5. Singularity formation and frontogenesis

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Mathematics of the atmosphere and oceans,
SUSTECH, Shenzhen, 2023

Plan

Geostrophic adjustment in 1.5D RSW, and the first idea of frontogenesis

- 1.5D RSW
- Lagrangian 1.5D RSW
- Fully nonlinear geostrophic adjustment
- The role of baroclinicity : 2-layer RSW

Fronts and frontogenesis in the PE model

- 2.5D PE
- Existence and uniqueness of adjusted state
- Lagrangian 2.5D EP
- Frontogenesis at zero PV
- Frontogenesis at constant PV

Cyclogenesis, a catastrophic cyclo-geostrophic adjustment

Conclusions

1.5D RSW

Dimension "1.5" : no dependence on y

Equations of the model :

$$\begin{aligned}
\partial_t u + u\partial_x u - fv + g\partial_x h &= 0\,,\\
\partial_t v + u\partial_x v + fu &= 0\,,\\
\partial_t h + u\partial_x h + h\partial_x u &= 0\,.
\end{aligned}$$

Frontal configurations : localized distributions of $v(x)$, $h(x)$ with common compact support in x of v, $\partial_x h$.

Lagrangian invariants

- Potential Vorticity :

$$Q = (\partial_x v + f)/h, \quad (1)$$

.

- Geostrophic Momentum :

$$M = v + fx \quad (2)$$

$$(\partial_t + u\partial_x)M = 0, \ \ (\partial_t + u\partial_x)Q = 0. \quad (3)$$

Inertia - gravity waves

Linearization with respect to the rest state $H = const$: zero mode (slow motions) and inertia- gravity waves (fast motions) with standard dispersion relation :

$$\omega = \pm(c_0^2 k^2 + f^2)^{\frac{1}{2}}, \ \ c_0 = \sqrt{gH}. \quad (4)$$

Geostrophic equilibrium

Exact solution of the equations of motion :

$$fv = g\partial_x h, \quad u = 0 \,, \tag{5}$$

(Infinitely) slow motions : vorticity is entirely determined by the perturbation of h and vice verse :

$$Q^{(g)} = \left(\frac{f + \frac{g}{f}\partial_{xx}^2 h}{h} \right) . \tag{6}$$

Geostrophic adjustment

Adjustment $\rightarrow$ Relaxation towards equilibrium state.
Equilibrium $\leftrightarrow$ minimum of energy $\Rightarrow$ necessity to evacuate energy. The only energy sink in the absence of dissipation : émission of inertia - gravity waves.

Equations of motion in Lagrangian coordinates

Lagrangian coordinates

Trajectories of "fluide parcels" $x \to X(x,t)$, where x is a position of the parcel at $t = 0$. $\dot{X} = u(X,t)$, $X' := \frac{\partial X}{\partial x}$.

Momentum equations

$$\begin{aligned} \ddot{X} - fv + g\partial_X h &= 0 , \\ \dot{v} + f\dot{X} &= 0 , \end{aligned} \quad (7)$$

where v is considered as a function of x and t.

Conservation of mass :

$$h(X,t)\, dX = h_I(x)\, dx, \;\Rightarrow\; h(X,t) = h_I(x)\partial_X x. \quad (8)$$

Reduction to a single equation

Integration of the equation for v :

$$v(x,t) + fX(x,t) = M(x)\,. \tag{9}$$

Determination of M from b.c. :

$$M(x) = fx + v_I(x). \tag{10}$$

Chaine differentiation :

$$\partial_X h = \partial_X \left(h_I(x)\partial_x X\right) = h_I' \left(X'\right)^{-2} - h_I(x)X'' \left(X'\right)^{-3}, \tag{11}$$

Closed equation for X :

$$\ddot{X} + f^2 X + g h_I' \left(X'\right)^{-2} + \frac{g h_I}{2}\left[\left(X'\right)^{-2}\right]' = fM\,. \tag{12}$$

Re-writing in terms of deviations of parcels from their initial positions : $X(x,t) = x + \phi(x,t)$:

$$\ddot{\phi} + f^2\phi + gh_I' \left(\frac{1}{(1+\phi')^2}\right) + \frac{gh_I}{2}\left(\frac{1}{(1+\phi')^2}\right)' = fv_I \ . \quad (13)$$

To be solved with b.c. :

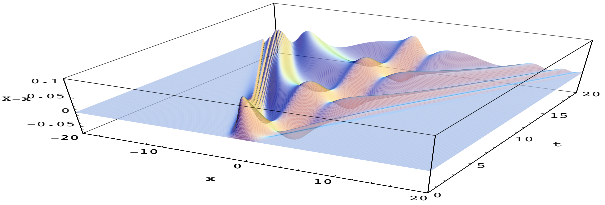
$$\phi(x,0) = 0, \quad \dot{\phi}(x,0) = u_I(x),$$

where u_I is the initial velocity in x direction.

