2 \pi_i + \mathcal{J}(\pi_i, \boldsymbol{\nabla}^2 \pi_i) + \partial_x \pi \right] = 0, \rightarrow$$

Equations for the pressures in the layers from boundary condition at the interface.

$$\frac{d_i^{(0)}}{dt} \left[\nabla^2 \pi_i - (-1)^i D_i^{-1} \eta + y \right] = 0\,,\ i = 1, 2. \qquad (43)$$

where

$$\frac{d_i^{(0)}}{dt} (\ldots) := \partial_t (\ldots) + J(\pi_i, \ldots)\,,\ i = 1, 2 \qquad (44)$$

Frequent hypothesis : weak stratification $\rho_2 \to \rho_1 \Rightarrow$
$\eta = \pi_2 - \pi_1$

Baroclinic and barotropic components of pressure :

$\eta = \pi_2 - \pi_1$ - baroclinic ; $\Pi = D_1 \pi_1 + D_2 \pi_2$ - barotropic.

- $\eta = 0$ - motion (velocity) identical in both layers
- $\Pi = 0$ - motion (velocity) opposite in the layers

Formal linearization of the 2-layer QG model :

$$\begin{aligned} \partial_t \left[\nabla^2 \pi_1 + D_1^{-1}(\pi_2 - \pi_1)\right] + \partial_x \pi_1 &= 0 \\ \partial_t \left[\nabla^2 \pi_2 - D_2^{-1}(\pi_2 - \pi_1)\right] + \partial_x \pi_2 &= 0 \end{aligned} \quad (45)$$

Wave solutions : $\pi_i = A_i e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$.

Condition of solvability :

$$\det \begin{pmatrix} \omega(\mathbf{k}^2 + D_1^{-1}) + k_x & -\omega D_1^{-1} \\ -\omega D_2^{-1} & \omega(\mathbf{k}^2 + D_2^{-1}) + k_x \end{pmatrix} = 0. \quad (46)$$

Dispersion relation :

$$\begin{aligned} \omega &= -\frac{k_x}{2\mathbf{k}^2(\mathbf{k}^2 + D_1^{-1} + D_2^{-1})} \left[(2\mathbf{k}^2 + D_1^{-1} + D_2^{-1})\right. \\ &\pm \left.(D_1^{-1} + D_2^{-1})\right] \end{aligned} \quad (47)$$

- Barotropic mode : $\omega_{bt} = -\frac{k_x}{\mathbf{k}^2}$ - faster.
- Baroclinic mode : $\omega_{bc} = -\frac{k_x}{(\mathbf{k}^2 + D_1^{-1} + D_2^{-1})}$ - slower.

Scaling and parameters

Characteristic scales

- Typical horizontal velocity : U
- Typical horizontal scale : L
- Time-scale : $T \sim L/U$ -turn-over time
- Typical vertical scale : H
- Typical vertical velocity : W $\frac{W}{H} \sim \lambda \frac{U}{L}$ -to confirm *aposteriori*
- Pressure scale : $\rho_0 g H$

Parameters

- Rossby number : $Ro = \frac{U}{f_0 L}$
- Typical dimensionless deviation of the isopycnal surfaces : λ
- Dimensionless gradient of the Coriolis parametre : $\tilde{\beta}$
- Stratification parametre : $N = \frac{\text{variable part of density}}{\text{constant part of density}}$
- Burger number : $Bu = \frac{R_d^2}{L^2}$, where R_d baroclinic deformation radius with reduced gravity $g' = Ng$.

Pressure and density related via hydrostatics :

$$
\begin{aligned}
\rho &= \rho_0 \left[1 + N\left(\rho_s(z) + \lambda\sigma(x, y, z; t)\right)\right], \Rightarrow \\
P &= \rho_0 g H \left[(1 - z) + N\left(p_s(z) + \lambda\pi(x, y, z; t)\right)\right] \quad (48)
\end{aligned}
$$

Non-dimensional primitive equations :

$$\epsilon \frac{d}{dt}\mathbf{v}_h + (1 + \tilde{\beta} y)\hat{z} \wedge \mathbf{v}_h = -\boldsymbol{\nabla}_h \pi. \quad (49)$$

$$\frac{d}{dt}\sigma + \rho_s' w = 0, \quad \partial_z \pi + \sigma = 0. \quad (50)$$

$$\boldsymbol{\nabla}_h \cdot \mathbf{v}_h + \lambda \partial_z w = 0; \quad (51)$$

$$\frac{d}{dt} = \partial_t + \mathbf{v}_h \cdot \boldsymbol{\nabla}_h + \lambda w \partial_z \quad (52)$$

Boundary conditions - rigid lid/flat bottom :

$$w|_{z=0,1} = 0. \quad (53)$$

QG regime. Asymptotic expansion order by order

$$\lambda \sim \tilde{\beta} \sim \epsilon \ll 1, \quad \Rightarrow \quad L \sim R_d, \; \frac{W}{H} = \lambda \frac{U}{L}. \qquad (54)$$

Order ϵ^0

$$u^{(0)} = -\partial_y \pi, \quad v^{(0)} = \partial_x \pi, \Rightarrow \partial_x u^{(0)} + \partial_y v^{(0)} = 0, \Rightarrow \partial_z w = 0. \qquad (55)$$

Consistent with the choice of the scale W.

Thermal wind

Geostrophic + hydrostatic equilibria :

$$u = -\partial_y \pi, \; v = \partial_x \pi, \; \sigma = -\partial_z \pi \Rightarrow \partial_z v = -\partial_x \sigma, \; \partial_z u = +\partial_y \sigma \qquad (56)$$

Horizontal density gradient $\leftrightarrow$ vertical shear of the horizontal wind. Atmosphere : $\sigma \to -\theta$.

Order ϵ^1

$$u^{(1)} = -\frac{d^{(0)}}{dt} v^{(0)} - yu^{(0)}, \quad v^{(1)} = \frac{d^{(0)}}{dt} u^{(0)} - yv^{(0)}, \Rightarrow \quad (57)$$

$$\partial_x u^{(1)} + \partial_y v^{(1)} = -\frac{d^{(0)}}{dt} \nabla_h^2 \pi - \partial_x \pi \equiv -\frac{d^{(0)}}{dt} \left(\nabla_h^2 \pi + y \right), \quad (58)$$

where $\frac{d^{(0)}}{dt} \cdots = \partial_t \cdots + \mathcal{J}(\pi, \dots)$ - horizontal geostrophic advection.

