Lagrangian theory of nonlinear gravity waves in shallow-water and related models

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International Center for Mathematics,
SUSTech, Shenzhen, 2024

Plan

1.5D RSW

Preliminary remarks

Standard first approach in studying waves : looking for plane waves i.e. 1D waves. Rotation mixes up 2 velocity components $\Rightarrow$ purely 1D approach impossible.
1.5D system : no dependence on one spatial coordinate, but corresponding velocity maintained.

Lagrangian 1.5D RSW

(Semi-) Lagrangian momentum equations with no y-dependence :

$$\begin{aligned} \ddot{X}(x,t) - fv + g\partial_X h(X,t) &= 0\,, \\ \dot{v}(X,t) + f\dot{X}(x,t) &= 0\,, \end{aligned} \tag{1}$$

Initial-value problems :
$X(x,0) = x,\ u(X,0) = u_I(x),\ v(X,0) = v_I(x),\ h(X,0) = h_I(x).$
Mass conservation :

$$h(X,t)\,dX = h_I(x)\,dx, \ \Rightarrow \ h(X,t) = h_I(x)\partial_X x. \tag{2}$$

Reduction to a single PDE

Straightforward integration of(53) :

$$v(X,t) + f\,X(x,t) = f\,x + v_I(x)\,, \tag{3}$$

Chain differentiation ($' \equiv \partial_x$) :
$\partial_X h = \partial_X\left(h_I(x)\partial_x X\right) = h_I'\left(X'\right)^{-2} - h_I(x)X''\left(X'\right)^{-3}, \rightarrow$
Closed 2nd order nonlinear "Master" equation for X :

$$\ddot{X} + f^2 X + g h_I'\left(X'\right)^{-2} + \frac{g h_I}{2}\left[\left(X'\right)^{-2}\right]' = f\left(fx + v_I(x)\right). \tag{4}$$

In terms of parcels' displacements : $\chi(x,t) = X(x,t) - x$:

$$\ddot{\chi} + f^2\chi + g h_I'\left(\frac{1}{(1+\chi')^2}\right) + \frac{g h_I}{2}\left(\frac{1}{(1+\chi')^2}\right)' = f v_I\,. \tag{5}$$

Focus on propagating waves : considered on the whole x-axis with rapid decay boundary conditions.

Scaling, non-dimensionalization, linearization

Starting with a state of rest (or using mass-weighted labels) :

$$h_l = H = \text{const},\ v_l = 0\ \Rightarrow h = H/(1+\chi'),\quad v = -f\,\chi. \tag{6}$$

Scaling adapted to wave motions :

$$(x,\chi) \sim L,\ t \sim L/\sqrt{gH} \to u \sim \sqrt{gH} \to \tag{7}$$

$$\ddot{\chi} + \gamma^2\chi + \frac{1}{2}\left(\frac{1}{(1+\chi')^2}\right)' = 0,\ \ \gamma = fL/\sqrt{gH}. \tag{8}$$

Control parameter ϵ for wave amplitudes : $\chi \to \epsilon\chi,\ \epsilon \ll 1$.
Linearization $\Leftrightarrow$ lowest order in $\epsilon \to$ Klein-Gordon equation :

$$\ddot{\chi} + \gamma^2\chi - \chi'' = 0. \tag{9}$$

Solutions : harmonic inertia-gravity waves with wavenumbers k and frequencies ω : $\chi \propto e^{i(kx-\omega t)} \;\to,$

$$\omega = \pm\sqrt{k^2+\gamma^2} \tag{10}$$

Long-wave dispersion, short waves non-dispersive.

General solution in non-rotating limit

Below : limit of weak rotation

No rotation : $f = 0 \leftrightarrow \gamma = 0 \Rightarrow$ no dispersion (phase velocity $c = \omega/k =$ const $\Rightarrow$ standard second-order wave equation for gravity waves :

$$\ddot{\chi} - \chi'' = 0 \leftrightarrow \frac{\partial^2 \chi}{\partial \xi_+ \partial \xi_-} = 0, \quad \xi_\pm := x \pm t, \tag{11}$$

$\xi_\pm$ - characteristic variables. General solution : rightward and leftward running wave-packets keeping their shape :

$$\chi(x, t) = F_+(\xi_+) + F_-(\xi_-), \tag{12}$$

where $F_\pm(\xi_\pm)$ are arbitrary functions - envelopes of superpositions of harmonic waves with phases $k(x \pm t)$ and different wavenumbers k.

Remarks : $\chi =$ const - trivial solution if b. c. are periodic in space. $\chi \propto t$ is admissible solution, if fluid parcels are allowed to move with a constant velocity.

Galilean invariance allows to exclude this.