Direct simulation with MATHEMATICA of the 1.5D adjustment

Initial configuration :

$$h_I(x) = 1 + e^{-x^2}, \; v_I(x) = -2(x + 0.2\sin(x))\, e^{-x^2}, \; u_I(x) = 0.1 e^{-x^2}$$



Wave breaking leads to numerical oscillations on the left in this non-shock-capturing simulation.

Example : adjustement of a "wind blow" (Rossby, 1936)

Initial condition : jet out of equilibrium :
$h_I = H = \text{const}, \; v_I \neq 0.$
Notation $J = \partial X / \partial x = H / h(X, t)$.

$$g \, \partial_X h = \partial_x P, \quad P = gH/(2J^2) - \text{Lagrangian pressure}$$

Lagrangian equations :

$$\dot{u} - fv + \partial_x P = 0, \tag{14}$$

$$\dot{v} + fu = 0, \tag{15}$$

$$\dot{J} - \partial_x u = 0. \tag{16}$$

Reduction to a single equation for J

$$\ddot{J} + f^2 J + \partial_{xx}^2 P = fHQ\,. \tag{17}$$

Here

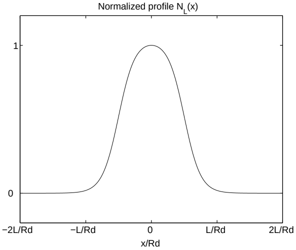
$$Q(x) = \frac{1}{H}\left(\partial_x v(x,t) + fJ(x,t)\right) = \frac{1}{H}\left(\partial_x v_I(x) + fJ_I(x)\right).$$

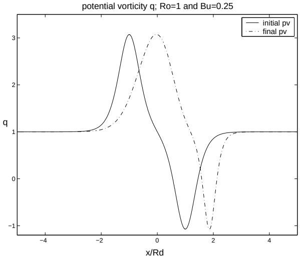
Stationary adjusted solution :

$$f^2 J + \partial_{xx}^2 P = fHQ \tag{18}$$

- entirely determined by Q

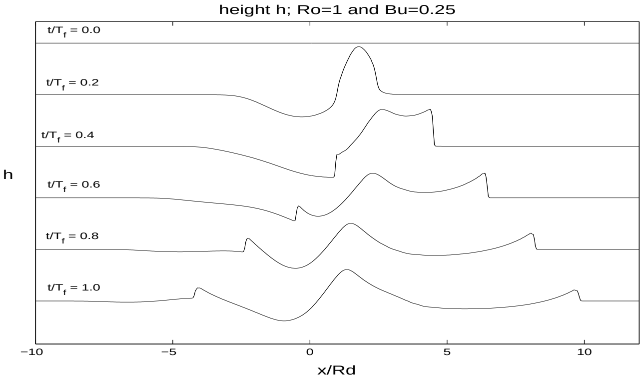
High-resolution numerical simulations of Rossby adjustment





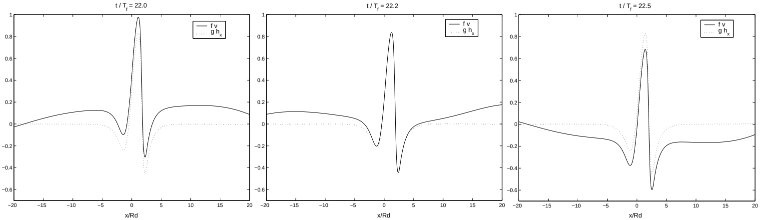
Initial jet profile (left) and initial and final distributions of PV (right). Notice the region of negative PV.

Process of adjustment as seen in the thickness field



Notice a discontinuity formation in the region of negative PV.

Check of balance in the adjusted state



Existence of adjusted state : question

Does physically acceptable, i.e. smooth with everywhere positive thickness, adjusted state exist for any initial PV, and is it unique ?

In original variables X : $dX = Jda$,

$$-\frac{g}{f}\,\frac{d^2h(X)}{dX^2} + h(X)\;Q(X) = f. \qquad (19)$$

where PV is a function of X, which is given by the inverse mapping $x = x(X,t)$:

$$Q(X) = \frac{1}{h_I(x(X))}\left(f + \frac{\partial v_I(x(X))}{\partial x}\right).$$

Existence and uniqueness of positive solutions for ODE (20) ?

