6. Nonlinear waves

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Plan

Weak dispersion/Weak nonlinearity

Weak dispersion/Strong nonlinearity

Strong dispersion/Weak nonlinearity

Strong dispersion/Strong nonlinearity

- Essentially non-linear Rossby waves : modons
- Strongly non-linear internal gravity waves : Long's solutions

Linear waves in Lagrangian 1.5D RSW

Equation for perturbations of Lagrangian coordinates :

$$\ddot{\phi} + f^2\phi + gh_l' \left(\frac{1}{(1+\phi')^2}\right) + \frac{gh_l}{2}\left(\frac{1}{(1+\phi')^2}\right)' = fv_l \ . \quad (1)$$

Initially balanced state : $h_l = H = \text{const},\ v_l = 0.$
Scaling : $(x, \phi) \sim L, t \sim L/\sqrt{gH} \to$

$$\ddot{\phi} + \gamma^2\phi + \frac{1}{2}\left(\frac{1}{(1+\phi')^2}\right)' = 0, \quad \gamma = fL/\sqrt{gH}. \quad (2)$$

Small deviations : $\phi \to \epsilon\phi$, $\epsilon \ll 1$. Linearization ($\equiv$ lowest order in ϵ) $\Rightarrow$ wave equation

$$\ddot{\phi} + \gamma^2\phi - \phi'' = 0. \quad (3)$$

No dispersion (constant phase velocity) if $f = 0 \leftrightarrow \gamma = 0$:

$$\ddot{\phi} - \phi'' = 0 \to \frac{\partial^2\phi}{\partial\xi_+\partial\xi_-} = 0, \quad \xi_\pm := x \pm t \quad (4)$$

General solution : rightward and leftward running waves

$$\phi(x,t) = f_+(\xi_+) + f_-(\xi_-), \quad f_\pm(\xi_\pm) - \text{arbitrary functions} \quad (5)$$

Weak nonlinearity, no rotation

Asymptotic expansion in ϵ, introducing slow time $T \sim \epsilon t$:

$$\phi = \phi_0(x,t,T) + \epsilon\phi_1(x,t,T) + \ldots \rightarrow$$

$$\ddot{\phi}_0 - \phi_0'' = 0. \tag{6}$$

$$\ddot{\phi}_1 - \phi_1'' = -\left(3\phi_0'\phi_0'' + 2\dot{\phi}_{0_T}\right) := -\mathcal{R}\left[\phi_0\right] \tag{7}$$

From (6) : $\phi_0(x,t) = f_+(\xi_+,T) + f_-(\xi_-,T) \Rightarrow$

$$2\frac{\partial^2\phi}{\partial\xi_+\partial\xi_-} = -\mathcal{R}\left[f_+\right] - \mathcal{R}\left[f_-\right] - 3(f_+'f_-'' + f_-'f_+'') \tag{8}$$

Absence of secular growth in $\xi_\pm \Rightarrow$

$$\pm 2f_{\pm_T}' + 3f_\pm' f_\pm'' = 0, \text{as } \dot{f}_\pm = \pm f_\pm'. \tag{9}$$

Recall $X = x + \epsilon\phi$, $\frac{\partial X}{\partial x} = \frac{H}{h} = (1+\epsilon\eta)^{-1} \Rightarrow \eta_0 = \phi_0'$ in non- dimensional terms $\Rightarrow$ simple-wave equations for left(right)-moving height perturbations (subscript omitted) :

$$\mp\eta_{\pm_T} + \frac{3}{2}\eta_\pm\eta_\pm' = 0 \Rightarrow \text{breaking in finite time.} \tag{10}$$

Weak nonlinearity, weak rotation/dispersion

Weak rotation hypothesis : $\gamma^2 \sim \epsilon \rightarrow$ r.h.s. of (7) acquires additional term :

$$\mathcal{R}\left[\phi_0\right] \rightarrow \mathcal{R}\left[\phi_0\right] + \phi_0 \Rightarrow \tag{11}$$

Condition of absence of secular growth changes :

$$\pm 2f'_{\pm_T} + 3f'_{\pm}f''_{\pm} + f_{\pm} = 0 \rightarrow \tag{12}$$

Ostrovsky - Hunter (OH) equation

$$\left(\mp\eta_{\pm_T} + \frac{3}{2}\eta_{\pm}\eta'_{\pm}\right)' - \frac{1}{2}\eta_{\pm} = 0. \tag{13}$$

Exhibits both breaking and finite-amplitude stationary waves.