Asymptotic expansion, no rotation

Solution of (8) : multi time scale asymptotic expansion in ϵ, with a slow time $T \sim \epsilon^{-1} t$:

$$\chi = \chi_0(x, t, T) + \epsilon \chi_1(x, t, T) + \dots .$$

No rotation, $\gamma \equiv 0$. First two orders in ϵ :

$$\ddot{\chi}_0 - \chi_0'' = 0, \tag{13}$$

$$\ddot{\chi}_1 - \chi_1'' = -\left(3\chi_0' \chi_0'' + 2\dot{\chi}_{0_T}\right) := -\mathcal{R}\left[\chi_0\right] \tag{14}$$

From (13) : $\chi_0(x, t) = F_+(\xi_+, T) + F_-(\xi_-, T) \Rightarrow$

$$2\frac{\partial^2 \chi}{\partial \xi_+ \partial \xi_-} = -\mathcal{R}\left[F_+\right] - \mathcal{R}\left[F_-\right] - 3(F_+' F_-' + F_-' F_+'') \tag{15}$$

Terms depending only on F_+ or on F_- in r.h.s. of (15) are resonant : independence on one of the variables $\Rightarrow$ linear growth of solution in this variable after integration over it. Mixed terms non-resonant if $F_\pm$ bounded and decaying at $\pm\infty$.

Killing resonances : modulation equations

Absence of secular growth in $\xi_\pm \Rightarrow$

$$\pm 2F'_{\pm_T} + 3F'_\pm F''_\pm = 0, \tag{16}$$

where $\dot{F}_\pm = \pm F'_\pm$ is used. Modulation equations describing slow evolution of the wave-packet envelopes.
Recall

$$X = x + \epsilon\chi, \frac{\partial X}{\partial x} = \frac{H}{h} = (1+\epsilon\eta)^{-1} \Rightarrow \eta_0 = -\chi'_0 \tag{17}$$

where $\epsilon\,\eta$ is a non-dimensional height perturbation $\rightarrow$ (16) $\equiv$ Hopf equation =inviscid Burgers = simple-wave equation (subscript 0 omitted) :

$$\mp\,\eta_{\pm_T} + \frac{3}{2}\eta_\pm\eta'_\pm = 0 \tag{18}$$

Describes wave-breaking and shock formation in finite time.

Slow rotation

Suppose $\gamma^2 \sim \epsilon \Rightarrow$ r.h.s. of (14) acquires additional term :

$$\mathscr{R}[\chi_0] \to \mathscr{R}[\chi_0] + \chi_0. \tag{19}$$

Conditions of absence of secular growth change, and give :

$$\pm 2F'_{\pm T} + 3F'_{\pm}F''_{\pm} + F_{\pm} = 0. \tag{20}$$

Differentiating once $\to$ reduced Ostrovsky = Ostrovsky-Hunter (OH) = Vakhnenko equation

$$\left(\mp\eta_{\pm T} + \frac{3}{2}\eta_{\pm}\eta'_{\pm}\right)' - \frac{1}{2}\eta_{\pm} = 0, \tag{21}$$

Exhibits both wave-breaking and shock formation and finite-amplitude stationary waves, depending on initial conditions.

Comment on inertial oscillations

Arbitrary constant $\bar{\chi}$ can be added to zeroth-order solution, if boundary conditions allow. Can be considered as a function of an intermediate slow time $\tau = \sqrt{\epsilon}$. Additional resonant term appears in the r.h.s. of the equation for χ_1, and should be "killed" $\Rightarrow$ harmonic oscillator equation for $\bar{\chi}$:

$$\frac{d^2\bar{\chi}}{d\tau^2} + \bar{\chi} = 0. \tag{22}$$

Recall

$$u = \dot{\bar{\chi}}, \quad v = -\bar{\chi} \;\rightarrow$$

(22) $\equiv$ equations for inertial oscillations :

$$\frac{d\bar{u}}{d\tau} - \bar{v} = 0, \quad \frac{d\bar{v}}{d\tau} + \bar{u} = 0, \tag{23}$$

$\bar{u}$ and $\bar{v}$ are x- and t- independent components of the velocity field. $\Rightarrow$ wave-packets in the presence of rotation propagate on a slowly oscillating background.

Eulerian 1.5D RSGN

$$\begin{cases} \partial_t u + u\partial_x u - fv + g\partial_x h + \frac{1}{3h}\partial_x\left(h^2(\partial_t + u\partial_x)^2 h\right) = 0 \ , \\ \partial_t v + u\partial_x v + fu = 0 \ , \\ \partial_t h + u\partial_x h + h\partial_x u = 0 \ , \end{cases} \quad (24)$$

Lagrangian RSGN and master equation

Mass and the y- momentum equations- same as in 1.5D RSW, x- momentum equation :

$$\ddot{X} - fv + g\partial_X h + \frac{1}{3h}\partial_X\left(h^2\ddot{h}\right) = 0. \quad (25)$$

Same scaling $\rightarrow$ RSGN master equation :

$$\ddot{\chi} + \gamma^2\chi + \frac{1}{2}\left(\frac{1}{(1+\chi')^2}\right)' + \frac{\delta^2}{3}\left[\frac{1}{(1+\chi')^2}\left(\frac{\ddot{1}}{(1+\chi')}\right)\right]' = 0, \quad (26)$$

Parameter $\delta = H/L$ controls non-hydrostatic effects.