Elimination of w

$$w^{(0)} = -\frac{1}{\rho_s'(z)} \frac{d^{(0)}}{dt} \sigma = \frac{1}{\rho_s'(z)} \frac{d^{(0)}}{dt} \partial_z \pi \quad (59)$$

Continuity equation :

$$
\begin{aligned}
-\frac{d^{(0)}}{dt}\left(\nabla_h^2\pi + y\right) + \partial_z\left(\frac{1}{\rho_s'(z)}\frac{d^{(0)}}{dt}\partial_z\pi\right) &= 0 \Rightarrow \\
\frac{d^{(0)}}{dt}\left(-\nabla_h^2\pi - y + \partial_z\left(\frac{1}{\rho_s'(z)}\partial_z\pi\right)\right) &= 0, \qquad (60)
\end{aligned}
$$

$$
\text{c.l. : } w|_{z=0,1} = 0 \Rightarrow \left.\frac{d^{(0)}}{dt}\partial_z\pi\right|_{z=0,1} = 0. \qquad (61)
$$

Physical meaning

Lagrangian conservation of quasi-geostrophic PV

$$
PVQG = -\nabla_h^2\pi - y + \partial_z\left(\frac{1}{\rho_s'(z)}\partial_z\pi\right) \qquad (62)
$$

+ advection of density perturbation along the boundaries.

Baroclinic Rossby waves : continuous stratification

Formal linearization

$$\partial_t \left[\nabla_h^2 \pi - \partial_z \left(\frac{1}{\rho_s'(z)} \partial_z \pi \right) \right] + \partial_x \pi = 0, \quad \partial_{tz}^2 \pi \big|_{z=0,1} = 0. \tag{63}$$

Separation of variables

$$\pi(x, y, z; t) = p(x, y; t) S(z) \Rightarrow \tag{64}$$

$$\partial_t \nabla_h^2 p(x, y; t) S(z) - \partial_t p(x, y; t) \left[\frac{1}{\rho_s'(z)} S'(z) \right]' + \partial_x p(x, y; t) S(z) = 0 \Rightarrow$$

$$\frac{1}{S(z)} \left[\frac{1}{\rho_s'(z)} S'(z) \right]' = \kappa^2 - \text{separation constant} \tag{65}$$

$$\partial_t \nabla_h^2 p(x, y; t) - \kappa^2 \partial_t p(x, y; t) + \partial_x p(x, y; t) = 0, \tag{66}$$

Vertical modes

Sturm - Liouville problem :

$$\left[\frac{1}{\rho_s'(z)}S'(z)\right]' - \kappa^2 S(z) = 0, \quad S'(z)\big|_{z=0,1} = 0 \qquad (67)$$

Eigenfunctions $S_n(z)$ and eigenvalues $\kappa_n, n = 0, 1, 2, \ldots$.

Rossby waves : $p(x, y; t) \propto e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$:

$$\omega = -\frac{k_x}{\mathbf{k}^2 + \kappa_n^2}. \qquad (68)$$

The larger is the vertical wavenumber n (stronger vertical shear) $\rightarrow$ the slower is the propagation.

Surface quasi-geostrophic model (SQG)

f- plane and constant stable stratification $\rho_s'(z) = \text{const} < 0$ $\Rightarrow$ $PVQG = \nabla^2 \pi$ after rescaling of the vertical coordinate $\Rightarrow$ any solution of the Laplace equation gives a solution of the full problem if the b.c. are verified $\Rightarrow$ dynamics is totally defined by evolution of density on the boundary $\Leftrightarrow$ surface quasi geostrophy (SQG).

Example : Solution of the 3D Laplace equation in the upper half-plane decaying at $z \to \infty$:

$$\pi(\mathbf{x}, z, t) = \int d\mathbf{k}\, \hat{\pi}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}} e^{-|\mathbf{k}|z},$$

where $\mathbf{x} = (x,\, y)$, $\mathbf{k} = (k,\, l)$. Therefore

$$\sigma(\mathbf{x}, z, t) = \int d\mathbf{k}\, |\mathbf{k}|\, \hat{\pi}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}} e^{-|\mathbf{k}|z}$$

Setting $z = 0$ and substituting to (61) produces the integro-differential equation for $\hat{\pi}(\mathbf{k}, t)$ on the boundary.

RSW at small *Ro*, and with two time-scales

Hypotheses :

- f- plane, infinite domain,
- Unique spatial scale L,
- Small Rossby number ϵ, regime QG : $\lambda \sim \epsilon$,
- Fast $t \sim f_0^{-1}$ and slow $t_1 \sim (\epsilon f_0)^{-1}$ time-scales

Non-dimensional equations :

$$(\partial_t + \epsilon\partial_{t_1})\mathbf{v} + \epsilon(\mathbf{v}\cdot\nabla\mathbf{v}) + \hat{\mathbf{z}}\wedge\mathbf{v} + \nabla h = 0\,, \qquad (69)$$

$$(\partial_t + \epsilon\partial_{t_1})h + (1+\epsilon h)\nabla\cdot\mathbf{v} + \epsilon\mathbf{v}\cdot\nabla h = 0\,, \qquad (70)$$

$$\partial_t Q + \epsilon\mathbf{v}\cdot\nabla Q = 0\,, \quad Q = \epsilon\frac{\zeta - h}{1+\epsilon h} \text{ — PV anomaly.} \qquad (71)$$

Geostrophic adjustment

Cauchy problem with localized initial conditions

$$u\big|_{t=0} = u_I \ , \ v\big|_{t=0} = v_I \ , \ h\big|_{t=0} = h_I \ . \tag{72}$$

Multi-scale asymptotic expansions

$$\begin{aligned} \mathbf{v} &= \mathbf{v}_0(x, y; t, t_1, \ldots) + \epsilon \mathbf{v}_1(x, y; t, t_1, \ldots) + \ldots \\ h &= h_0(x, y; t, t_1, \ldots) + \epsilon h_1(x, y; t, t_1, \ldots) + \ldots, \end{aligned} \tag{73}$$

Slow - fast decomposition order by order in ϵ :

$$h_i = \bar{h}_i(x, y; t_1, \ldots) + \tilde{h}_i(x, y; t, t_1, \ldots), \ i = 0, 1, 2, \ldots \tag{74}$$

$$\bar{h}_i(x, y; t_1, \ldots) = \lim_{T \to \infty} \frac{1}{T} \int_0^T h_i(x, y, t, t_1, \ldots) \, dt, \tag{75}$$