Existence of adjusted state : answer

Theorem

For positive $Q(X)$ with compact support derivatives and arbitrary constant asymptotics (front) equation (20) has unique bounded and everywhere positive solution $h(X)$ at $-\infty \leq X \leq +\infty$.

Frontogenesis : "catastrophic" adjustment leading to a non-smooth adjusted state.

2-layer rigid-lid RSW

$$\partial_t u_1 + u_1 \partial_x u_1 - f v_1 + \rho_1^{-1} \partial_x \pi = 0 \; , \tag{20a}$$

$$\partial_t v_1 + u_1 (f + \partial_x v_1) = 0 \; , \tag{20b}$$

$$\partial_t u_2 + u_2 \partial_x u_2 - f v_2 + \rho_2^{-1} \partial_x \pi + g' \partial_x \eta = 0 \; , \tag{20c}$$

$$\partial_t v_2 + u_2 (f + \partial_x v_2) = 0 \; , \tag{20d}$$

$$\partial_t (H_1 - \eta) + \partial_x ((H_1 - \eta) u_1) = 0 \; , \tag{20e}$$

$$\partial_t (H_2 + \eta) + \partial_x ((H_2 + \eta) u_2) = 0 \; , \tag{20f}$$

g' - reduced gravity, $H_1 + H_2 = H$. PV layer-wise :

$$Q_1 = \frac{f + \partial_x v_1}{h_1} \, , \quad \text{and} \quad Q_2 = \frac{f + \partial_x v_2}{h_2} \, . \tag{21}$$

Geostrophic equilibria : exact solutions :

$$v_1 = \frac{1}{f \, \rho_1} \partial_x \pi \; , \tag{22a}$$

$$v_2 = \frac{1}{f \rho_2} \partial_x \pi + \frac{g'}{f} \partial_x \eta \; . \tag{22b}$$

Existence and uniqueness of adjusted states

Combining (22) and (23a) $\rightarrow$ ODEs for equilibrium heights :

$$\frac{g'}{f} h_1'' - (Q_2 + r\,Q_1)\,h_1 = -\left(-f(1-r) + H\,Q_2\right)\;, \quad (23a)$$

$$\frac{g'}{f} h_2'' - (Q_2 + r\,Q_1)\,h_2 = -\left(f(1-r) + rH\,Q_1\right)\;, \quad (23b)$$

where $r = \rho_1/\rho_2 < 1$. Both of the type

$$h'' - R(x)\,h = -S(x)$$

Similar to 1-layer case : $R > 0, S > 0 \Rightarrow$ existence, uniqueness, and positiveness of solutions.
$\pm\infty$. $\Rightarrow$ adjusted state $\exists$ for initial states with localized PV anomalies with

$$Q_1 \geq 0\;, \quad \text{and} \quad Q_2 \geq (1-r)\,f/H\;, \qquad (24)$$

$\rightarrow$ additional restrictions.

Equations of 2.5D PE

"Dimension 2.5" : no dependence of y :

$$\frac{Du}{Dt} - fv + \phi_x = 0\,, \tag{25}$$

$$\frac{Dv}{Dt} + fu = 0\,, \tag{26}$$

$$\phi_z = g\frac{\theta}{\theta_r}\,, \tag{27}$$

$$u_x + w_z = 0\,, \tag{28}$$

$$\frac{D\theta}{Dt} = 0\,, \tag{29}$$

$$\frac{D}{Dt} = \partial_t + u\partial_x + w\partial_z \tag{30}$$

Lagrangian invariants

- Potential temperature θ,
- Potential vorticity :

$$q = (\partial_x v + f)\theta_z - v_z\theta_x, \tag{31}$$

- Geostrophic momentum

$$M = v + fx \tag{32}$$

$$\frac{D}{Dt}(\theta,\ M,\ q) = 0. \tag{33}$$

Expression of q in terms of M :

$$q\ =\ \frac{\partial(M,\theta)}{\partial(x,z)}\ . \tag{34}$$

Inertia - gravity waves

Linearization about state of rest with linear stratification

$$\theta = \theta_r \frac{N^2}{g} z$$

with constant Brunt - Vaisala frequency $N = const$: zero mode (slow motions) and inertia- gravity waves (fast motions) with standard dispersion relation :

$$\omega = \pm (N^2 \frac{k^2}{m^2} + f^2)^{\frac{1}{2}} , \quad (35)$$

where wavenumber in (x, z) space is :

$$\mathbf{k} = k\hat{\mathbf{x}} + m\hat{\mathbf{z}}. \quad (36)$$

Thermal wind and geostrophic potential

Stationary states

$$u = w = 0, \quad fv = \phi_x, \quad g\frac{\theta}{\theta_r} = \phi_z. \tag{37}$$