No rotation

The non-hydrostatic effects : weak by construction, we take $\delta^2 = \mathcal{O}(\epsilon)$, no rotation : $\gamma^2 \equiv 0 \; \to$ r.h.s. of (14) becomes

$$\mathcal{R}\left[\chi_0\right] \to \left(-\ddot{\chi}''_0/3 + 3\chi_0'\chi_0'' + 2\dot{\chi}_{0_T}\right) \tag{27}$$

Killing resonances $\to$:

$$\pm 2F'_{\pm_T} + 3F'_{\pm}F''_{\pm} + \frac{1}{3}F''''_{\pm} = 0, \tag{28}$$

equivalent to Korteweg - deVries (KdV) equations for $\eta_{\pm}$:

$$\mp \eta_{\pm_T} + \frac{3}{2}\eta_{\pm}\eta'_{\pm} - \frac{1}{6}\eta'''_{\pm} = 0. \tag{29}$$

If considered on the whole x- axis :
no wave-breaking, completely integrable, and has solitary wave (soliton), and multi-soliton solutions.
If considered on the finite interval :
exact periodic finite-amplitude wave solutions.

Slow rotation

If $\gamma^2 = \mathcal{O}(\epsilon)$, r.h.s of (14) becomes

$$\mathcal{R}\left[\chi_0\right] \to \left(\chi_0 - \ddot{\chi}''_0/3 + 3\chi_0'\chi_0'' + 2\dot{\chi}_{0T}\right) \qquad (30)$$

Killing resonances $\to$:

$$\pm 2F'_{\pm T} + 3F'_{\pm}F''_{\pm} + \frac{1}{3}F''''_{\pm} + F_{\pm} = 0. \qquad (31)$$

Ostrovsky = rotation-modified KdV equation. Extra differentiation and passage to η variables :

$$\left(\mp\eta_{\pm T} + \frac{3}{2}\eta_{\pm}\eta'_{\pm} - \frac{1}{6}\eta'''_{\pm}\right)' - \frac{1}{2}\eta_{\pm} = 0. \qquad (32)$$

Admit no soliton solutions, nor shock formation, and is not completely integrable.

If periodic b.c., slow inertial oscillations accompany even slower inertia-gravity wave packets.

Comments on "zero-mass paradox"

"Zero-mass paradox" : constraint $\int dx\, \eta = 0$ not imposed in derivation. Lagrangian view : no paradox.

$$\int_{-L}^{L} \eta dx = -\int_{-L}^{L} \chi' dx = \chi|_L - \chi|_{-L} \equiv 0.$$

Same constraint for χ : $\mathcal{X}_1 \equiv \int dx\, \chi = 0 \Leftrightarrow \Leftrightarrow \int dx\, x\, \eta = 0$ for (32). Integrate (26) over the domain of the flow :

$$\ddot{\mathcal{X}}_1 + \gamma^2 \mathcal{X}_1 = 0, \tag{33}$$

harmonic oscillator equation. Asymptotic expansion :

$$\left(\frac{\partial^2}{\partial t^2} + 2\epsilon \frac{\partial^2}{\partial t\, \partial T} + \epsilon^2 \frac{\partial^2}{\partial T^2}\right)(\mathcal{X}_1^{(0)} + \epsilon \mathcal{X}_1^{(1)} + \dots) + \epsilon\, (\mathcal{X}_1^{(0)} + \epsilon \mathcal{X}_1^{(1)} + \dots) = 0. \tag{34}$$

Lowest order : $\mathcal{X}_1^{(0)}$ does not depend on t. Next order :

$$\frac{\partial^2}{\partial t^2}\mathcal{X}_1^{(1)} + 2\frac{\partial^2}{\partial t\, \partial T}\mathcal{X}_1^{(0)} + \mathcal{X}_1^{(0)} = 0. \tag{35}$$

$\mathcal{X}_1^{(0)}$ does not depend on t $\Rightarrow$ to exclude secular growth of $\mathcal{X}_1^{(1)}$ the lowest-order $\mathcal{X}_1^{(0)}$ should vanish $\Rightarrow$ singular character of the asymptotic expansion in γ^2.

Lagrangian 2.5D equations for the deep layer

Volume-preserving flow map :
$(x, z) \to (X(x, z, t), Z(x, z, t)) \to$

$$\ddot{X} - fv = -\frac{1}{\rho}\frac{\partial P}{\partial X} = -\frac{1}{\rho_2}\frac{\partial(P, Z)}{\partial(x, z)} \quad (36)$$

$$\dot{v} + f\dot{X} = 0 \quad (37)$$

$$\ddot{Z} + g = -\frac{1}{\rho}\frac{\partial P}{\partial Z} = -\frac{1}{\rho_2}\frac{\partial(X, P)}{\partial(x, z)} \quad (38)$$

$$\frac{\partial(X, Z)}{\partial(x, z)} = 1. \quad (39)$$

Flat bottom : $Z = z = -H_2$, and interface top : $Z = \eta$ - material surfaces.

Solving for v : $v = v_I - f(X - x)$ and introducing the deviations of fluid parcels from initial positions, and deviation of pressure from its hydrostatic value :

$$X(x, z, t) = x + \chi(x, z, t), \quad Z = z + \zeta(x, z, t), \quad P = -\rho_2 g Z + p.$$

Interface deviation : $\eta(x, t) \equiv \zeta(x, 0, t)$.