Lowest order in ϵ : velocity

$$\partial_t \mathbf{v}_0 + \hat{\mathbf{z}} \wedge \mathbf{v}_0 = -\nabla h_0 \,, \tag{76}$$

$$\partial_t (\zeta_0 - h_0) = 0 \,, \tag{77}$$

where $\zeta_0 = \hat{\mathbf{z}} \cdot \nabla \wedge \mathbf{v}_0$ - relative vorticity, and equation for PV is used. Initial conditions :

$$u_0|_{t=0} = u_I \,,\; v_0|_{t=0} = v_I \,,\; h_0|_{t=0} = h_I \,. \tag{78}$$

Re-writing (76) in terms of relative vorticity ζ and divergence $D = \nabla \cdot \mathbf{v}_0$:

$$\partial_t \zeta_0 + D_0 = 0 \,, \tag{79}$$

$$\partial_t D_0 - \zeta_0 = -\nabla^2 h_0 \,. \tag{80}$$

Immediate integration of (77) in fast time t :

$$\zeta_0 - h_0 = \Pi_0 \,, \tag{81}$$

where Π_0 is yet unknown function of x, y, t_1 (integration "constant").

Lowest order in ϵ : height

Elimination of ζ_0 and $D_0 \rightarrow$ linear inhomogeneous equation for h_0 :

$$-\frac{\partial^2 h_0}{\partial t^2} - h_0 + \nabla^2 h_0 = \Pi_0(x, y; t_1, t_2, ...) . \qquad (82)$$

Solution : slow + fast :

$$h_0 = \tilde{h}_0(x, y; t, ...) + \bar{h}_0(x, y; t_1, ...) \qquad (83)$$

$$-\frac{\partial^2 \tilde{h}_0}{\partial t^2} - \tilde{h}_0 + \nabla^2 \tilde{h}_0 = 0 ; \qquad (84)$$

$$-\bar{h}_0 + \nabla^2 \bar{h}_0 = \Pi_0 \qquad (85)$$

Klein - Gordon (KG) and Helmholtz equations.
Π_0 : geostrophic PV constructed from the slow component $\bar{h}_0$.

Initialization problem :

How to split the i.c.for $\bar{h}_0$ in slow and fast ?

Response (unique at $\epsilon \to 0$)

- By definition :

$$\Pi_0(x, y; 0) = \partial_x v_I - \partial_y u_I - h_I \equiv \Pi_I(x, y) \qquad (86)$$

- Determination of the initial value $\bar{h}_{0I}$ of $\bar{h}_0$ by inversion :

$$-\bar{h}_{0I} + \nabla^2 \bar{h}_{0I} = \Pi_I, \Rightarrow \bar{h}_{0I} = -(\nabla^2 - 1)^{-1}\Pi_I. \qquad (87)$$

- Determination of the initial value $\tilde{h}_{0I}$ of $\tilde{h}_0$:

$$\tilde{h}_{0I} = h_I - \bar{h}_{0I}. \qquad (88)$$

- Second i.c. for $\tilde{h}_0$ (PV and ζ - D) :

$$\partial_t \tilde{h}_0\Big|_{t=0} = -D_I \equiv \partial_x u_I + \partial_y v_I\,. \qquad (89)$$

Slow/fast decomposition for velocity :

$$\mathbf{v}_0 = \tilde{\mathbf{v}}_0(x, y; t, ...) + \bar{\mathbf{v}}_0(x, y; t_1, ...) \,, \tag{90}$$

slow components verify the geostrophic relation :

$$\bar{\mathbf{v}}_0 = \hat{\mathbf{z}} \wedge \nabla \bar{h}_0 \tag{91}$$

and the fast ones obey the equations

$$\partial_t \tilde{\mathbf{v}}_0 + \hat{\mathbf{z}} \wedge \tilde{\mathbf{v}}_0 = -\nabla \tilde{h}_0 \tag{92}$$

with i.c. :

$$\tilde{u}_I^{(0)} = u_I - \bar{u}_{0I} \,; \;\; \tilde{v}_I^{(0)} = v_I - \bar{v}_{0I} \,, \tag{93}$$

where $\bar{u}_{0I}, \bar{v}_{0I}, \bar{h}_{0I}$ verify (91). Linearized PV $\tilde{\zeta}_0 - \tilde{h}_0$ of the fast component is identically zero.

Fast component : solution for h :

Inertia-gravity waves propagating out of the initial perturbation ; created by its non-balanced par $\tilde{u}_I^{(0)}, \tilde{v}_I^{(0)}, \tilde{h}_{0I}$:

$$\tilde{h}_0(\mathbf{x};t) = \sum_{\pm} \int d\mathbf{k}\, H_0^{(\pm)}(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x} \pm \Omega_{\mathbf{k}} t)}, \tag{94}$$

where

$$H_0^{(\pm)}(\mathbf{k}) = \frac{1}{2}\left(\hat{\tilde{h}}_{0I}(\mathbf{k}) \pm i\,\frac{\hat{D}_I(\mathbf{k})}{\Omega_{\mathbf{k}}}\right), \tag{95}$$

and the notation $\hat{...}$ is used for the Fourier transformations of the corresponding quantities.

Fast component : solution for $\mathbf{v}$:

KG equation for complex velocity $\mathcal{U} = u + iv$:

$$-\frac{\partial^2 \tilde{\mathcal{U}}_0}{\partial t^2} - \tilde{\mathcal{U}}_0 + \nabla^2 \tilde{\mathcal{U}}_0 = 0 \tag{96}$$

with i.c. :

$$\tilde{\mathcal{U}}_0\big|_{t=0} = \tilde{u}_I^{(0)} + i\,\tilde{v}_I^{(0)} \equiv \tilde{\mathcal{U}}_{0I}\,, \tag{97}$$

$$\partial_t \tilde{\mathcal{U}}_0\big|_{t=0} \equiv \mathcal{W}_I = -i\,\tilde{\mathcal{U}}_{0I} - \left(\partial_x \tilde{h}_{0I} + i\,\partial_y \tilde{h}_{0I}\right) \tag{98}$$

is solved by

$$\tilde{\mathcal{U}}_0(\mathbf{x}; t) = \sum_{\pm} \int d\mathbf{k}\, U_0^{(\pm)}(\mathbf{k})\, e^{i(\mathbf{k}\cdot\mathbf{x} \pm \Omega_{\mathbf{k}} t)} \tag{99}$$

$$U_0^{(\pm)}(\mathbf{k}) = \frac{1}{2}\left(\hat{\tilde{\mathcal{U}}}_{0I}(\mathbf{k}) \pm i\,\frac{\hat{\mathcal{W}}_I(\mathbf{k})}{\Omega_{\mathbf{k}}}\right) \tag{100}$$