Elimination of ϕ, use of M :

$$f\frac{\partial M}{\partial z} = \frac{g}{\theta_r}\frac{\partial \theta}{\partial x} \,, \tag{38}$$

$\Rightarrow$ "geostrophic potential" Φ may be introduced for equilibrium states :

$$M = f^{-1}\frac{\partial \Phi}{\partial x} \,, \tag{39a}$$

$$\theta = \frac{\theta_r}{g}\frac{\partial \Phi}{\partial z} \,. \tag{39b}$$

Monge-Ampère equation for adjusted state

Lagrangian conservation of PV $\Rightarrow$ same PV in initial and adjusted states :

$$q = \frac{\partial(M,\theta)}{\partial(x,z)} = \frac{\partial^2\Phi}{\partial x^2}\frac{\partial^2\Phi}{\partial z^2} - \left(\frac{\partial^2\Phi}{\partial x\partial z}\right)^2 = \frac{gf}{\theta_r}q \qquad (40)$$

Monge-Ampère (MA) equation.
Localized fronts on the background of linear stratification $N =$ const in the whole (x,z)- plane :

$$\Phi|_{|x|,|z|\to\infty} = \frac{1}{2}(f^2x^2 + N^2z^2) \Rightarrow$$

$\Phi =$ const at a distant ellipse $\Rightarrow$ Dirichlet problem in a convex domain.
Existence and uniqueness of solution : $\Leftrightarrow$ $q > 0$.

Complications with vertical boundaries

Flow in a strip $z_- \leq z \leq z_+$, $z_\pm =$ const. $q > 0 \Rightarrow$ $(x, z) \to (M, \theta)$ well defined, strip $\mapsto$ non-convex domain.
Thermal wind relation :

$$f \frac{\partial x}{\partial \theta} = \frac{g}{\theta_r} \frac{\partial z}{\partial M} \quad \to \tag{41}$$

"Potential" Ψ for (x, z) :

$$x = \frac{\theta_r}{g} \frac{\partial \Psi}{\partial M} \quad z = f^{-1} \frac{\partial \Psi}{\partial \theta} \quad \to \tag{42}$$

Monge-Ampère equation :

$$\frac{\partial^2 \Psi}{\partial M^2} \frac{\partial^2 \Psi}{\partial \theta^2} - \left(\frac{\partial^2 \Psi}{\partial M \partial \theta} \right)^2 = \frac{\theta_r}{gf} \frac{1}{q}. \tag{43}$$

Neumann boundary conditions at vertical boundaries :

$$f^{-1} \frac{\partial \Psi}{\partial \theta} \bigg|_{M_\pm, \theta_\pm} = z_\pm \tag{44}$$

No existence and uniqueness results for MA equation in non-convex domain with Neumann b.c.

Equations of motion in Lagrangian coordinates

Lagrangian coordinates

Trajectories of fluid "parcels"
$(x,\, z) \to (X(x, z, t),\, Z(x, z, t))$, where (x, z) is a position of a parcel at $t = 0$. $(\dot{X}, \dot{Z}) = (u(X, Z, t), w(X, Z, t))$.

Incompressibility equation - conservation of volume :

$$\frac{\partial(X, Z)}{\partial(x, z)} = 1\,. \tag{45}$$

Hydrostatic equation

$$\partial_Z \phi \equiv \frac{\partial(X, \phi)}{\partial(x, z)} = g\frac{\theta_l}{\theta_r}. \tag{46}$$

Horizontal momentum equations

$$\begin{aligned} \ddot{X} - fv + \partial_X \phi &= 0 , \\ \dot{v} + f\dot{X} &= 0 , \end{aligned} \tag{47}$$

Elimination of v :

Conservation of M and b. c. :

$$M(x) = v + fX = fx + v_I(x). \tag{48}$$

Elimination of ϕ by cross-differentiation :

$$\frac{\partial(X, \ddot{X} - fv_I - f^2 x)}{\partial(x, z)} + \frac{g}{\theta_r} \frac{\partial(\theta_I, Z)}{\partial(x, z)} = 0 \tag{49}$$

Stationary adjusted states :

$$\frac{\partial(X, -fv_I - f^2 x)}{\partial(x,z)} + \frac{g}{\theta_r}\frac{\partial(\theta_I, Z)}{\partial(x,z)} = 0 \qquad (50)$$

$$\frac{\partial(X, Z)}{\partial(x,z)} = 1 \qquad (51)$$

These equations can be solved analytically for configurations with constant PV, for example for a layer of the fluid between a flat bottom (at $z = 0$) and a rigid lid (at $z = H = 1$), with b. c. :

$$Z(x, 0) = 0\,, Z(x, 1) = 1\,. \qquad (52)$$

Localized fronts/jets correspond to $X|_{x\to\pm\infty} = x$.