Scaling and non-dimensionalization

Scaling :

$$(x, \chi) \sim L, \ (z, \zeta) \sim H_2, \ t \sim L/\sqrt{gH_2}, \ p \sim \rho_2 g H_2, \ v_I \sim \sqrt{gH_2}.$$

Deep layer : $H_2 \sim L \Rightarrow \delta_2 = \frac{H_2}{L} = \mathcal{O}(1) \Rightarrow$ hydrostatics out.

Non-dimensional system for deviations, $\gamma_2^2 = \frac{f^2 L}{g}$:

$$\ddot{\chi} + \gamma_2^2 \chi + \frac{\partial p}{\partial x} + \frac{\partial (p, \zeta)}{\partial (x, z)} = \gamma_2 v_I, \quad (40)$$

$$\ddot{\zeta} + \frac{\partial p}{\partial z} + \frac{\partial (\chi, p)}{\partial (x, z)} = 0, \quad (41)$$

$$\frac{\partial \chi}{\partial x} + \frac{\partial \zeta}{\partial z} + \frac{\partial (\chi, \zeta)}{\partial (x, z)} = 0, \quad (42)$$

Small-amplitude motions in the deep layer

Setup

Amplitudes of the displacements small : $|\chi|, |\zeta| = \mathcal{O}(\epsilon) \ll 1$.
Waves over the state of rest : $v_l = 0$. Slow rotation : $\gamma_2^2 \ll \epsilon$.

Consistent with the previous $\gamma_1^2 = \dfrac{f^2 L^2}{g H_1} \sim \epsilon$, $\delta_1^2 = \dfrac{H_1^2}{L^2} \sim \epsilon$

for a thin layer choice is $\gamma_2^2 \sim \epsilon^{\frac{3}{2}}$, as $\gamma_2^2 = \gamma_1^2 \, \delta$.

Equations in the leading order in ϵ :

$$\ddot{\chi}^{(1)} + \frac{\partial p^{(1)}}{\partial x} = 0, \tag{43}$$

$$\ddot{\zeta}^{(1)} + \frac{\partial p^{(1)}}{\partial z} = 0, \tag{44}$$

$$\frac{\partial \chi^{(1)}}{\partial x} + \frac{\partial \zeta^{(1)}}{\partial z} = 0. \tag{45}$$

Solving for pressure

Cross-differentiating and using (45) → Laplace equation :

$$\frac{\partial^2 p^{(1)}}{\partial x^2} + \frac{\partial^2 p^{(1)}}{\partial z^2} = 0. \tag{46}$$

To be solved, within same accuracy, in the strip $-1 \leq z \leq 0$, with the Neumann boundary conditions following from (44) :

$$\left.\frac{\partial p^{(1)}}{\partial z}\right|_{z=-1}, \quad \left.\frac{\partial p^{(1)}}{\partial z}\right|_{z=0} = -\ddot{\zeta}^{(1)}(x,0,t). \tag{47}$$

Solution well-known, can be obtained by Fourier transformations :

$$p^{(1)}(x,z,t) = -\int_{-\infty}^{+\infty} d\bar{x} \int_{-\infty}^{+\infty} dk\, e^{ik(x-\bar{x})} \frac{\cosh k(z+1)}{k \sinh k} \ddot{\zeta}^{(1)}(\bar{x},0,t). \tag{48}$$

Relevant for upper layer quantity is

$$\left.\frac{\partial p^{(1)}}{\partial x}\right|_{z=0} = -\frac{1}{2}\int_{-\infty}^{+\infty} d\bar{x}\, \ddot{\zeta}^{(1)}(\bar{x},0,t) \coth \frac{\pi(\bar{x}-x)}{2}. \tag{49}$$

Eulerian 1.5D equations for upper-layer

$$\begin{cases} \partial_t u_1 + u_1 \partial_x u_1 - f v_1 = +g \partial_x h_1 - \frac{1}{\rho_1} \partial_x \left. P \right|_{z=H_1-h_1} , \\ \partial_t v_1 + u_1 \partial_x v_1 + f u_1 = 0 , \\ \partial_t h_1 + u_1 \partial_x h_1 + h_1 \partial_x u_1 = 0 , \end{cases} \tag{50}$$

P - pressure at the interface.
Displacement of the upper surface of the deep layer $\zeta(x, 0, t)$ in terms of the thickness of the upper layer :

$$\zeta(x, 0, t) = Z(x, 0, t) = H_1 - h_1(x, t). \tag{51}$$

Restoring dimensions and omitting superscripts :

$$\begin{aligned} \left. \frac{\partial p}{\partial x} \right|_{z=0} &= -\frac{\rho_2}{2H_2} \int_{-\infty}^{+\infty} d\bar{x}\, \ddot{\zeta}(\bar{x}, 0, t) \coth \frac{\pi (\bar{x} - x)}{2H_2} \\ &\equiv -\rho_2 \mathcal{F} \left[\ddot{\zeta}(\bar{x}, 0, t) \right] . \end{aligned}$$