Periodic nonlinear waves in OH equation

Renormalized equation for steady-moving waves $\eta = \eta(x - cT)$:

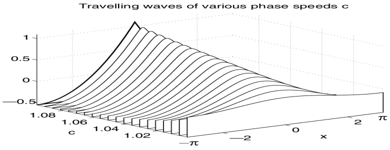
$$(\eta_T + \eta\eta')' - \eta = 0 \;\rightarrow\; (-c\,\eta + \eta\eta')' - \eta = 0, \qquad (14)$$

Solvable in elliptic functions. Continuous 2π-periodic solutions exist only if

$$1 \leq c \leq \frac{\pi^2}{9}, \qquad (15)$$

with the limiting-amplitude cusp wave

$$\eta(x) = \frac{\pi^2}{9} - \frac{\pi}{3}|x| + \frac{1}{2}x^2 \qquad (16)$$



(from J. Boyd, EJAM, 2005)

Integrability and breaking

Observation :

$$F = \sqrt[3]{1 - 3u_{xx}} \qquad (17)$$

obeys a conservation law, which can be exploited in Lagrangian variables $x \to X(\tau, x)$, $J := \partial X/\partial x$:

$$F_T + (uF)_X = 0 \Rightarrow (FJ)_\tau = 0 \Rightarrow FJ = F_I. \qquad (18)$$

$$u_{XX} = J^{-1}(J^{-1}u_x)_x = J^{-1}(J^{-1}J_\tau)_x = J^{-1}(\log J)_{\tau x}. \qquad (19)$$

Combining (17), (18), (19)

$$(\log J)_{\tau x} = \frac{J}{3}\left(1 - \frac{F_I^3}{J^3}\right) \leftrightarrow 3(\log F)_{\tau x} = \frac{F_I}{F}(F^3 - 1) \qquad (20)$$

$\to$ integrable Bullough-Dodd equation

$$3G_{\tau\chi} = e^{2G} - e^{-G} = 0,$$

after changes of variables $G = \log F$, $\chi = \int^X F_I(X')\, dX'$. The latter requires $F \neq 0$, otherwise breaking

Adding non-hydrostatic dispersion

Serre- Green- Naghdi version of (2), with $\delta = H/L$:

$$\ddot{\phi} + \gamma^2\phi + \frac{1}{2}\left(\frac{1}{(1+\phi')^2}\right)' + \frac{\delta^2}{3}\left[\frac{1}{(1+\phi')^2}\left(\frac{\ddot{1}}{(1+\phi')}\right)\right]' = 0 \quad (21)$$

Small perturbations : $\phi \to \epsilon\phi$. Small dispersion : $\delta = \mathcal{O}(\epsilon)$

No rotation $\gamma^2 = 0$: R.h.s of (7) becomes

$$\mathcal{R}\left[\phi_0\right] \to \left(-\ddot{\phi}'_0/3 + 3\phi_0'\phi_0'' + 2\dot{\phi}_{0_T}\right) \quad (22)$$

Killing resonances $\to$ Korteweg - deVries (KdV) equations for $\eta_\pm$, with solitary wave (soliton) solutions, and no breaking :

$$\mp\, \eta_{\pm_T} + \frac{3}{2}\eta_\pm\eta_\pm' - \frac{1}{6}\eta_\pm''' = 0. \quad (23)$$

Weak rotation $\gamma^2 = \mathcal{O}(\epsilon)$: R.h.s of (7) becomes

$$\mathcal{R}\left[\phi_0\right] \to \left(\phi_0 - \ddot{\phi}'_0/3 + 3\phi_0'\phi_0'' + 2\dot{\phi}_{0_T}\right) \quad (24)$$

Killing resonances $\to$ Ostrovsky equation with no breaking :

$$\left(\mp\eta_{\pm_T} + \frac{3}{2}\eta_\pm\eta_\pm' - \frac{1}{6}\eta_\pm'''\right)' - \frac{1}{2}\eta_\pm = 0. \quad (25)$$