Resumé of the first approximation

- Slow and fast components are defined unambiguously
- Fast and slow motions are separated dynamically (non-interacting)
- Fast part completely resolved : inertia-gravity waves propagating out of the initial perturbation
- Evolution of the slow part is still to determine

Next order in ϵ

Momentum equations :

$$\partial_t \mathbf{v}_1 + \hat{\mathbf{z}} \wedge \mathbf{v}_1 = -\nabla h_1 - (\partial_{t_1} + \mathbf{v}_0 \cdot \nabla)\, \mathbf{v}_0 \,. \tag{101}$$

Equation for PV in first order :

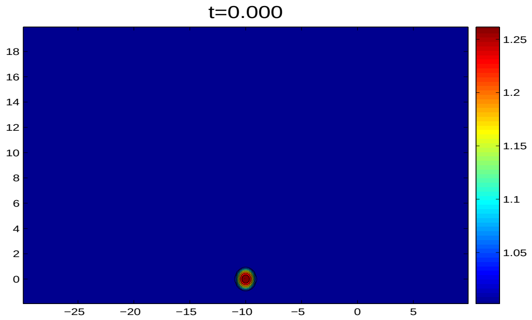
$$\partial_t \left(\zeta_1 - h_1\right) - \Pi_0\, \partial_t \tilde{h}_0 + \tilde{u}^{(0)} \partial_x \Pi_0 + \tilde{v}^{(0)} \partial_y \Pi_0 = -\partial_{t_1} \Pi_0 - J(\bar{h}_0, \Pi_0)\,. \tag{102}$$

Integrability condition $\Leftrightarrow$ averaging over t :

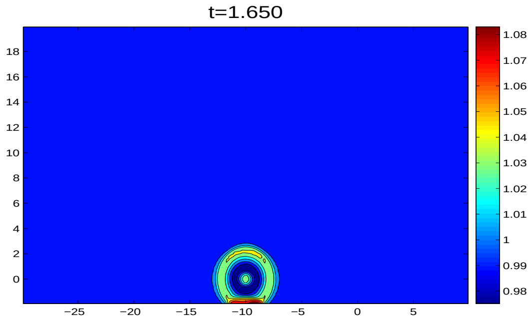
$$\partial_{t_1} \Pi_0 + J(\bar{h}_0, \Pi_0) \equiv \partial_{t_1}(\nabla^2 \bar{h}_0 - \bar{h}_0) + J(\bar{h}_0, \nabla^2 \bar{h}_0) = 0\,. \tag{103}$$

$\Rightarrow$ QG equation.
Arises from elimination of resonances in the equation for fast component at order 1.

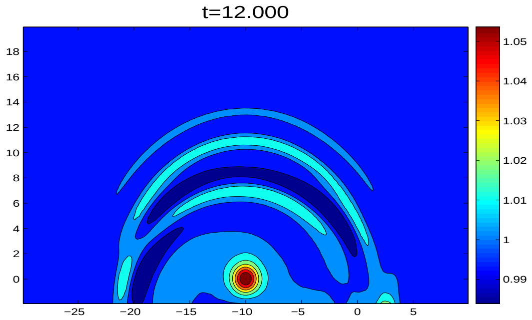
Adjustment near the wall



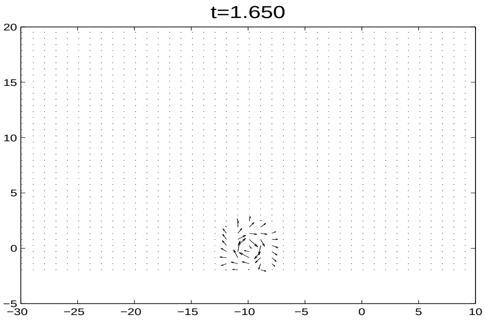
Initial stage of adjustment, *h*.



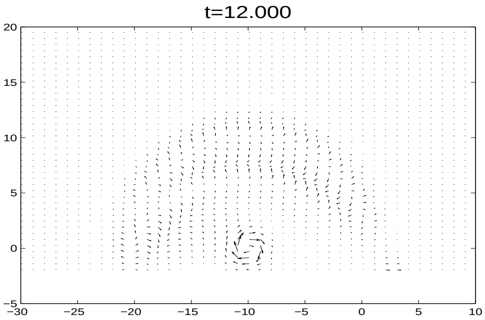
Advanced stage of adjustment, h.



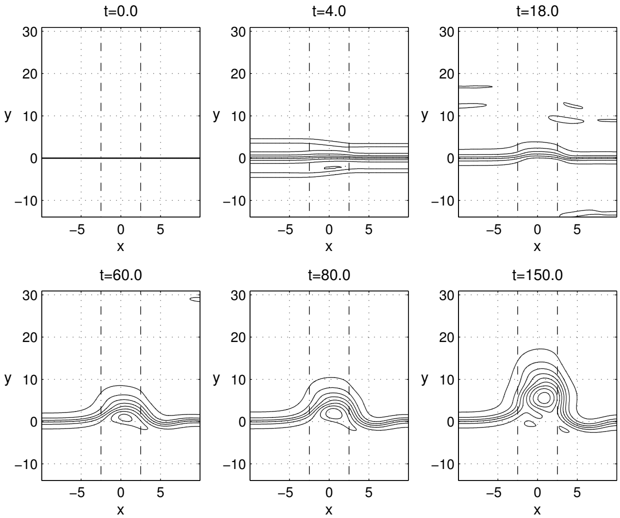
Initial stage of adjustment, velocity.



Advanced stage of adjustment, velocity.