Closed system for upper layer

Eulerian form :

$$\begin{cases} \partial_t u_1 + u_1 \partial_x u_1 - f v_1 = g \left(1 - \frac{\rho_2}{\rho_1}\right) \partial_x h_1 + \frac{\rho_2}{\rho_1} \mathcal{F}\left[\ddot{h}_1(\bar{x}, t)\right] , \\ \partial_t v_1 + u_1 \partial_x v_1 + f u_1 = 0 , \\ \partial_t h_1 + u_1 \partial_x h_1 + h_1 \partial_x u_1 = 0 . \end{cases} \tag{52}$$

$\mathcal{F} \to$ Hilbert transform for $H_2 \to \infty$:

$\mathcal{F}[h] \to \int_{-\infty}^{+\infty} d\bar{x} \, \frac{h(\bar{x})}{(\bar{x} - x)} \equiv \mathcal{H}[h].$

Lagrangian form :

$$\begin{aligned} \ddot{X}_1 - f v_1 + g \left(\frac{\rho_2}{\rho_1} - 1\right) \partial_X h_1 &= \frac{\rho_2}{\rho_1} \mathcal{F}\left[\ddot{h}_1(\bar{x}, t)\right] , \\ h_1 \, dX = h_{1_I} \, dx; \quad \dot{v}_1 + f \dot{X}_1 &= 0 . \end{aligned} \tag{53}$$

Master equation for the upper layer

Same scaling with $H \to H_1$, and same manipulations as before $\to$ non-dimensional master equation for deviations $\chi = X_1 - x$:

$$\ddot{\chi} + \gamma^2 \chi + \frac{1}{2}\left(\frac{1}{(1+\chi')^2}\right)' = \frac{\delta}{2}\frac{\rho_2}{\rho_1}\mathcal{F}\left[\frac{\ddot{1}}{(1+\chi')}\right]. \quad (54)$$

Keeping nonlinear in χ terms in r.h.s. is over the accuracy $\to$

$$\ddot{\chi} + \gamma^2 \chi + \frac{1}{2}\left(\frac{1}{(1+\chi')^2}\right)' = -\frac{\delta}{2}\frac{\rho_2}{\rho_1}\mathcal{F}[\ddot{\chi}']. \quad (55)$$

Remark : r.h.s. is $\mathcal{O}(\delta)$, while non-hydrostatic correction is $\mathcal{O}(\delta^2)$, which explains the omission of the latter.

Setup

$\chi \to \epsilon\chi$, $\epsilon \ll 1$, $\gamma^2 \lesssim \epsilon$. Weak coupling : $\delta \sim \epsilon \to$ wave equation in the lowest order $\Rightarrow$

$$\chi_0 = F_+(x+t) + F_-(x-t)$$

Further steps - as before, with modified dispersive correction

Killing the resonances

R.h.s of the wave equation for χ_1 :

$$\mathcal{R}\left[\chi_0\right] \to \left(-\frac{\rho_2}{2\,\rho_1}\mathcal{F}[\ddot{\chi}'] + 3\chi_0'\chi_0'' + 2\dot{\chi}_{0_T}\right) \qquad (56)$$

Killing the resonances $\to$:

$$\pm 2F'_{\pm_T} + 3F'_\pm F''_\pm - \frac{\rho_2}{2\,\rho_1}\mathcal{F}[F'''_\pm] + F_\pm = 0. \qquad (57)$$

Magenta term disappears in the absence of rotation.

Modulation equations

Recalling $\eta = -\chi'$ (lowest order) $\Rightarrow$
(rotating) intermediate long waves (ILW) equations :

$$\left(\mp\eta_{\pm T} + \frac{3}{2}\eta_{\pm}\eta'_{\pm} + \frac{\rho_2}{4\,\rho_1}\mathcal{F}[\eta''_{\pm}]\right)' - \frac{1}{2}\eta_{\pm} = 0. \tag{58}$$

In the limit $H_2 \to \infty$ (rotating) Benjamin-Ono (BO) equations :

$$\left(\mp\eta_{\pm T} + \frac{3}{2}\eta_{\pm}\eta'_{\pm} + \mathcal{H}[\eta''_{\pm}]\right)' - \frac{1}{2}\eta_{\pm} = 0. \tag{59}$$

In the absence of rotation (terms in magenta) both ILW and BO equations admit soliton solutions, and BO is completely integrable.
No soliton solutions in the presence of rotation.

Eulerian 2-layer 1.5D RSW system

$$\partial_t u_1 + u_1 \partial_x u_1 - f v_1 + \rho_1^{-1} \partial_x p_1 = 0 \,, \tag{60a}$$
$$\partial_t v_1 + u_1 (f + \partial_x v_1) = 0 \,, \tag{60b}$$
$$\partial_t u_2 + u_2 \partial_x u_2 - f v_2 + \rho_2^{-1} \partial_x p_2 = 0 \,, \tag{60c}$$
$$\partial_t v_2 + u_2 (f + \partial_x v_2) = 0 \,, \tag{60d}$$
$$\partial_t h_1 + \partial_x (h_1 u_1) = 0 \,, \tag{60e}$$
$$\partial_t h_2 + \partial_x (h_2 u_2) = 0 \,, \tag{60f}$$
$$p_2 = p_1 + g \Delta\rho\, \eta, \; \Delta\rho := \rho_2 - \rho_1 . \tag{60g}$$

Rigid lid : $h_1 + h_2 = H = \text{const} \Rightarrow h_{1(2)} = H_{1(2)} - (+)\eta$.
η - interface displacement, $H_{1,2}$ - unperturbed thicknesses.
Summing up (60e), (60f), and using (51) $\rightarrow$:

$$\partial_x (h_1 u_1 + h_2 u_2) = 0 \Rightarrow h_1 u_1 + h_2 u_2 = H U(t), \tag{61}$$

$U(t)$ - arbitrary function.