Traveling waves in 1.5D RSW

Looking for periodic wave solutions of 1.5D RSW

$$\begin{aligned} \partial_t u + u\partial_x u - fv + g\partial_x h &= 0\,, \\ \partial_t v + u\partial_x v + fu &= 0\,, \\ \partial_t h + u\partial_x h + h\partial_x u &= 0\,. \end{aligned}$$

depending on $\xi = x - ct$ only $((...)' \equiv \partial_\xi(...))$:

$$\begin{aligned} -cu' + uu' - fv + g\partial_x h &= 0\,, \\ -cv' + uv' + fu &= 0\,, \\ -ch' + uh' + h\partial_x u &= 0\,, \end{aligned} \qquad (26)$$

From (26) : $u = c + K/h, K = \text{const}$. Total mass-flux $\int d\xi hu = 0$ over the (yet unknown) wavelength $\Rightarrow K = -cH$, where H is the mean (rest) height, $\Rightarrow u = c\left(1 - \frac{H}{h}\right)$. Elimination of $v \to$ equation for non-dimensional height $\eta = h/H$, where $M = c/\sqrt{gH}$:

$$\left(\frac{M^2}{2\eta^2} + \eta\right)'' - \frac{f^2}{c^2}(\eta - 1) = 0. \qquad (27)$$

Exact solutions - periodic waves

Condition of existence : at the wave-crest $\eta' = 0, \eta'' < 0, \eta > 1$. From (27) : $1 < \eta^3 < M^2 \Rightarrow$ Mach number $M > 1$ ("supersonic" waves).

Multiplying (27) by $\left(\frac{M^2}{2\eta^2} + \eta\right)'$ and integrating once $\rightarrow$ "particle-in-a-well" equation at zero energy :

$$\eta'^2 + \mathcal{U}(\eta; M, A) = 0, \tag{28}$$

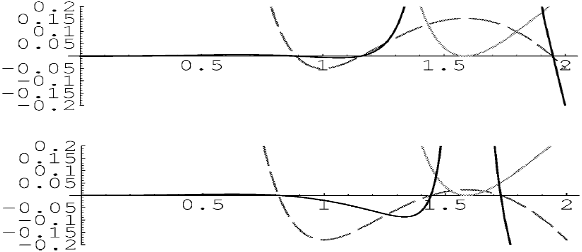
with a "potential" :

$$\mathcal{U}(\eta; M, A) = \frac{f^2}{c^2} \frac{(\eta - 1)^2 \left(\frac{M^2}{\eta^2} - 1\right) - A}{\left(\frac{M^2}{\eta^3} - 1\right)^2}, \tag{29}$$

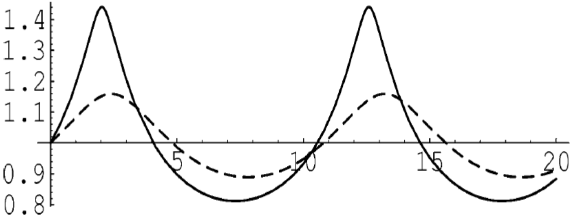
where A - integration constant.
Oscillatory solutions : between zeroes of "potential".

Profiles of nonlinear waves

Zeros of potential (solid line) at $A = 0.05, A = 018$:

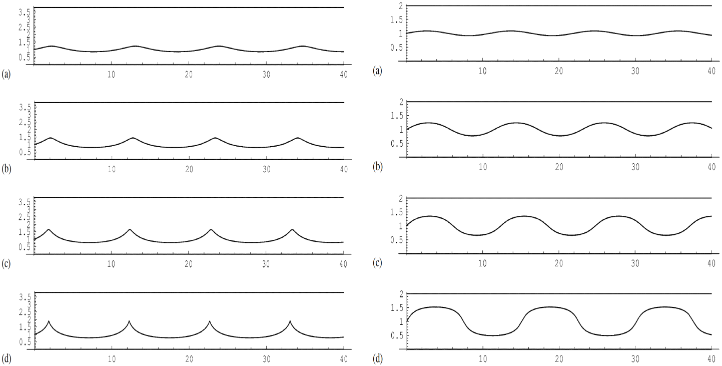


Corresponding profiles of height of nonlinear waves :