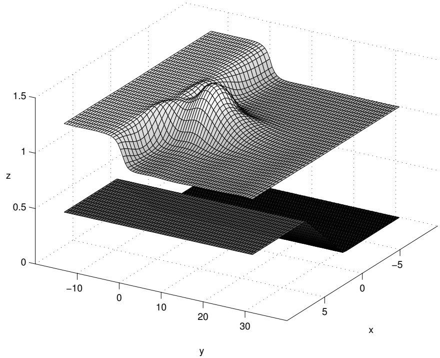


Adjustment of a pressure front over escarpment

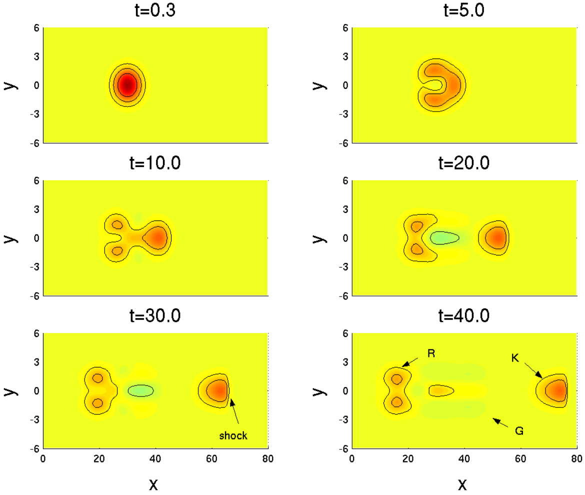


Front/jet deflected over escarpment by a slow moving along it large-scale coherent structure.

3D snapshot of the adjustment over escarpment



Adjustment of a long-wave pressure anomaly at the Equator



Slow-motion scaling on the equatorial β- plane

RSW equations at the Equator :

$$\begin{cases} \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \beta\, y\, \hat{\mathbf{z}} \wedge \mathbf{v} + g \nabla h = 0\,, \\ \partial_t h + \nabla \cdot (\mathbf{v} h) = 0\,, \end{cases} \tag{104}$$

Thickness perturbation : $h = H(1 + \lambda\, \eta)$,
QG-like scaling : $(x, y) \sim L,\ (u, v) \sim U,\ t \sim L/U$.
$\lambda \to 0$, and $gH\lambda/U^2 = \mathcal{O}(1) \Rightarrow U << \sqrt{gH}$ = maximal phase velocity of waves $\Rightarrow$ slow motion.
Non-dimensional equations with $\overline{\beta} = \beta L^2/U$:

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \overline{\beta}\, y\, \hat{\mathbf{z}} \wedge \mathbf{v} + \nabla \eta = 0\,, \tag{105}$$

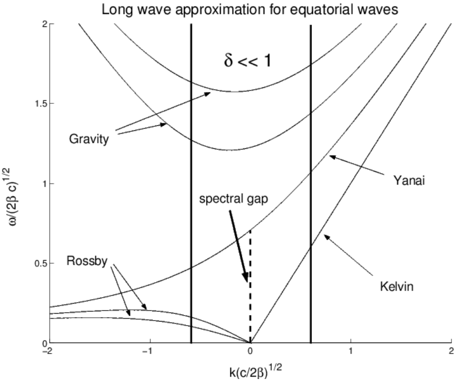
$$\lambda(\partial_t \eta + \mathbf{v} \cdot \nabla \eta) + (1 + \lambda \eta)\, \nabla \cdot \mathbf{v} = 0, \tag{106}$$

Leading order in λ $\nabla \cdot \mathbf{v}_0 = 0 \Rightarrow$ non-divergent motion $\Rightarrow$ $u_0 = -\partial_y \psi,\ v_0 = \partial_x \psi$. Cross-differentiation $\to$

$$\nabla^2 \psi_t + \mathcal{J}(\psi, \nabla^2 \psi) + \overline{\beta} \psi_x = 0, \tag{107}$$

Standard QG with $R_d \to \infty$. Linearization $\Rightarrow$ Rossby waves, all other equatorial waves filtered out.

Long-wave spectrum in RSW at the Equator



Long Kelvin and Rossby waves are well separated from the others.

Long-wave scaling

- Disparity of spatial scales :
  $x \sim L_x,\ y \sim L_y,\ L_y \ll L_x\ \Rightarrow \delta = \frac{L_y}{L_x} \ll 1$
- Disparity of velocity scales : $u \sim U,\ v \sim \delta U$ (motivation : $v = 0$ for Kelvin waves)
- Slow time scale : $t = t_1 \sim \delta\,(\beta R_e)^{-1}$
- Rossby number $\epsilon \sim \delta^2$

$R_e = \sqrt{\frac{\sqrt{gH_0}}{\beta}}$ - equatorial deformation radius.

Rescaled RSW equations on the equatorial β-plane

$$\begin{cases} u_{t_1} + \delta^3(uu_x + vu_y) - y\,v = -h_x, \\ \delta^2 v_{t_1} + \delta^4(uv_x + vv_y) + y\,u = -\delta h_y, \\ h_{t_1} + u_x + v_y + \delta^2\left[(hv)_y + (hu)_x\right] = 0, \end{cases} \tag{108}$$

Boundary condition : $(u,\ v,\ h)|_{y\to\pm\infty} \to 0.$

Lowest order in δ

$$\begin{cases} u_{t_1}^{(0)} - yv^{(0)} + h_x^{(0)} = 0, \\ yu^{(0)} + h_y^{(0)} = 0, \\ h_{t_1}^{(0)} + u_x^{(0)} + v_y^{(0)} = 0. \end{cases} \quad (109)$$

Two kinds of wave solutions, as in full linearised RSW on the equatorial β- plane :

- solutions with $v^{(0)} \neq 0$,
- solutions with $v^{(0)} \equiv 0$.

$v \neq 0$: mean-flow and Rossby waves

Eliminating variables in favour of v :

$$\left(y^2 v^{(0)} - v_{yy}^{(0)}\right)_{t_1 t_1} - v_{x t_1}^{(0)} = 0, \qquad (110)$$

Two types of solutions : time-independent and propagative ones.

- Time-independent mean flow (equatorial jet) in geostrophic equilibrium :

$$u_M^{(0)} = u^{(0)}(y), \ \ v_M^{(0)} = 0, \ \ h_M^{(0)} = h^{(0)}(y), \ \ y u^{(0)} + h^{(0)} = 0. \qquad (111)$$

  (also solution of the full system)

- Propagating long Rossby waves (RW) :

$$v_R^{(0)} = \sum_{n=1}^{\infty} v_{n_R}^{(0)} = \sum_{n=1}^{\infty} V_n(x + c_n t_1)\phi_n(y), \ c_n = -\frac{1}{2n+1}. \qquad (112)$$

Zonal velocity and pressure in RW

Obtained from

$$\begin{cases} u^{(0)}_{t_1} + h^{(0)}_x = y v^{(0)}, \\ h^{(0)}_{t_1} + u^{(0)}_x = -v^{(0)}_y, \end{cases} \tag{113}$$

$$u^{(0)}_{n_R} = \frac{1}{2}\left(\frac{\sqrt{2(n+1)}}{1+c_n}\mathcal{V}_n(x+c_n t_1)\phi_{n+1}(y) + \frac{\sqrt{2n}}{1-c_n}\mathcal{V}_n(x+c_n t_1)\phi_{n-1}(y)\right),$$

$$h^{(0)}_{n_R} = \frac{1}{2}\left(\frac{\sqrt{2(n+1)}}{1+c_n}\mathcal{V}_n(x+c_n t_1)\phi_{n+1}(y) - \frac{\sqrt{2n}}{1-c_n}\mathcal{V}_n(x+c_n t_1)\phi_{n-1}(y)\right),$$

where $\mathcal{V}_n$ denotes the primitive of V_n : $\mathcal{V}_n = \int^x V_n(x')\,dx'$.