Eliminating barotropic pressure

(60a) $h_1 +$ (60c) $h_2 \rightarrow$

$$\frac{\partial p_2}{\partial x} = \left(\frac{h_1}{\rho_1} + \frac{h_2}{\rho_2}\right)^{-1} \Big(f(h_1 v_1 + h_2 v_2) - \frac{\partial}{\partial x}\left(h_1 u_1^2 + h_2 u_2^2\right) - H\dot{U}(t) + g\Delta\rho h_1 \frac{\partial h_2}{\partial x}\Big) \Rightarrow \tag{62}$$

System of 4 equations $h_1 = H - h_2, \quad u_1 = \frac{HU(t) - h_2 u_2}{H - h_2}$:

$$\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} - f v_2 + \frac{\rho_1}{\rho_2 h_1 + \rho_1 h_2}\Big(f(h_1 v_1 + h_2 v_2) - \frac{\partial}{\partial x}\left(h_1 u_1^2 + h_2 u_2^2\right) + \frac{g\Delta\rho}{\rho_1} h_1 \frac{\partial h_2}{\partial x} - H\dot{U}(t)\Big) = 0, \tag{63}$$

$$\frac{\partial h_2}{\partial t} + u_2 \frac{\partial h_2}{\partial x} + h_2 \frac{\partial u_2}{\partial x} = 0, \tag{64}$$

$$\frac{\partial v_2}{\partial t} + u_2 \frac{\partial v_2}{\partial x} + f u_2 = 0, \tag{65}$$

$$\frac{\partial v_1}{\partial t} + u_2 \frac{\partial v_1}{\partial x} + (u_1 - u_2) \frac{\partial v_1}{\partial x} + f u_1 = 0, \tag{66}$$

Semi-Lagrangian form of equations

"Oceanographic" limit : $\frac{\rho_2}{\rho_1} \to 1$. Lagrangian coordinate $X(x,t)$ of a fluid parcel in lower layer. Lagrangian derivative is $\frac{d}{dt} = \frac{\partial}{\partial t} + u_2 \frac{\partial}{\partial x}$. Mass-conservation : $h_{2_i}(x)dx = h_2(X,t)dX$.

$$v_2(x,t) + f\chi(x,t) = v_{2_I}(x). \tag{67}$$

2-layer semi-Lagrangian system, $h = \frac{h_I(x)}{H_2 X'}$:

$$\ddot{X} - f(1-h)(v_I - v_1 - f(X-x)) - \frac{1}{X'}\left[\frac{h(\dot{X}^2 - 2U(t)\dot{X})}{1-h} + \frac{U^2(t)}{1-h}\right]' + g'H(1-h)\frac{h'}{X'} - \dot{U}(t) = 0, \quad = \quad 0, \tag{68}$$

$$\dot{v}_1 - \frac{\dot{X}}{1-h}\left(\frac{v_1'}{X'} + fh\right) + \frac{U(t)}{1-h}\left(\frac{v_1'}{X'} + f\right) = 0, \tag{69}$$

$g' = g\frac{\rho_2 - \rho_1}{\rho_2}$- reduced gravity, prime over g omitted below.

Master equations

Displacements $\chi = X - x \to X' = 1 + \chi', \dot{X} = \dot{\chi}$, "wave" configurations : $v_l = 0, h_l == H_2$. Non-dimensional unperturbed thickness : $D_2 = \frac{H_2}{H} \to h = \frac{D_2}{1+\chi'}$. Equations (69), (69) with $D_1 = 1 - D_2$:

$$\ddot{\chi} + f\left(\frac{D_1 + \chi'}{1+\chi'}\right)(v_1 + f\chi) - \quad -$$

$$\frac{1}{1+\chi'}\left[\frac{D_2(\dot{\chi}^2 - 2U(t)\dot{\chi}) + U^2(t)(1+\chi')}{D_1+\chi'}\right]' \quad + \tag{70}$$

$$\frac{gH}{2}(D_1 + \chi')\left(\frac{D_2}{(1+\chi')^2}\right)' - \dot{U}(t) \quad = \quad 0,$$

$$\dot{v}_1 - \frac{\dot{\chi}}{(D_1+\chi')(1+\chi')}(v_1' + fD_2) \quad +$$

$$U(t)\left(\frac{1+\chi'}{D_1+\chi'}\right)\left(\frac{v_1'}{1+\chi'} + f\right) = 0. \tag{71}$$