Nonlinear periodic waves in 2-layer RSW

Same procedure in 2-layer RSW with a rigid lid $\rightarrow$ two families of nonlinear waves at the interface



Waves of two families of increasing amplitude/phase speed (from top to bottom)

Weakly nonlinear Rossby waves in the QG model

Non-dimensional equations of motion on the β-plane :

$$\nabla^2\psi_t - \psi_t + \epsilon\mathcal{J}(\psi, \nabla^2\psi) + \psi_x = 0,$$

ϵ - non-linearity parameter, $\epsilon = o(1)$. Asymptotic expansion in $\epsilon \to$ solution - asymtotic series :

$$\psi = \psi^{(0)} + \epsilon\psi^{(1)} + \dots$$

Order zero : linear Rossby waves.

$$\nabla^2\psi_t^{(0)} - \psi_t^{(0)} + \psi_x^{(0)} = 0, \Rightarrow$$

$$\psi^{(0)} = \sum_i A_i e^{i(\mathbf{k}_i\cdot\mathbf{x} - \omega(\mathbf{k}_i)t)} + c.c., \ \omega(\mathbf{k}) = -\frac{k}{k^2 + l^2 + 1}, \ \mathbf{k} = (k, l). \tag{30}$$

Order one : first non-linear correction :

$$\nabla^2 \psi_t^{(1)} - \psi_t^{(1)} + \psi_x^{(1)} = -\mathcal{J}(\psi^{(0)}, \nabla^2 \psi^{(0)}), \qquad (31)$$

Term in the r.h.s. :

$$\sum_{i,j} A_i A_j \left[\left(k_i l_j - k_j l_i \right) \mathbf{k}_j^2 \right] e^{i\left[(\mathbf{k}_i + \mathbf{k}_j)\cdot \mathbf{x} - (\omega(\mathbf{k}_i) + \omega(\mathbf{k}_j))t \right]} \qquad (32)$$

$$- \sum_{i,j} A_i A_j^* \left[\left(k_i l_j - k_j l_i \right) \mathbf{k}_j^2 \right] e^{i\left[(\mathbf{k}_i - \mathbf{k}_j)\cdot \mathbf{x} - (\omega(\mathbf{k}_i) - \omega(\mathbf{k}_j))t \right]} + c.c.$$

Integrability conditions : solution $\psi^{(1)}$ should be bounded

Integrability conditions :

$$\begin{aligned} \forall \hat{\psi} : \nabla^2 \hat{\psi}_t - \hat{\psi}_t + \hat{\psi}_x &= 0, \\ \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \hat{\psi}^* \left(\nabla^2 \psi_t^{(1)} - \psi_t^{(1)} + \psi_x^{(1)} \right) &= 0. \end{aligned}$$

Therefore :

$$\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \hat{\psi}^* \left(\mathcal{J}(\psi, \nabla^2 \psi) \right) = 0. \qquad (33)$$

- orthogonality of the r.h.s. to the eigenvectors of the zero-order linear operator.

Resonances and resonant triads

Necessarily $\hat{\psi} \propto e^{i(\hat{\mathbf{k}}\cdot\mathbf{x}-\omega(\hat{\mathbf{k}})t)}$, and (33) becomes :

$$\sum_{i,j} A_i A_j \left[(k_i l_j - k_j l_i)\, \mathbf{k}_j^2 \right] \int_{-\infty}^{\infty} dt\, dx\, dy e^{i\left[(\mathbf{k}_i+\mathbf{k}_j-\hat{\mathbf{k}})\cdot\mathbf{x}-(\omega(\mathbf{k}_i)+\omega(\mathbf{k}_j)-\omega(\hat{\mathbf{k}}))t\right]} -$$
$$\sum_{i,j} A_i A_j^* \left[(k_i l_j - k_j l_i)\, \mathbf{k}_j^2 \right] \int_{-\infty}^{\infty} dt\, dx\, dy e^{i\left[(\mathbf{k}_i-\mathbf{k}_j-\hat{\mathbf{k}})\cdot\mathbf{x}-(\omega(\mathbf{k}_i)-\omega(\mathbf{k}_j)-\omega(\hat{\mathbf{k}}))t\right]} = 0$$