$v \equiv 0$: Kelvin waves

Solutions of

$$\begin{cases} u^{(0)}_{t_1} + h^{(0)}_x = 0, \\ y u^{(0)} + h^{(0)}_y = 0, \\ h^{(0)}_{t_1} + u^{(0)}_x = 0. \end{cases} \tag{114}$$

Eastward-propagating equatorial Kelvin waves :

$$(u^{(0)}_K, v^{(0)}_K, h^{(0)}_K) = (K(x-t),\, 0,\, K(x-t))\, \phi_0(y). \tag{115}$$

General solution in the lowest order is a combination of a mean flow (equatorial jet) and long Rossby and Kelvin waves :

$$(u^{(0)}, v^{(0)}, h^{(0)}) = (u^{(0)}_M, 0, h^{(0)}_M) + (u^{(0)}_K, 0, h^{(0)}_K) + \sum_{n=1}^{\infty} (u^{(0)}_{nR}, v^{(0)}_{nR}, h^{(0)}_{nR}). \tag{116}$$

Nonlinear slow dynamics

Consistency (absence of secular growth) conditions in the second order in δ give :

- Rossby-wave sector : Korteweg - de Vries (KdV) equation

$$\left(\mathcal{V}_{n_{t_3}} + \alpha_n \mathcal{V}_{n_{xxx}} + \beta_n \mathcal{V}_n \mathcal{V}_{n_x}\right)_x = 0, \qquad (117)$$

  α_n, β_n are determined from the meridional structure of the mode.

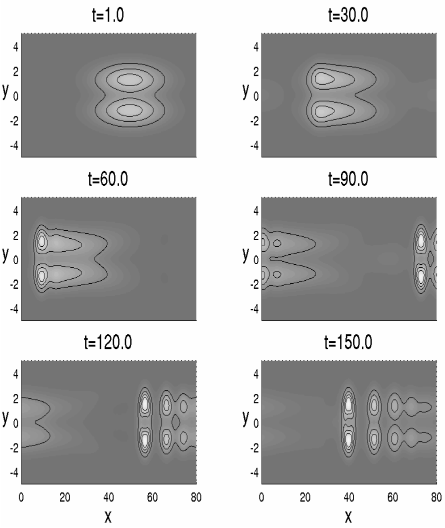
- Kelvin wave sector : simple wave equation

$$K_{t_3} + \gamma K K_x = 0, \qquad (118)$$

  $\gamma = \frac{3}{2}\int_{-\infty}^{\infty} dy \phi_0^3(y)$.

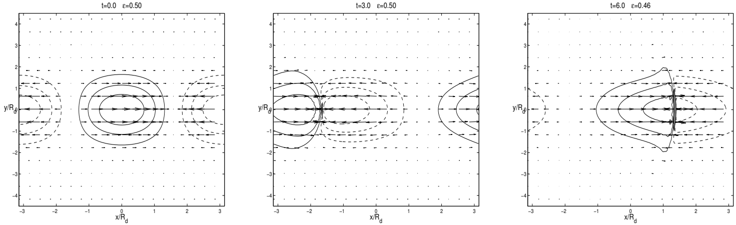
Here $t_3 \sim \delta^3 (\beta R_e)^{-1}$.

Equatorial Rossby solitons



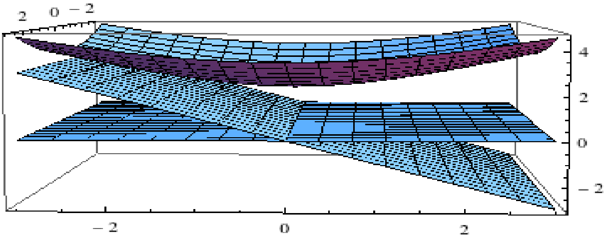
Evolution of a long-wave Rossby-wave packet with formation of a soliton tail. Levels of gray : thickness (pressure).

Breaking of an equatorial Kelvin wave



Breaking of an equatorial Kelvin wave. Snapshots at $t = 0, 3, 6(\beta R_e)^{-1}$.

Long-wave spectrum of RSW in a half-plane



Dispersion relation for internal-gravity (upper surface), coastal Kelvin waves (inclined plane), and vortex motions, $\omega = 0$ (horizontal plane).
Frequencies of IGW and KW are well separated at small l.

Long-wave scaling

- Disparity of spatial scales : $x \sim L_x,\ y \sim L_y,\ L_x \ll L_y \ \Rightarrow \delta = \frac{L_x}{L_y} \ll 1$
- Disparity of velocity scales : $v \sim V,\ u \sim \delta V$ (motivation : $u = 0$ for Kelvin waves)
- Slow time scale : $t = t_1 \sim \delta f^{-1}$
- Rossby number $\epsilon = \frac{V}{fL_x} \sim \delta$
- Burger number $Bu = \frac{R_d^2}{L_x^2} \sim 1.$

Rescaled RSW equations on the half f-plane

$$\begin{cases} \delta^2 u_{t_1} + \delta^3 (u u_x + v u_y) - v = -h_x, \\ \delta v_{t_1} + \delta (u v_x + v v_y) + \delta u = -\delta h_y, \\ h_{t_1} + u_x + v_y + \delta \left[(hv)_y + (hu)_x \right] = 0, \end{cases} \tag{119}$$

Boundary condition : $u|_{x=0} = 0.$

Lowest order in δ

$$\begin{cases} -v^{(0)} + h_x^{(0)} = 0, \\ v_{t_1}^{(0)} + u^{(0)} + h_y^{(0)} = 0, \\ h_{t_1}^{(0)} + u_x^{(0)} + v_y^{(0)} = 0. \end{cases} \quad (120)$$

whence

$$\left(h_{xx}^{(0)} - h^{(0)}\right)_{t_1} = 0, \quad (121)$$

Physical meaning : conservation of QG potential vorticity at small δ.