Non-dimensional master equations

$\sqrt{gH}$ scaling for U and v_1, and the same as before "wave" scaling for time $\rightarrow$

$$
\begin{aligned}
\ddot{\chi} + \gamma \left(\frac{D_1 + \chi'}{1+\chi'} \right) (v_1 + \gamma \chi) \quad & - \\
\frac{1}{1+\chi'} \left[\frac{D_2(\dot{\chi}^2 - 2U(t)\dot{\chi}) + U^2(t)(1+\chi')}{D_1 + \chi'} \right]' \quad & + \\
\frac{1}{2} (D_1 + \chi') \left(\frac{D_2}{(1+\chi')^2} \right)' - \dot{U}(t) \quad & = \quad 0, \qquad (72)
\end{aligned}
$$

$$
\begin{aligned}
\dot{v}_1 - \frac{\dot{\chi}}{(D_1 + \chi')(1+\chi')} (v_1' + \gamma D_2) \quad & + \\
U(t) \left(\frac{1+\chi'}{D_1 + \chi'} \right) \left(\frac{v_1'}{1+\chi'} + \gamma \right) = 0. \quad & = \quad 0. \qquad (73)
\end{aligned}
$$

No rotation, $\gamma \equiv 0$

Parcel displacementssmall : $\chi \to \epsilon\chi$, as well as global flux $U \to \epsilon U$. Suppose $U = U(T)$, $T = \mathcal{O}(\epsilon^{-1}) \leftrightarrow U$ - velocity of a tide. Master equations uncouple :

$$\ddot{\chi} - \frac{\epsilon}{1+\epsilon\chi'}\left[\frac{D_2\left(\dot{\chi}^2 - 2U(T)\dot{\chi}\right) + U^2(T)(1+\epsilon\chi')}{D_1 + \epsilon\chi'}\right]' - (D_1 + \epsilon\chi')\left(\frac{D_2}{(1+\epsilon\chi')^3}\right)\chi'' - \epsilon\dot{U}(T) = 0. \quad (74)$$

Asymptotic expansion $\chi = \chi_0(x,t,T) + \epsilon\chi_1(x,t,T) + \dots$. Leading order - wave equation for gravity waves at the interface with phase velocity $c = \sqrt{D_1 D_2}$:

$$\ddot{\chi}_0 - D_1 D_2 \chi_0'' = 0 \Rightarrow \quad (75)$$

$$\chi_0 = F_+(x+ct) + F_-(x-ct),$$

$F_\pm$ - arbitrary localized functions.

Modulation equations

Killing resonances in the next order⇒

$$\pm 2\dot{F}_{\pm T} + 2U(T)\frac{D_2}{D_1}\dot{F}'_{\pm} - 2\frac{D_2}{D_1}\dot{F}_{\pm}\dot{F}'_{\pm} + (3D_1D_2 - D_2)F'_{\pm}F''_{\pm} = U_T. \tag{76}$$

Right-moving waves $(\dot{\ldots}) = -c(\ldots)' \Rightarrow$:

$$\dot{F}_{-T} + U(T)\frac{D_2}{D_1}\dot{F}'_{-} + A\dot{F}_{-}\dot{F}'_{-} = \frac{1}{2}U_T, \tag{77}$$

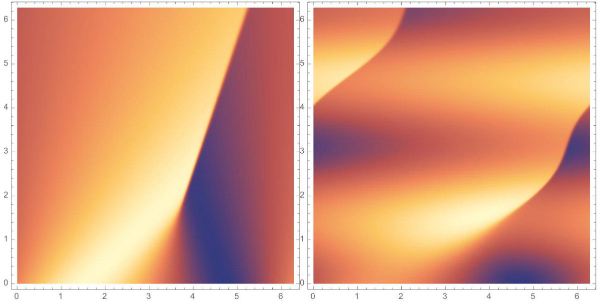
where $A = \left(\frac{3}{2} - \frac{1}{2D_1} - \frac{D_2}{D_1}\right)$. Nglobal flux $U = 0 \to$ Hopf equation. U brings in advection and forcing.Advection by U can be eliminated : $w := \dot{F}_{-} - \frac{D_2}{AD_1}U \to$
Forced Hopf equation with time-dependent forcing.

$$w_T + Aww' = CU_T, \tag{78}$$

$C = \frac{1}{2} + \frac{D_2}{AD_1} = \frac{1}{2D_1}$.

Periodic forcing and shock formation

Forcing does not remove breaking inherent to the Hopf equation, just modulates the propagating shock.



Evolution of initial $w = 1/3 + 2/3 \sin x$ in the $x - t$ plane of unforced (left), and periodically forced (right panel) Burgers equations for w with very small viscosity 0.001 and periodic boundary conditions. Lighter(darker) colors : higher (lower) values of w.