Reminder on Dirac's delta-function :

$$\int_{-\infty}^{\infty} dk e^{ik(x-x')} = \delta(x-x') : \int_{-\infty}^{\infty} dx' F(x')\delta(x-x') = F(x) \tag{34}$$

Non-zero contributions :

$$\mathbf{k}_i \pm \mathbf{k}_j = \hat{\mathbf{k}}, \quad \omega(\mathbf{k}_i) \pm \omega(\mathbf{k}_j) = \omega(\hat{\mathbf{k}}). \tag{35}$$

three-wave resonances, resonant triads.

Elimination of resonances by amplitude modulation

If $\exists\, \hat{\mathbf{k}}$ which verifies (35) the first non-linear correction is not bounded $\Rightarrow$ asymptotic procedure is not self-consistent : resonances should be "killed".

Introducing slow evolution of the amplitudes :

$$\begin{aligned} \partial_t \to \partial_t + \epsilon \partial_T \quad &\Rightarrow \\ \nabla^2 \psi_t^{(1)} - \psi_t^{(1)} + \psi_x^{(1)} \quad &= \quad -\nabla^2 \psi_T^{(0)} - \psi_T^{(0)} - \mathcal{J}(\psi^{(0)}, \nabla^2 \psi^{(0)}) \end{aligned}$$

New contribution in the r.h.s. :

$$\sum_i A_{i_T} e^{i(\mathbf{k}_i \cdot \mathbf{x} - \omega(\mathbf{k}_i)t)} + c.c. \quad \Rightarrow \tag{36}$$

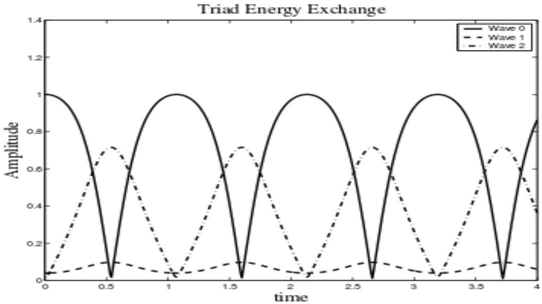
Possibility of compensation of resonant contributions by slow evolution of amplitudes.

Modulation equations for a resonant triad

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3, \quad \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) = \omega(\mathbf{k}_3), \tag{37}$$

$$\begin{aligned} \dot{A}_3 &= c(\mathbf{k}_1, \mathbf{k}_2) A_1 A_2, \\ \dot{A}_2 &= c(\mathbf{k}_3, -\mathbf{k}_1) A_3 A_1^*, \\ \dot{A}_1 &= c(\mathbf{k}_3, -\mathbf{k}_2) A_3 A_2^*, \end{aligned} \tag{38}$$

$c(\mathbf{k}_1, \mathbf{k}_2) = \hat{\mathbf{z}} \cdot (\mathbf{k}_1 \wedge \mathbf{k}_2) \mathbf{k}_2^2$ - interaction coefficients.
System integrable in elliptic functions.



Strongly non-linear Rossby waves

Starting point : QG equations on the β- plane

$$\nabla^2\psi_t - \psi_t + \mathcal{J}(\psi, \nabla^2\psi) + \psi_x = 0, \tag{39}$$

Solutions propagating without change of form :

$$\psi(x, y, t) = \psi(x - Ut, y), \Rightarrow \mathcal{J}(\psi + Uy, \nabla^2\psi + (1 + U)y) = 0 \Rightarrow$$

$$\nabla^2\psi + (1 + U)y = F(\psi + Uy), \tag{40}$$

F - arbitrary function. Domains with different F admitted $\Rightarrow$ matching. Localized solutions :

$$r = \sqrt{x^2 + y^2} \to \infty \Rightarrow \psi \to 0 \tag{41}$$

$\Rightarrow$ "external" F - linear function :

$$F(\psi + Uy) = p^2(\psi + Uy), \; p^2 = \frac{1 + U}{U} > 0 \Rightarrow \tag{42}$$

Either $U > 0$ or $U < -1$ out of the interval of linear Rossby waves' phase velocities