Solutions :

- Stationary - geostrophic balance,
- t_1- dependent, with $h^{(0)} \propto e^{\pm x} \rightarrow$ Kelvin wave (KW) :

$$u = 0, \quad h^{(0)} = K(t_1 + y)e^{-x}, \quad v^{(0)} = -K(t_1 + y)e^{-x}. \quad (122)$$

Next order in δ

Introducing slower time scale $t_2 \sim \delta^2 f^{-1}$:

$$\begin{cases} -v^{(1)} + h_x^{(1)} = 0, \\ v_{t_1}^{(1)} + u^{(1)} + h_y^{(1)} = \mathcal{R}_v, \\ h_{t_1}^{(1)} + u_x^{(1)} + v_y^{(1)} = \mathcal{R}_h, \end{cases} \tag{123}$$

Here

$$\mathcal{R}_v = -\left(v_{t_2}^{(0)} + u^{(0)} v_x^{(0)} + v^{(0)} v_y^{(0)}\right), \tag{124}$$

$$\mathcal{R}_h = -\left(h_{t_2}^{(0)} + (u^{(0)} h^{(0)})_x + (v^{(0)} h^{(0)})_y\right) \tag{125}$$

Two parts of the solution : QG (vortex motions), with $u \neq 0$, and KW with $u \equiv 0$.

QG solution

Combining equations in (123) :

$$\left(h^{(1)}_{xx} - h^{(1)}\right)_{t_1} = -\mathcal{R}_h + \mathcal{R}_{v_x}. \tag{126}$$

t_1 - average of the r.h.s. should vanish, otherwise secular growth $\Rightarrow$

$$\left(h^{(0)}_{xx} - h^{(0)}\right)_{t_2} + \mathcal{J}\left(h^{(0)}, h^{(0)}_{xx} - h^{(0)}\right) = 0. \tag{127}$$

$\rightarrow$ standard QG equation on the f- plane in the limit $\frac{L_x}{L_y} \rightarrow 0$

KW solution

Combining equations in (123) with $u \equiv 0$:

$$\left(h_x^{(1)} - h^{(1)}\right)_{t_1} - \left(h_x^{(1)} - h^{(1)}\right)_y = -\mathcal{R}_h + \mathcal{R}_v. \quad (128)$$

For KW the l.h. side integrated with e^{-x} over x vanishes $\Rightarrow$

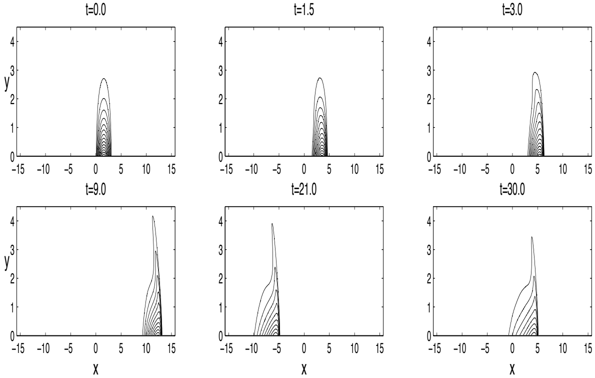
$$\int_0^\infty dx\, e^{-x} \left(-\mathcal{R}_h + \mathcal{R}_v\right) = 0. \quad (129)$$

Injecting (122) $\rightarrow$

$$h_{t_2}^{(0)} + \frac{3}{4} h^{(0)} h_y^{(0)}\Big|_{x=0} = 0. \quad (130)$$

simple wave equation $\rightarrow$ breaking in finite time

Kelvin wave breaking as seen in the thickness field



Breaking of a localised packet of coastal Kelvin waves propagating along the straight boundary along the x- axis. Time in units of f^{-1}, distances in units of R_d.

Potential vorticity in the presence of topography

PV conservation :

$$\frac{d}{dt}\left(\frac{\zeta + f}{h - b}\right) = 0. \tag{131}$$

Topography of weak amplitude $|b| \sim Ro$ the QG equation on the β-plane :

$$\nabla^2 \eta_t - \eta_t + \eta_x + \mathcal{J}(\eta, \nabla^2 \eta + b) = 0. \tag{132}$$

Stationary solutions

Stationarity :

$$\mathcal{J}(\eta, \nabla^2\eta + b + y) = 0. \quad (133)$$

General solution :

$$\nabla^2\eta + b + y = \mathcal{F}(\eta), \quad (134)$$

$\mathcal{F}$ - arbitrary function. Zonal flow U plus any perturbation : $\eta = -Uy + \psi$,

$$\nabla^2\psi + b + y = \mathcal{F}(\psi - Uy). \quad (135)$$

Looking for waves generated by localised topgraphy$\Rightarrow$ far upstream, at $x \to +\infty$ for $U < 0$, and $x \to -\infty$ for $U > 0$, the perturbation ψ vanishes and (135) becomes

$$y = \mathcal{F}(-Uy), \quad (136)$$

$\Rightarrow \mathcal{F}(x) = -\frac{x}{U} \Rightarrow$ linear equation for ψ :

$$U\nabla^2\psi + \psi = -Ub. \quad (137)$$

One-dimensional topography

Ridge : $b = b(x) \Rightarrow \psi = \psi(x)$. Equation (137) becomes

$$\psi''(x) + \frac{1}{U}\psi(x) = -b(x). \tag{138}$$

Solution : inversion of the operator, Green's function :

$$\psi(x) = -U \int_{-\infty}^{+\infty} dx' \, G(x - x') b(x'), \tag{139}$$

$$G''(x - x') + \frac{1}{U} G(x - x') = \delta(x - x'). \tag{140}$$

Fourier transformation :

$$G(x - x') = \int_{-\infty}^{+\infty} dk \, \frac{e^{i(k(x-x')}}{k^2 - \frac{1}{U}} \tag{141}$$

Dirac's delta-function (unity in function space) :
$\delta(x) = \int_{-\infty}^{+\infty} dk \, e^{ikx}$.

Calculation of Green's function

- easterly flow $U < 0$, calculation straightforward

$$G(x - x') = \pi\sqrt{U} e^{-\frac{|x-x'|}{\sqrt{U}}}, \tag{142}$$

- decaying at both sides of the "ridge"

- westerly flow $U > 0$, integrand is singular, method of residues. Upstream decay $\rightarrow$ singularity shifted to the upper half-plane of complex l.

$$G(x - x') = \begin{cases} \pi\sqrt{U} \sin\frac{(x-x')}{\sqrt{U}}, & x - x' > 0 \\ 0, & x - x' < 0. \end{cases} \tag{143}$$

- oscillating (waves) behind the "ridge".