Slow rotation, $\gamma \to \sqrt{\epsilon}\gamma$

Transverse velocity correspondingly small : $v_1 \to \epsilon^{3/2} v_1$.
Master equations (72) and (73) coupled, and in the leading order :

$$\dot{v}_{1_0} - \gamma \left(\frac{D_2}{D_1} \dot{\chi}_0 - \frac{1}{D_1} U \right) = 0 \Rightarrow \tag{79}$$

$U = U(T)$: inconsistent, leads to incurable secular growth in v_1. Reason : constant (in fast time) zonal velocity $U \to$ Coriolis force $\to$ growth of the meridional velocity.Stoppable only by a pressure gradient in y, forbidden in 1.5D setup $\Rightarrow$ only reasonable setting : $U = U(t)$.
New variable $\bar{\chi} = \chi + \mathcal{U}(t)$, leading order in $\epsilon \to$

$$\chi_0 = F_+(x + ct) + F_-(x - ct) - \mathcal{U}(t). \tag{80}$$

$t = \frac{\xi_+ - \xi_-}{2c}$, where $\xi_\pm = (x \pm ct) \to \mathcal{U}(t)$ is a function of both $\xi_\pm \Rightarrow$ terms containing $\mathcal{U}(t)$ in the r.h.s. of the equation for the first correction χ_1 non-resonant, and do not enter the reduced Ostrovsky equations for $F_\pm$.

RMCC system and pressure elimination

$$\partial_t u_1 + u_1 \partial_x u_1 - f v_1 + \rho_1^{-1} \partial_x p_1 + \frac{1}{3h_1} \partial_x \left(h_1^2 (\partial_t + u_1 \partial_x)^2 h_1 \right) = 0 \,,$$

$$\partial_t v_1 + u_1 (f + \partial_x v_1) = 0 \,,$$

$$\partial_t u_2 + u_2 \partial_x u_2 - f v_2 + \rho_2^{-1} \partial_x p_2 + \frac{1}{3h_2} \partial_x \left(h_2^2 (\partial_t + u_2 \partial_x)^2 h_2 \right) = 0 \,,$$

$$\partial_t v_2 + u_2 (f + \partial_x v_2) = 0 \,,$$

$$\partial_t h_1 + \partial_x (h_1 u_1) = 0 \,,$$

$$\partial_t h_2 + \partial_x (h_2 u_2) = 0$$

$$p_2 = p_1 + g \Delta \rho \eta$$

Eliminating $p_{1,2} \to$ (62) with additional terms in big parentheses in its r.h.s.

$$\frac{1}{3} \partial_x \left(h_1^2 (\partial_t + u_1 \partial_x)^2 h_1 \right) + \frac{1}{3} \partial_x \left(h_2^2 (\partial_t + u_2 \partial_x)^2 h_2 \right) . \quad (82)$$

Same addition (82) in big parentheses in (63), other equations the same.

Master momentum equation

Equation for X with SGN-type addition :

$$\ddot{X} - f\,(1-h)\,(v_l - v_1 - f(X-x)) - \quad (83)$$

$$\frac{1}{X'}\left[\frac{h(\dot{X}^2 - 2U(t)\dot{X})}{1-h} + \frac{U^2(t)}{1-h}\right]' \quad +$$

$$g'H\,(1-h)\,\frac{h'}{X'} - \dot{U}(t) + \frac{H}{3X'}\left(\ddot{h}(1-2h)\right)' \quad = \quad 0, \quad (84)$$

Correspondingly,

$$\ddot{\chi} + f\left(\frac{D_1+\chi'}{1+\chi'}\right)(v_1 + f\chi) \quad -$$

$$\frac{1}{1+\chi'}\left[\frac{D_2(\dot{\chi}^2 - 2U(t)\dot{\chi}) + U^2(t)(1+\chi')}{D_1+\chi'}\right]' \quad +$$

$$\frac{gH}{2}\,(D_1+\chi')\left(\frac{D_2}{(1+\chi')^2}\right)' - \dot{U}(t) \quad +$$

$$\frac{\delta^2}{3}\,\frac{D_2}{1+\chi'}\left[\left(\ddot{\frac{1}{1+\chi'}}\right)\frac{D_2-D_1-\chi'}{1+\chi'}\right]' \quad = \quad 0,(85)$$

No rotation

$\delta^2 = \mathcal{O}(\epsilon)$, $U = U(T)$ - safe choice $\Rightarrow$, (77) acquires addition

$$-\frac{1}{3}D_2(D_2 - D_1)\ddot{F}''_{\pm}, \tag{86}$$

KdV with time-dependent forcing in ivariable w :

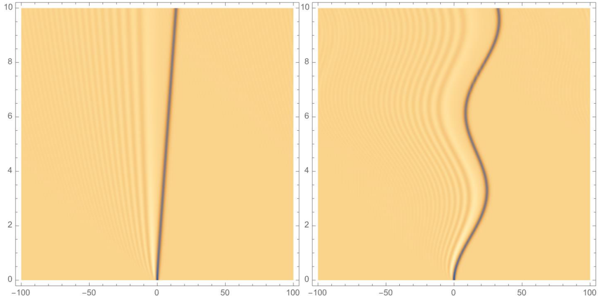
$$w_T + Aww' + Gw''' = CU_T, \tag{87}$$

where $G = -\frac{1}{6}D_2(D_2 - D_1)$.

Slow rotation

Same reasoning as above $\Rightarrow$ only self-consistent choice $U = U(t) \Rightarrow$
Ostrovsky equations with no contributions from U.

Unforced vs periodically forced KDV solutions



Characteristic $(x - t)$ diagrams of the evolution of an initial localized bump in the non-forced (left), and periodically forced (right) KdV equations. Darker color = higher amplitude.

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