Modons

Solutions with F linear outside and inside a circle $r = a$:

$$\begin{aligned} \nabla^2\psi &= p^2(\psi + Uy) - (1+U)y, \ r > a, \\ \nabla^2\psi &= -k^2(\psi + Uy) - (1+U)y, \ r < a, \end{aligned} \quad (43)$$

Solutions in polar coordinates $x = r\cos\phi$, $y = r\sin\phi \rightarrow$ Bessel functions :

$$\begin{aligned} \psi &= BK_1(pr)\sin\phi, \ r > a, \\ \psi &= \left[AJ_1(kr) - \frac{r}{k^2}(1 + U + Uk^2)\right]\sin\phi, \ r < a \end{aligned} \quad (44)$$

where J_1 - Bessel function (oscillating), K_1 - modified Bessel function (decaying), A, B - constants to be determined from the conditions of matching and b.c..

Matching

Continuity of $\psi, \partial_r\psi, \partial^2_{rr}\psi$:

$$\begin{aligned} \psi + Uy|_{a-} &= \psi + Uy|_{a+} = 0, \\ \partial_r\psi|_{a-} &= \partial_r\psi|_{a+} . \end{aligned} \tag{45}$$

2 first conditions giving A, B :

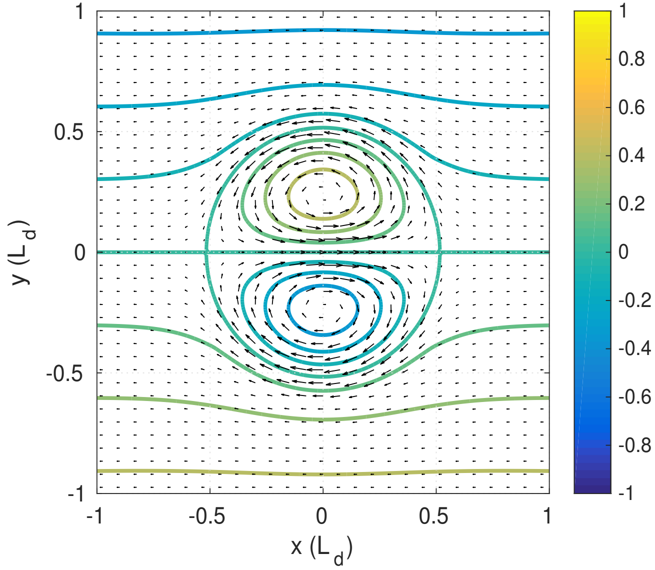
$$A = \frac{a(1+U)}{k^2 J_1(ka)}, \quad B = -\frac{Ua}{K_1(pa)}. \tag{46}$$

3rd condition : determining k :

$$\frac{J_1'(ka)}{J_1(ka)} = \frac{1}{ka}\left(1 + \frac{k^2}{p^2}\right) - \frac{k}{p}\frac{K_1'(pa)}{K_1(pa)}. \tag{47}$$

$\forall(a,p)$ infinite series of solutions for k. Minimal value of k - dipolar structure.

