1. Lagrangian approach in hydrodynamics

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Plan

Reminder on hydrodynamics

Where the governing equations come from :

- Mechanical system $\Rightarrow$ Newton's 2nd law $\leftrightarrow$ momentum conservation.
- Continuous medium $\Rightarrow$ local mass conservation
- Thermodynamical system $\Rightarrow$ 1st and 2nd laws of thermodynamics, equation of state
- Dissipative effects $\Rightarrow$ flux-gradient relations.

Description in terms of instantaneous positions in 3D space of fluid parcels $\boldsymbol{X}(\boldsymbol{x}, t)$, along their trajectories, where $\boldsymbol{x}$ are initial positions (Lagrangian labels).
Newton's 2nd law :

$$\rho(\boldsymbol{X}, t)\frac{d^2\boldsymbol{X}}{dt^2} = -\boldsymbol{\nabla} P(\boldsymbol{X}, t), \tag{1}$$

where ρ and P are density and pressure in the fluid.
Continuity equation :

$$\rho_i(x)d^3\boldsymbol{x} = \rho(\boldsymbol{X}, t)d^3\boldsymbol{X}, \leftrightarrow \rho_i(x) = \rho(\boldsymbol{X}, t)\mathcal{J} \tag{2}$$

where ρ_i is initial distribution of density, $\mathcal{J} = \frac{\partial(X,Y,Z)}{\partial(x,y,z)}$ is the Jacobi determinant (Jacobian).
Fluid velocity : $\boldsymbol{v}(\boldsymbol{X}, t) = \frac{d\boldsymbol{X}}{dt} \equiv \dot{\boldsymbol{X}}$.

Fluid dynamics according to Euler :

Description in terms of instantaneous values of the velocity, density and pressure fields at the fixed point of space : $\boldsymbol{v}(\boldsymbol{x}, t)$, $\rho(\boldsymbol{x}, t)$, $P(\boldsymbol{x}, t)$. Euler-Lagrange duality : there is a fluid parcel at any point $\boldsymbol{x}$, and at any time : $\boldsymbol{X} \leftrightarrow \boldsymbol{x}$

Newton's 2nd law :

$$\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} \right) = -\boldsymbol{\nabla} P. \tag{3}$$

Continuity equation :

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0. \tag{4}$$

Lagrangian derivative :

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}. \tag{5}$$

Closure of the system : equation of state

General equation of state

$$P = P(\rho, s), \tag{6}$$

where s - entropy per unit mass ;

- Barotropic(isentropic) fluid :

$$P = P(\rho) \leftrightarrow s = \text{const}, \tag{7}$$

- Baroclinic fluid :

$$P = P(\rho, s), \Rightarrow \tag{8}$$

Equation for s neccessary. Perfect fluid :

$$\frac{ds}{dt} = \frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} s = 0. \tag{9}$$

Particular case of the barotropic fluid - incompressible fluid :

Volume conservation :

$$\mathcal{J} = 1 \leftrightarrow \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \Rightarrow . \tag{10}$$

pressure is not an independent variable.

1. If in addition, $\rho = const$:

$$\boldsymbol{\nabla} \cdot (\boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}) = -\frac{1}{\rho} \boldsymbol{\nabla}^2 P. \tag{11}$$

2. Otherwise

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \rho = 0. \tag{12}$$

and

$$\boldsymbol{\nabla} \cdot (\boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}) = -\boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{\nabla} P}{\rho} \right). \tag{13}$$

Thermodynamics : reminder

1st principle, "dry" thermodynamics

$$\delta\epsilon = T\delta s - P\delta v, \tag{14}$$

where ϵ - internal energy per unit mass, $v = \frac{1}{\rho}$ - volume per unit mass, δ denote small variations.
Enthalpy per unit mass : $h = \epsilon + Pv$:

$$\delta h = T\delta s + v\delta P. \tag{15}$$

Energy density of the fluid :

$$e = \frac{\rho \boldsymbol{v}^2}{2} + \rho\epsilon. \tag{16}$$

Local conservation of energy :

$$\frac{\partial e}{\partial t} + \boldsymbol{\nabla} \cdot \left[\rho \boldsymbol{v} \left(\frac{\boldsymbol{v}^2}{2} + h\right)\right] = 0. \tag{17}$$