Initial configuration :

θ_I varying only horizontally, no vertical shear in v_I :

$$\theta_I = \theta_I(x), \;\; v_I = v_I(x) \;\Rightarrow q \equiv 0. \tag{53}$$

Horizontal momentum equation :

$$\frac{\partial X}{\partial z} f(v_I' + f) + \frac{\partial Z}{\partial z}\frac{g\theta_I'}{\theta_r} = 0\,, \tag{54}$$

where prime denotes differentiation with respect to x.

Integration in z :

$$X = \frac{\mathcal{F}(x)}{fv_I' + f^2} - \frac{g\theta_I'/\theta_r}{fv_I' + f^2} Z\,, \tag{55}$$

$\mathcal{F}(x)$ - integration "constant" .

Using the incompressibility equation :

$$Z^2 \left(\frac{g\theta_I'/\theta_r}{fv_I' + f^2} \right)' - 2 \left(\frac{\mathcal{F}}{fv_I' + f^2} \right)' Z + 2(\mathcal{G}(x) + z) = 0 \, , \tag{56}$$

where $\mathcal{G}(x)$ - another integration "constant" after integration in z.

Applying b. c.

$$\mathcal{G}(x) = 0, \tag{57}$$

$$\left(\frac{\mathcal{F}}{fv_I' + f^2} \right)' = 1 + \frac{1}{2} \left(\frac{g\theta_I'/\theta_r}{fv_I' + f^2} \right)' \equiv 1 + \frac{1}{2}\mathcal{A}'(x). \tag{58}$$

Explicit form of stationary solutions :

$$X_s = x + \mathcal{A}(x)\left(\frac{1}{2} - Z\right), \; \mathcal{A} = \frac{g\theta_I'/\theta_r}{fv_I' + f^2} \tag{59}$$

$$Z_s = \frac{1}{\mathcal{A}'(x)}\left[1 + \frac{1}{2}\mathcal{A}'(x) - \sqrt{\left(1 + \frac{1}{2}\mathcal{A}'(x)\right)^2 - 2z\mathcal{A}'(x)}\right], \tag{60}$$

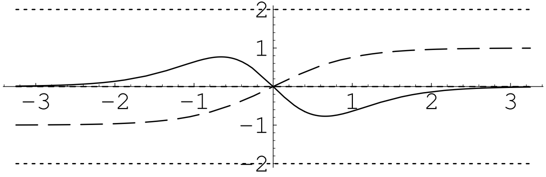
This mapping $(x, z) \to (X_s, Z_s)$ can be singular $\equiv$ not bijective, if $\exists (x, z)$: $\frac{\partial X_s}{\partial x} = 0$.
Singularity appears at the boundaries $\Rightarrow$ criterion : $1 \pm \frac{\mathcal{A}'}{2} = 0$, or :

$$\frac{g}{f\theta_r}\left(\frac{g\theta_I'/\theta_r}{f + v_I'}\right)' = \pm 2\,. \tag{61}$$

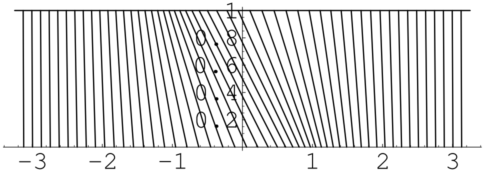
Illustrations : zero PV, localized anomaly of θ without initial jet $v_I \equiv 0$.