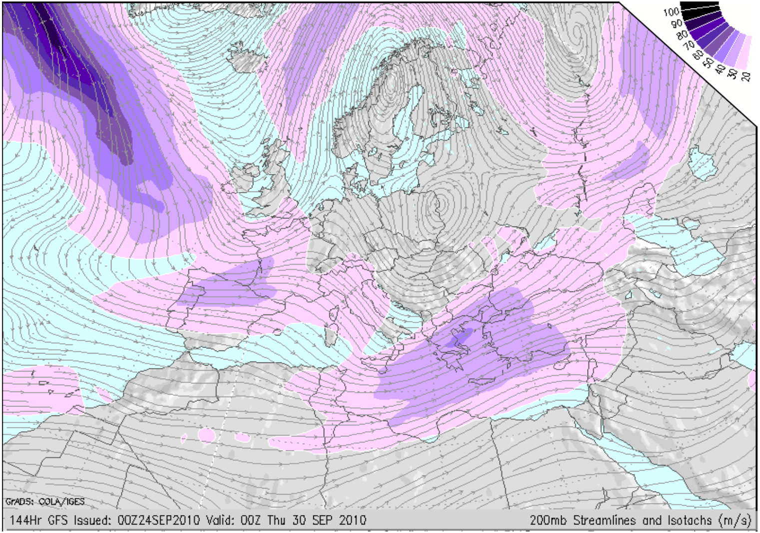
Streamfunction of a modon



Generalizations

- Equations of RSW on the equatorial beta-plane in Charney regime are equivalent to (39) in the limit $R_d \to \infty \Rightarrow$ the same procedure holds giving only eastward propagating modons
- QG equations on the f-plane admit the same procedure giving modons propagating in any directions $\leftrightarrow$ dual character of modons, which are, simultaneously, waves and vortices.
- TQG equations, both on f- and β- planes admit the same procedure giving thermal modons, which carry buoyancy anomaly in their cores.

Modons and atmospheric blockings



Stratified non-rotating fluid in Boussinesq equations

$$\begin{aligned} u_t + uu_x + wu_z + \phi_x &= 0, \\ w_t + uw_x + ww_z + \Xi + \phi_z &= 0, \\ u_x + w_z = 0, \quad \Xi_t + u\Xi_x + w\Xi_z &= 0. \end{aligned} \tag{48}$$

$\Xi = \frac{g(\rho(z)+\sigma)}{\rho_0}$ - buoyancy variable, including the effects of background stratification $\rho(z)$, $\phi = \frac{P}{\rho_0}$ - geopotential, ρ_0 - constant normalisation density, σ - density perturbation. Equations in streamfunction - buoyancy variables :

$$\begin{aligned} \Delta\psi_t + \mathcal{J}(\psi, \Delta\psi) + \Xi_x &= 0, \\ \Xi_t + \mathcal{J}(\psi, \Xi) &= 0. \end{aligned} \tag{49}$$

ψ - streamfunction, $\zeta = -\Delta\psi$ - horizontal vorticity, Δ - Laplacian, $\mathcal{J}$ - Jacobian.

Hydrostatic limit (long waves)

Replacement of the equation for w by hydrostatic equation $-\Xi = \phi_z$:

$$\begin{aligned} \psi_{zzt} + \mathcal{J}(\psi, \psi_{zz}) + \Xi_x &= 0, \\ \Xi_t + \mathcal{J}(\psi, \Xi) &= 0. \end{aligned} \quad (50)$$

Stationary solutions (change of reference frame ⇒ propagation at constant speed) :

$$\begin{aligned} \mathcal{J}(\psi, \Delta\psi) + \Xi_x &= 0, \\ \mathcal{J}(\psi, \Xi) &= 0. \end{aligned} \tag{51}$$

Therefore $\Xi = \Xi(\psi)$ and

$$\begin{aligned} \mathcal{J}(\psi, \Delta\psi) + \Xi'(\psi)\psi_x = 0 \quad &\Rightarrow \\ \mathcal{J}(\psi, \Delta\psi + \Xi'(\psi)z) = 0 \quad &\Rightarrow \\ \Delta\psi + \Xi'(\psi)z = F(\psi), & \end{aligned} \tag{52}$$

where $\Xi(\psi)$ and $F(\psi)$ - arbitrary functions.

Long's waves

Upstream ($x \to \infty$) : b.c. of constant velocity $\to \psi = cz$, and of a given stratification $\Xi = \Xi_0(z) \Rightarrow$

$$\Xi(\psi) = \Xi_0(\frac{\psi}{c}), \quad F(\psi) = \Xi'(\psi)\frac{\psi}{c} = \frac{\psi}{c^2}\Xi_0'(\frac{\psi}{c}) \tag{53}$$

Example : linear stratification upstream : $\Xi_0 = \text{const} + \alpha z$ $\to$ Long's equation (linear !) for a non-linear stationary wave :

$$\Delta\psi + \frac{\alpha}{c}(z - \frac{\psi}{c}) = 0. \tag{54}$$

New variable - deviation of streamlines :

$$\phi = \psi - cz, \Rightarrow \Delta\phi - \frac{\alpha}{c^2}\phi = 0. \tag{55}$$

B.c. in z : $\psi|_{z=h(x)} = h(x)$, h - topography.