Mountain Rossby waves in the westerly flow, as seen in the data.

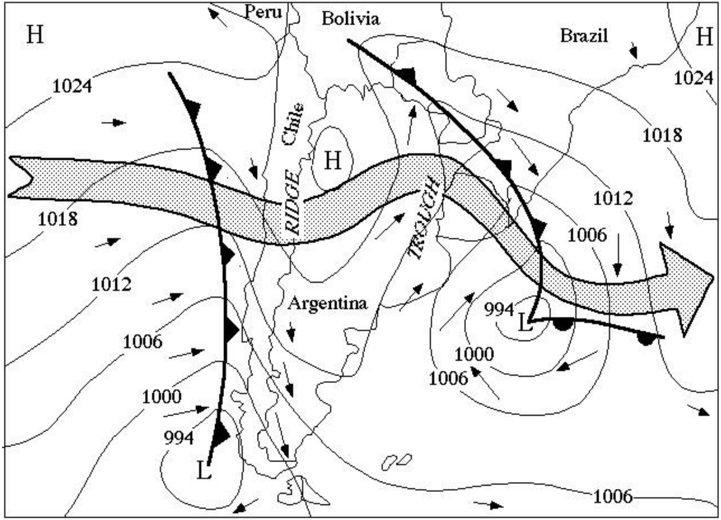


Fig 12.K.5 The effect of the Andes on the upper westerly winds, in terms of the isobar pattern (in hPa) on 4 June 1995. The small arrows show the direction and speed of surface winds, and the bold wavy band is the jet stream. There is a ridge in the 300hPa flow near the mountains and a lee trough to the east, which promote a Rossby wave whose northward swing cradles a low on the right of the diagram.

2-dimensional topography

Green's function $G(x-x', y-y')$:

$$U\nabla^2 G + G = \delta(x-x')\delta(y-y'); \qquad (144)$$

Fourier-transform : $G(x-x', y-y') \to$
$G(k,l), \left[-U(k^2+l^2)+1\right] G(k,l) = 1$

$$G(x-x', y-y') = \int_{-\infty}^{+\infty} dkdl \, \frac{e^{i(k(x-x')+l(y-y'))}}{-U(k^2+l^2)+1}$$

Polar coordinates in Fourier-space $\to$

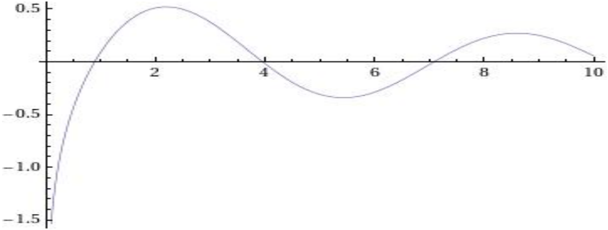
$$\int_0^{+\infty} |\mathbf{k}| d|\mathbf{k}| \int_0^{2\pi} d\theta \, \frac{e^{i|\mathbf{k}||\mathbf{x}-\mathbf{x}'|\cos\theta}}{-U\mathbf{k}^2+1} = 2\pi \int_0^{+\infty} \frac{|\mathbf{k}| d|\mathbf{k}|}{-U\mathbf{k}^2+1} J_0\left(|\mathbf{k}||\mathbf{x}-\mathbf{x}'|\right) := \mathcal{I} \qquad (145)$$

J_0 - Bessel function

$$U > 0: \ \mathcal{I} = -\frac{\pi}{2U} Y_0\left(\frac{|\mathbf{x}-\mathbf{x}'|}{U}\right), \ U < 0: \mathcal{I} = \frac{2\pi}{|U|} K_0\left(\frac{|\mathbf{x}-\mathbf{x}'|}{|U|}\right)$$

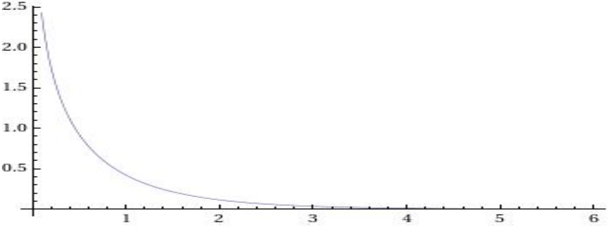
Bessel functions

- Bessel function Y_0 : oscillating, and weakly decaying



Inconsistent with b.c. of strong decay upstream - ??.

- Modified Bessel function K_0 : exponentially decaying



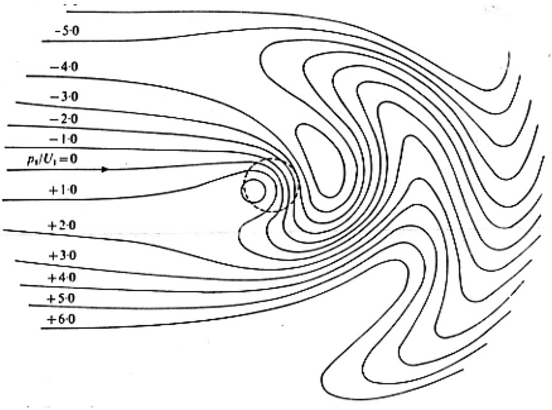
Exponentially decaying perturbation in the easterly flow.

Recipe for correcting westerly flow result

Solution for westerly flow :

Green's function to be corrected by a solution of the homogeneous problem which "kills" the oscillations far upstream. The correction can not be found in closed form, it is expressed as a series of Bessel functions $\sum_{n=1}^{\infty} \frac{1}{2n-1} J_{2n-1}\left(\frac{|\mathbf{x}-\mathbf{x}'|}{\sqrt{U}}\right) \cos(2n-1)\phi$, where ϕ is the polar angle on the $x-y$ plane.

Mountain Rossby waves in the westerly flux over a circular mountain



Conclusions

- "Vortex" scaling and $Ro \to 0 \leftrightarrow$ filtering of gravity waves $\to$ simplified QG models $\Leftrightarrow$ PV conservation in the limit of small Ro
- QG RSW on the f-plane $\leftrightarrow$ 2D Euler with modified streamfunction-vorticity relation
- QG on the β- plane : barotropic and baroclinic Rossby waves
- QG with topography $\to$ mountain waves
- QG in coastal and equatorial wave-guides to be completed with Kelvin waves
- Nonlinearity of slow wave-guide motions : wave breaking (in hydrostatic approximation) and soliton formation.