Initial configuration

Profiles of $\theta_I = tanh(x)$ (dashed), of $\mathcal{A}'$ (continuous), and discontinuity thresholds $\mathcal{A}' = \pm 2$ (dotted) :

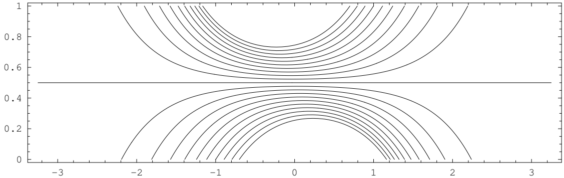


Isentropes in the adjusted state

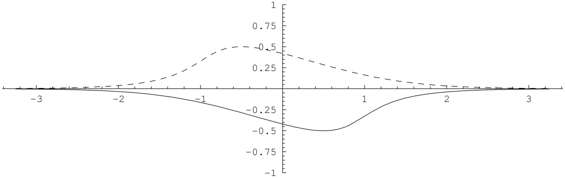


Adjusted state

Isotachs of the velocity field



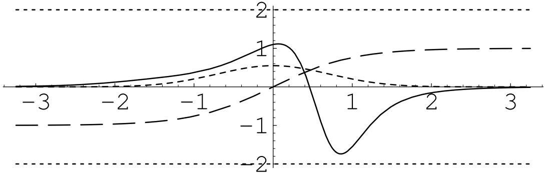
Velocity as a function of X on the vertical boundaries



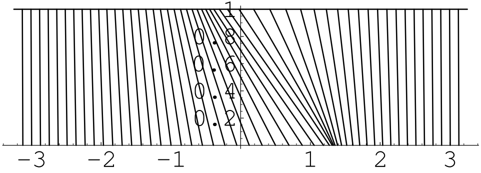
Illustrations : zero PV, localized anomaly of θ with an initial jet $v_I = 0.55e^{-x^2}$.

Initial configuration

Profiles of $\theta_I = tanh(x)$ (dashed) and of $\mathcal{A}'$ (continuous), and discontinuity thresholds $\mathcal{A}' = \pm 2$ (dotted) :

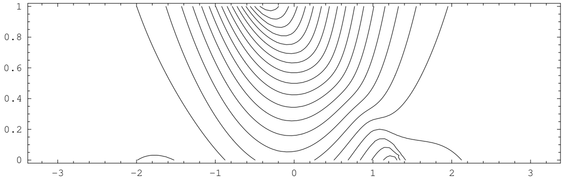


Isentropes in the adjusted state

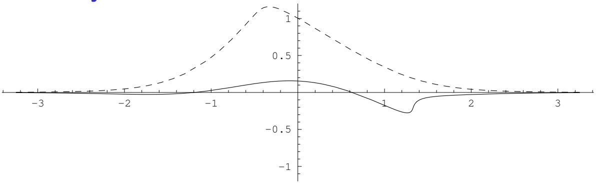


Adjusted state

Isotachs of the velocity field

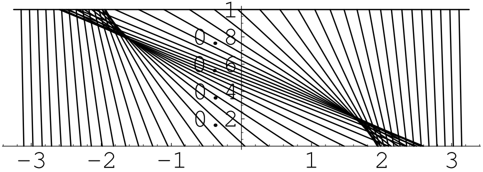


Velocity as a function of X at the vertical boundaries

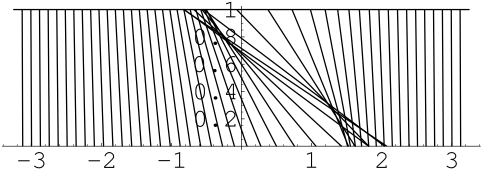


Beyond the singularity :

Configuration with $v_I = 0$, isentropes :



Configuration with $v_I \neq 0$, isentropes :



Initial conditions and equation for adjusted state

Initial configuration :

Flow in a strip : $-\infty \leq x \leq +\infty$, $0 \leq z \leq H$.
Horizontal anomaly of linear profile potential temperature, no v_I :

$$\theta_I = \frac{\theta_r}{g}\left(N^2 z + g\psi(x)\right), \quad v_I = 0 \; N = \text{constant}. \tag{62}$$

Horizontal momentum equation :

$$-\frac{\partial(X, f^2 x)}{\partial(x, z)} + g\psi'(x)\frac{\partial Z}{\partial z} - N^2\frac{\partial Z}{\partial x} = 0\,, \tag{63}$$

New variables and equivalent Laplace equation

New variables : $(x, z) \to (x, Z)$. Momentum equation $\to$

$$-\frac{\partial(X, f^2 x)}{\partial(x, Z)} + g\psi'(x) - N^2 \frac{\partial z}{\partial x} = 0 \qquad (64)$$

Incompressibility condition :

$$\frac{\partial(X, Z)}{\partial(x, z)} = 1 \;\to\; \frac{\partial X}{\partial x} - \frac{\partial z}{\partial Z} = 0. \qquad (65)$$

Elimination of $z \to$ equation for X :

$$f^2 \frac{\partial^2 X}{\partial^2 Z} + N^2 \frac{\partial^2 X}{\partial^2 x} = 0, \quad \left.\frac{\partial X}{\partial Z}\right|_{Z=0,H} = \frac{g}{f^2}\psi'(x). \qquad (66)$$

After rescaling - standard Laplace equation with Neumann boundary conditions in a unit strip.

Use of the theory of complex variables

Neumann problem $\leftrightarrow$ Dirichlet problem for conjugate analytic function.

Schwarz integral : general solution $F(\zeta) = a(\zeta) + ib(\zeta)$, $\zeta = x + iZ$ of Dirichlet problem in a unit strip in the complex x, Z plane with b.c. $a = a_{0,1}$ at $Z = (0, 1)$

$$F(\zeta) = \frac{i}{2}\int_{-\infty}^{+\infty} dt\, a_1(t) \tanh \frac{\pi(t-\zeta)}{2} - \frac{i}{2}\int_{-\infty}^{+\infty} dt\, a_0(t) \coth \frac{\pi(t-\zeta)}{2}.$$

Our case : $a_0 = a_1 = \psi(x) \Rightarrow$

$$F(\zeta) = -i\int_{-\infty}^{+\infty} dt \frac{\psi(t)}{\sinh \pi(t-\zeta)}. \tag{67}$$

Solution of the adjustment problem

$$X(x, Z) = x + \int_{x}^{+\infty} dt \left(1 + \frac{\partial a}{\partial Z}\right), \qquad (68)$$

where

$$a(x, Z) = \mathcal{R}\left[F(\zeta)\right] = \int_{-\infty}^{+\infty} dt\, \psi(t) \frac{\cosh \pi(t - x)\, \sin \pi Z}{\sinh^2 \pi(t - x) + \sin^2 \pi Z}. \qquad (69)$$

Singularity : $\exists (x, Z) : \frac{\partial X}{\partial x} \leq 0$. Appears at strong enough amplitudes of density perturbation $|\psi(x)|$ at the boundaries.

Example : $\psi(x) = a \tanh(\pi x)$

$$X(x,\, Z) = x + a \frac{\cos(\pi Z)}{\cosh(\pi x) + \sin(\pi Z)} \rightarrow$$

Singularity at $z = Z = 0$ if $a \geq 2$.

Axisymmetric PE

$$\frac{du}{dt} - v\,(f + \frac{v}{r}) = -\partial_r \phi\,, \tag{70a}$$

$$\frac{dv}{dt} + u\,(f + \frac{v}{r}) = 0\,, \tag{70b}$$

$$\frac{db}{dt} = 0, \quad \partial_z \phi = b\,, \tag{70c}$$

$$\frac{1}{r}\partial_r(r\,u) + \partial_z w = 0, \tag{70d}$$

where $\frac{d}{dt} = \partial_t + u\,\partial_r + w\,\partial_z$.
Stationary solution : cyclo-geostrophic equilibrium :

$$v\,(f + \frac{v}{r}) = \partial_r \phi, \quad b = \partial_z \phi. \tag{71}$$

Lagrangian picture

Lagrangian mapping : $(r, z) \mapsto (R(r,z), Z(r,z))$, $u = \dot{R}$.
Advection of buoyancy : $\dot{b}(R,Z) = 0 \Rightarrow b(R,Z) = b_I(r,z)$
From (71b)

$$\frac{d}{dt}\left(rv + f\frac{r^2}{2}\right) = 0 \;\Rightarrow\; Rv + f\frac{R^2}{2} = rv_I + f\frac{r^2}{2} \equiv M,$$

where $(...)_I$ - initial values. M - angular momentum $\rightarrow$

$$v = \frac{1}{R}\left(M - f\frac{R^2}{2}\right). \tag{72}$$

Incompressibility $\leftrightarrow R\,dR\,dZ = r\,dr\,dz \leftrightarrow \frac{\partial(R^2, Z)}{\partial(r^2, z)} = 1$
Lagrangian equations :

$$\ddot{R} + \frac{f^2}{4}R - \frac{M^2}{R^3} + \partial_R\phi, \tag{73a}$$

$$\partial_Z\phi = b, \quad \frac{\partial\left(R^2,\, Z\right)}{\partial(r^2, z)} = 1. \tag{73b}$$

Equation for adjusted state

Elimination of ϕ by cross-differentiation :

$$\frac{\partial\left(R, \ddot{R} - \frac{M^2}{R^3}\right)}{\partial\,(R, Z)} - \frac{\partial(b, Z)}{\partial\,(R, Z)} = 0, \tag{74a}$$

$$\frac{R}{r}\frac{\partial\,(R,\ Z)}{\partial(r, z)} = 1. \tag{74b}$$

Stationary solution$\leftrightarrow$ adjusted state resulting from evolution of initial state with M_I, b_I :

$$-\frac{1}{R^3}\frac{\partial(R, M_I^2(r,z))}{\partial(r,z)} + \frac{\partial(b_I(r,z), Z)}{\partial(r,z)}, \quad \frac{R}{r}\frac{\partial\,(R,\ Z)}{\partial(r,z)} = 1. \tag{75}$$

Barotropic initial state

z- independent initial configuration $b_I(r)$, $M_I(r) \Rightarrow$

$$R^{-3}\partial_z R \partial_r M_I^2 + \partial_r b_I \partial_z Z = 0 \Rightarrow \tag{76}$$

Integrating in z allows to get an expression for R :

$$R^2 = A_I (Z - \mathcal{H}(r))^{-1}, \tag{77}$$

where $\mathcal{H}$ is arbitrary function ("integration constant" divided by b_I'), to be determined from b.c., prime denotes r differentiation, and we introduced $A_I(r) := \frac{M_I M_I'}{b_I'}$
Injection into (75b) and integration once more in $z \rightarrow$

$$\frac{1}{2r}\left(A_I' \log\left(Z - \mathcal{H}\right) - A_I \mathcal{H}'(Z - \mathcal{H})^{-1}\right) = z + \mathcal{G}(r), \tag{78}$$

where $\mathcal{G}(r)$ - another arbitrary function to be determined.

Using boundary conditions

Vertical boundary conditions :

1) $Z|_{z=0} = 0$, 2) $Z|_{z=H} = H \Rightarrow$

1. Expression of $\mathcal{G}$ via $\mathcal{H}(r)$

$$\frac{1}{2r}\left(A_l' \log\left(-\mathcal{H}\right) + A_l \mathcal{H}' \mathcal{H}^{-1}\right) = \mathcal{G}(r). \tag{79}$$

2. Equation for $\mathcal{H}$

$$\frac{1}{2r}\left(A_l' \log\left(H - \mathcal{H}\right) - A_l \frac{\mathcal{H}'}{H - \mathcal{H}}\right) = H + \mathcal{G}(r), \tag{80}$$

   where expression for $\mathcal{F}$ (81) to be injected

Using b.c. $R(\infty, z) = r$

$$\mathcal{H} = H\left[1 - \exp\left(\frac{H}{A_l} r^2\right)\right]^{-1} \Rightarrow \tag{81}$$

z as a function of (r, Z) from (79) :

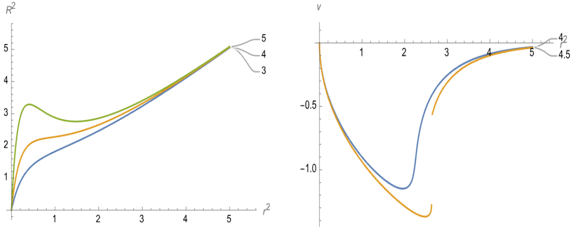
$$z = \frac{1}{2r}\left(A_l'(r) \log\left(1 - \frac{Z}{\mathcal{H}(r)}\right) + A_l(r) \frac{\mathcal{H}'(r)}{\mathcal{H}(r)} \frac{\frac{Z}{\mathcal{H}(r)}}{1 - \frac{Z}{\mathcal{H}(r)}}\right). \tag{82}$$

Simple example of cyclogenesis

Scaling : $z \sim H, r \sim R_d = \frac{gH}{f^2}, b \sim g$. Initial motionless state with Gaussian distribution of (non-dimensional) buoyancy : $v_I = 0,\ b_I = Be^{-r^2/4}$.
At the upper boundary (78) :

$$R^2\big|_{Z=1} = \left(r^2 \exp(r^2/B)\left(\exp(B\exp(-r^2)) - 1\right)\right) \qquad (83)$$

becomes non-monotonous $\leftrightarrow \exists r : \partial R/\partial r = 0$ at $B \geq B_{cr} \approx 4.3$.



R^2 (left) and v (right) as functions of r^2 at $z = Z = H$.

Conclusions

- Geostrophic adjustment in the absence of dissipation can lead to singularity in Lagrangian mapping
- Singularity manifests itself as formation of sharp gradients of buoyancy and velocity ⇔ fronto- or cyclo-genesis
- Sufficient condition of the absence of singularity in the unbounded domain : positiveness of potential vorticity (PV)
- Absence of criteria in the presence of boundaries
- Zero-PV configurations allow for analytic solutions of the adjustment problem and direct proof of singularity formation.